SELECTED WORKS OF S.L. SOBOLEV

Volume I: Mathematical Physics, Computational Mathematics, and Cubature Formulas

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Volume I: Mathematical Physics, Computational Mathematics, and Cubature Formulas

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Preface

The Russian edition of this book was dated for the 95th anniversary of the birth of Academician S. L. Sobolev (1908–1989), a great mathematician of the twentieth century. It includes S. L. Sobolev's fundamental works on equations of mathematical physics, computational mathematics, and cubature formulas.

S. L. Sobolev's works included in the volume reflect scientific ideas, approaches, and methods proposed by him. These works laid the foundations for intensive development of modern theory of partial differential equations and equations of mathematical physics, and were a gold mine for new directions of functional analysis and computational mathematics.

The book starts with the paper "Academician S. L. Sobolev is a founder of new directions of functional analysis" by Academician Yu. G. Reshetnyak. It was written on the basis of his lecture delivered at the scientific session devoted to S. L. Sobolev in the Institute of Mathematics (Novosibirsk, October, 2003).

The book consists of two parts. Part I includes selected articles on equations of mathematical physics and Part II presents works on computational mathematics and cubature formulas. All works are given in chronological order.

Part I consists of 11 fundamental works of S. L. Sobolev devoted to the study of classical problems of elasticity and plasticity theory, and a series of hydrodynamic problems that arose due to active participation of S. L. Sobolev in applied investigations carried out in the 1940s.

The first mathematical articles by S. L. Sobolev were written during his work in the Theoretical Department of the Seismological Institute of the USSR Academy of Sciences (Leningrad). Five articles from this cycle are included in this book (papers [1–5] of Part I). These works are devoted to solving a series of important applied problems in the theory of elasticity.

In the first paper included in the volume, S. L. Sobolev solves the classical problem posed in the famous article by H. Lamb (1904) on propagation of elastic vibrations in a half-plane and a half-space. At first, he considers H. Lamb's plane problem, then for this case studies reflection of longitudinal and transverse elastic plane waves from the plane. Using the theory of functions of complex variable, he proposes a method for finding plane waves falling at different angles on the boundary. In particular, he points out a method for finding the Rayleigh waves. Then, using H. Lamb's formulas and applying the method of superposition of plane waves, he gets integral formulas for longitudinal and transverse waves at any internal point of the medium. With these results he studies H. Lamb's space problem.

The next two papers by S. L. Sobolev and his teacher V. I. Smirnov are devoted to more general problems of H. Lamb type. In these articles the authors propose a new method for the study of problems of the theory of elasticity. Using the method, the authors get totally new results in the theory of elasticity and point out a series of problems which can be solved by the method. In the literature the method is known as the method of functionally invariant solutions. The main advantage of the method is that there is no need to use Fourier integrals as did H. Lamb. The method has visual geometric character and allows one to apply the theory of functions of a complex variable. The set of functionally invariant solutions contains important solutions of the wave equation (the Volterra solution, plane waves). This set is closed with respect to reflection and refraction. Using functionally invariant solutions, the authors solve H. Lamb's generalized problem on vibrations of an elastic half-space under the action of a force source inside the half-space. In these papers V. I. Smirnov and S. L. Sobolev obtain formulas for components of displacements at arbitrary point of the space. The authors give a physical interpretation of the obtained formulas. In particular, they conclude that, at infinity, elastic vibrations cause a wave of finite amplitude, and the wave moves with the velocity of the Rayleigh waves.

It should be noted that the first three works are practically unknown to readers because they were published in sources which are difficult to access.

In the paper [4] of Part I the problem on propagation of elastic vibrations in a half-plane and an elastic layer is considered. Unlike all preceding investigations, S. L. Sobolev studies the problem in the case of arbitrary initial conditions. For solving this problem he applies the Volterra method and the method of functionally invariant solutions. The main result of the author is integral formulas for components of displacements at arbitrary points of the medium at any point of time. In particular, the formulas clarify the reason for appearance of the Rayleigh space waves in the general case.

The Smirnov–Sobolev method found numerous applications in subsequent investigations. A review of results obtained by the method at the Seismological Institute of the USSR Academy of Sciences (Leningrad) is given in the paper [5] of Part I.

The paper [6] contains an exhaustive explanation of the Smirnov–Sobolev method of functionally invariant solutions for the wave equation. S. L. Sobolev proves that all functionally invariant solutions to the two-dimensional wave equation can be obtained by this method.

The paper [7] of Part I is devoted to the theory of diffraction of waves on Riemann surfaces. Solving the problem, the author comes to the necessity of using functions which are solutions to the wave equation

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

in a generalized sense. S. L. Sobolev introduces a notion of weak solution of the wave equation. He says that a function u is called a weak solution of the wave equation in a domain D, if the function is the limit of a sequence of classical solutions of the equation in L_1. S. L. Sobolev studies properties of weak solutions and elaborates the method of average functions. Using properties of weak solutions, the author proposes a method for solving the problem of diffraction of waves on Riemann surfaces.

In his subsequent works S. L. Sobolev developed the notion of weak solution, introduced a notion of generalized derivative, defined functional spaces W_p^l called Sobolev spaces, and proved embedding theorems. These works laid the foundations of the modern theory of generalized functions. A series of works devoted to the subject will be included in the next volume of selected works of S. L. Sobolev.

In the paper [8] of Part I, S. L. Sobolev solves the important problem of propagation of a plastic state in an infinite plane, with a circular hole, exposed to the action of symmetrical forces causing displacements on the boundary. S. L. Sobolev indicates the method of computation of all quantities characterizing the motion, i.e., the displacement components at any point of time in the plastic and elastic zones, the stress tensor components in both zones, and the flow lines in the plastic zone.

The last three papers [9–11] of Part I are devoted to the problem of small oscillations of a rotating fluid. The problem is classical. The study of this problem began with the famous article “Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation” by H. Poincaré (1885). Papers [9, 10] contain results of investigations carried out by S. L. Sobolev in the 1940s.

In the paper [9] S. L. Sobolev considers a system of partial differential equations of the form

$$\begin{aligned} &\frac{\partial \overrightarrow{v}}{\partial t} - [\overrightarrow{v} \times \mathbf{k}] + \nabla p = \overrightarrow{F}, \\ &\operatorname{div} \overrightarrow{v} = g. \end{aligned} \tag{1}$$

This system arises when studying small oscillations of a rotating ideal fluid. The main aim of the author is to research the Cauchy problem, the first and second boundary value problems for system (1) in a bounded domain. Using methods of functional analysis developed by him, S. L. Sobolev proves well-posedness of the problems, and proposes a method for construction of solutions. He establishes also a close connection between system (1) and the non-classical equation

$$\varDelta \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = f. \tag{2}$$

By the method of potentials, S. L. Sobolev obtains explicit formulas of solutions to the Cauchy problem for system (1) and for equation (2).

System (1) can be written as

$$A_0 \frac{\partial}{\partial t} \begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_x \frac{\partial}{\partial x} \begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_y \frac{\partial}{\partial y} \begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_z \frac{\partial}{\partial z} \begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + B \begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} = \begin{pmatrix} \overrightarrow{F} \\ g \end{pmatrix},$$

where the matrix A_0 is singular, i.e., system (1) is not a Cauchy–Kovalevskaya system. Probably, equations and systems not solvable with respect to the highest-order derivative were first studied by H. Poincaré (1885). Subsequently, they were considered in a number of articles by mathematicians and mechanicians. This was connected initially with research into certain hydrodynamics problems. Particularly, the most intense interest in equations and systems not solvable with respect to the highest-order derivative arose in connection with the investigation of the Navier–Stokes system by C. W. Oseen (1927), F. K. G. Odqvist (1930), J. Leray and J. Schauder (1934), E. Hopf (1950) and the study of the problem on small oscillations of a rotating fluid by S. L. Sobolev. The paper [9] was one of the first deep investigations of equations and systems not solvable with respect to the highest-order derivative. This paper originated intense research into such equations and systems. At present, system (1) is called the Sobolev system, equation (2) is called the Sobolev equation in the literature.

The paper [10] was written by S. L. Sobolev in 1943, but it was published only in 1960. In the work he considers the problem of stability of motion of a heavy symmetric top with a cavity filled with a fluid. It is assumed that the top rotates around its axis, and its foot is immovable. The author reduces the research into stability of motion to solving the differential equation

$$\frac{dR}{dt} = iBR + R_0,$$

where B is a linear operator self-conjugate with respect to a Hermitian form Q. This form depends on parameters characterizing mechanical properties of the shell and the fluid. It is interesting that the form Q can be positive definite or indefinite depending on values of the parameters. Since solutions to the equation are written by means of the resolvent of the operator B, the author studies the solutions in a space with indefinite metric.

It should be noted that the theory of differential equations in spaces with indefinite metric began to develop in the 1940s. Therefore the paper [10] is one of the first works in this direction.

The main results of the paper [10] follow from established properties of the resolvent of the operator B in a space with inner product defined by the form Q. In particular, if the form Q is positive definite, then the motion is stable; if the form Q is indefinite, then the motion can be unstable. S. L. Sobolev studies in detail the cases when the cavity filled with the fluid has the form of an ellipsoid or a cylinder. The author points out angular velocities under which the motion is stable, and he describes the cases when the resonance phenomenon is to be observed.

The paper [11] is a survey of S. L. Sobolev's lecture delivered at the International Symposium on Applied Analysis and Mathematical Physics (Cagliari-Sassari, Italy, 1964). In the paper he discusses mathematical problems connected with the research into system (1) and equation (2) arising when studying small oscillations of a rotating ideal fluid. The study of properties of solutions of system (1), equation (2) and more general equations began with appearance of S. L. Sobolev's famous article (1954; see [9] of Part I). In his lecture he gives a survey of results obtained in this direction for the past 10 years. In particular, he notes a series of unexpected results on spectral properties of operators generated by the problems. From the results it follows that, as a rule, solutions to the boundary value problems have non-compact trajectories. S. L. Sobolev points out that asymptotic properties of solutions depend essentially on domain geometry. On the other hand, in the case of a boundary time interval, S. L. Sobolev proves that solutions of many boundary value problems depend continuously on deformations of the domain boundary. He notes also some connections of many boundary value problems with various problems of mathematical analysis and other problems of partial differential equations. He emphasizes that the class of the problems under discussion is at an initial stage of study.

Part II of the book includes 29 articles on computational mathematics and cubature formulas. It starts with an early paper which is devoted to the Schwartz method for approximate solution of boundary value problems for partial differential equations of elasticity theory. The next five works were written by S. L. Sobolev as part of his active participation in applied investigations carried out in the Soviet Union in the 1940-50s. These articles are devoted to computational methods in difference and integral equations, and problems of approximation of linear operators. In these works S. L. Sobolev actively advocated the use of functional analysis in computational mathematics, and pointed out close interconnections between computational mathematics, differential equations and functional analysis. He emphasized that the use of computers for solving complex applied problems will be more effective under active collaboration of mathematicians and engineers.

A noticeable place in the scientific legacy of S. L. Sobolev is occupied by his contributions to the theory of approximate multidimensional integration which were accomplished during his stay of 25 years in Novosibirsk. His first article in this direction was published in 1961 and the last in 1985 and there are two dozen of these papers in this volume. In these papers S. L. Sobolev mainly pursue a functional-analytical approach. This implies that, first, the integrands are combined in a Banach space and, second, the difference between the integral and the approximative combination of the values of the integrand is treated as the result of applying some linear functional. This functional, called the error of a cubature formula, is usually continuous. Knowledge of the value of its norm allows us to derive guaranteed estimates for the accuracy of the cubature formula under study on the elements of the chosen space. In addition to describing the construction of the formulas under consideration, i.e., indicating their nodes and weights or algorithms for their determination, the functional-analytical approach implies the study of the norms of the respective errors in a chosen Banach space. In particular, two-sided estimates for these norms are derived. In papers [7, 8, 9] of Part II S. L. Sobolev addresses the main problems of the theory of cubature formulas and the theory of interpolation.

In the theory of cubature formulas, a term coined by S. L. Sobolev, four principal directions are specified. All are exposed in the present edition.

The first in chronological order of the directions consists in studying the cubature formulas in three-dimensional space which possess high polynomial degree and are invariant under the action of the rotation group of some regular polyhedron. The requirement that a cubature formula with fixed nodes be exact for polynomials up to a certain degree reduces the problem of constructing the weights of the formula to solution of a system of linear equations. The higher the desired order is and the larger are the number of nodes, the greater becomes the size of this system. However, in the case when the integration domain possesses some symmetry and we use an invariant cubature formula for approximate integration, it is possible to diminish substantially the size of the system to be solved. Papers [10, 11] of Part II address the question of how to achieve this.

The second direction in the theory, which seems to be most advanced, consists in studying asymptotically optimal cubature formulas on the spaces of functions of finite smoothness (papers [12, 14, 16, 18, 23] of Part II). In this respect S. L. Sobolev himself considered the Hilbert $L_2^{(m)}$ spaces. The construction of a regular boundary layer which he proposed makes it possible to find the weights of a cubature formula with arbitrarily many nodes by solving only a few standard systems of linear equations of size depending only on the order m (papers [13, 18, 21] of Part II). The central place in this direction is occupied by derivation of an asymptotic expansion of the $L_2^{(m)*}$ norm of an error with regular boundary layer. The expansion contains two summands. The first is written explicitly via the so-called generalized Bernoulli numbers, whereas the second is negligible as compared with the first, provided that the small mesh-size h of the lattice of integration is sufficiently small. The expansion implies that the norm of an error with regular boundary layer decreases like h^m as $h \to 0$. It is a rather deep analytical fact enabling us to give not an algebraic but rather a functional-analytical definition of the order of a cubature formula on some function class (paper [29] of Part II).

The expansion of the $L_2^{(m)*}$ norm of an error with regular boundary layer gives solid grounds for choosing a numerical integration formula with nodes comprising a lattice. Indeed, given N nodes, we may pose the problem of finding a cubature formula whose error has $L_2^{(m)*}$ norm minimal, with the minimum taken over not only the weights but also the nodes of the formula.

However, the ratio of the $L_2^{(m)*}$ norm of the error of such an optimal formula to the $L_2^{(m)*}$ norm of the error with regular boundary layer and the same number N of nodes is bounded from below by a positive quantity independent of N. This is immediate from the Bakhvalov Theorem (paper [14] of Part II). Increasing N, we could however hardly expect large gain from using formulas with arbitrary disposition of nodes instead of those with nodes comprising a parallelepipedal lattice. Moreover, to optimize a formula over nodes is a difficult problem involving solution of simultaneous nonlinear equations of high order. This is in sharp contrast to the formulas with regular boundary layer whose nodes are explicit and need no calculation at all.

Note that the theory of formulas with regular boundary layer actually presents the function summation problem pertinent to the calculus of finite differences. From this point of view, every cubature formula with regular boundary layer is a multidimensional analog of the classical quadrature formula of Gregory. Constructing such a cubature formula, we thus take account of the behavior of an integrand near to the boundary of the integration domain by especially selecting the weights of the formula at the nodes belonging to some boundary layer. All remaining weights coincide.

Remarkable is the method proposed by S. L. Sobolev for finding the norm of an error $l(x)$ and his use of the concept of extremal function $u(x)$ (papers [7, 12, 14, 15] of Part II). Such function is considered as a weak solution to the many-dimensional polyharmonic equation with a special right side

$$\Delta^m u(x) = (-1)^m l(x).$$

A solution to this equation on the real axis is a piecewise-polynomial function of the class $W_2^{(m)}$, i.e., a spline. In many dimensions, this approach enabled S. L. Sobolev to apply the methods he invented in the theory of partial differential equations to study of the classical problems of analysis.

The third direction of the theory comprises the S. L. Sobolev contribution to cubature formulas on the classes of infinitely differentiable functions (papers [17, 22, 29] of Part II). As such he considered the spaces of periodic functions of many variables with prescribed behavior of the integral norms in the $L_2^{(m)}$ spaces as m tends to infinity. The classification he proposed embraces the conventional spaces of entire functions of given type and order, spaces of analytic functions and the Gevrey classes containing quasianalytic functions. Considering the action on this space of the error of a lattice formula with equal weights, S. L. Sobolev obtained an asymptotic expansion of the logarithm of the norm of the error. In exact analogy with the case of the spaces of finite smoothness, the respective formula comprises two summands. One of them is explicitly expressed through the parameters of the initial class, whereas the other is negligible as compared with the first at a small mesh-size h. This research demonstrated in particular that a noteworthy effect accompanies the transition from functions of finite smoothness to infinitely differentiable functions. Namely, the norm of the error of a cubature formula, decreasing not faster than some power of the lattice mesh-size in the first case, decreases exponentially in the second case. S. L. Sobolev suggested that in the second case the order of a cubature formula be assumed infinite. More exactly, a cubature formula possesses infinite order in a Banach space provided that the norm of the corresponding error in the dual space vanishes faster than any degree of the mesh-size of the integration lattice. S. L. Sobolev exhibited one example of the sort in the case of many dimensions.

Finally, the fourth direction of the theory comprises the S. L. Sobolev research in $L_2^{(m)}$-optimal lattice cubature formulas (papers [20] and [24] of Part II). A central place is occupied here by description of some analytic algorithm for determining weights of such formulas. To this end, S. L. Sobolev defined and studied a special finite-difference operator whose action on a function of a discrete argument may be written as convolution with a special kernel in analogy with the action of the polyharmonic operator Δ^m on a continuously differentiable function (paper [19] of Part II).

The problem of calculating the convolution kernel for an arbitrary m turns out rather involved. It was partly solved in the one-dimensional case: here a formula is available expressing the desired values through the roots of the Euler–Frobenius polynomials of degree $2m$. The weights of optimal formulas are conveniently treated as the values at the appropriate points of some compactly-supported function of a many-dimensional discrete argument. This function happens to satisfy a linear finite-difference equation with a special right side. Applying to this right side a discrete convolution analog of the polyharmonic operator, S. L. Sobolev obtained an analytical formula for the sought weights (paper [24] of Part II). To use it in the one-dimensional case, he revealed many properties of the roots of the Euler–Frobenius polynomials (papers [25–28] of Part II). In particular, he obtained asymptotic formulas for the roots of these polynomials. The results by S. L. Sobolev on the weights of optimal cubature formulas generalized some results by A. Sard, I. Meyers, I. Schoenberg and S. Silliman derived by the method of splines.

The method of S. L. Sobolev for studying cubature formulas is deeply rooted in such fields of theoretical mathematics as mathematical analysis, the theory of differential equations and functional analysis. At the same time, the specific subject of research, a cubature formula for approximate integration, is traditionally ascribed to numerical analysis which the modern computational mathematics stems from. As a result, a theory has emerged which has undeniable import for applications. This order of events seems by far not random but rather an inevitable phenomenon of modern mathematics.

We would like to say a few words about selected works of S. L. Sobolev. In 2001 the Scientific Council of the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences (Novosibirsk) made a decision to publish selected works of Academician S. L. Sobolev in many volumes. An editorial board was formed, consisting of Academician Yu. G. Reshetnyak, Prof. G. V. Demidenko, Prof. S. S. Kutateladze, Prof. V. L. Vaskevich, and Prof. S. K. Vodop'yanov. As mentioned above, the

Russian edition of the first volume came out in 2003. Prof. G. V. Demidenko and Prof. V. L. Vaskevich are the editors of this volume. The second volume will be published in Russian in 2006. It will include fundamental works of S. L. Sobolev on functional analysis and differential equations. The editors of the second volume are Prof. G. V. Demidenko and Prof. S. K. Vodop'yanov.

Selecting S. L. Sobolev's works for the first volume, the editors used the chronology of his works. It was composed by V. M. Pestunova and published in the Sobolev Institute of Mathematics in 1998. A big help in search of early works of S. L. Sobolev was given by the employees of the library of the Sobolev Institute of Mathematics: L. G. Gulyaeva, L. A. Mikuta, and V. G. Ponomarchuk.

Many people actively participated in the preparation of the manuscript: members of the Sobolev Institute of Mathematics L. V. Alekseeva and Dr. I. I. Matveeva; members of the Lavrentiev Institute of Hydrodynamics Prof. N. I. Makarenko and Dr. A. E. Mamontov; students of Novosibirsk State University L. N. Buldygerova, V. G. Demidenko, Yu. E. Khropova, T. V. Kotova, A. A. Kovalenko, M. A. Kuklina, A. V. Mudrov, A. M. Popov, and E. A. Samuilova.

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Novosibirsk,
December 2005

Gennadii Demidenko
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Academician S. L. Sobolev is a Founder of New Directions of Functional Analysis

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S. L. Sobolev – one of the most prominent mathematicians of the 20th century – was born on October 6, 1908 in Petersburg. His father, Lev Alexandrovich Sobolev, was a public attorney. Lev Alexandrovich studied at Petersburg University, but was expelled because of his participation in the revolutionary movement and sent to the army as a soldier. Afterwards, he passed, as external student, the state examinations at the Law Department of Kharkov University. Sergei Sobolev's paternal grandfather was a hereditary Siberian kazak.

It was in his early youth when Sergei Sobolev lost his father; he was brought up by his mother, Natalia Georgievna, a most educated woman, teacher of literature and history. Natalia Georgievna also had a second specialty: she graduated from a Medical Institute and worked as associate professor at the First Leningrad Medical Institute. She inculcated in Sergei Sobolev such personality features as fidelity to principle, honesty and purposefulness, which characterized him as scientist and person.

Sergei Sobolev mastered the high school program by himself, being particularly fond of mathematics. In the years of the Civil War, he lived in Kharkov with his mother. There he studied for one semester at preparatory courses to a labor technical night school. By 15 years of age, he knew the complete course of mathematics, physics, chemistry, and other sciences according to the high school curriculum, had read many books of classic Russian and foreign literature as well as books on philosophy, medicine, biology, etc. Having moved from Kharkov to Petrograd in 1923, Sergei Sobolev was enrolled in the final school year of School 190 and finished it with excellence in 1924. After finishing school, he could not enter a university because of his young age (he was under 16), so he began to study at the First State Art Studio, in a piano class.

In 1925, Sergei Sobolev entered the Physics and Mathematics Department of Leningrad State University, proceeding with his studies at the Art Studio. In Leningrad State University, he attended lectures by Professors N. M. Gyunter, V. I. Smirnov, G. M. Fikhtengol'ts and others. He wrote his diploma thesis on analytical solutions of a system of differential equations with two independent variables under the academic supervision of Prof. N. M. Gyunter.

In those years, Leningrad University was a major scientific mathematical center, which retained the remarkable traditions of the Petersburg mathematical school. It was famous for its great discoveries in mathematics and connected with the names of P. L. Chebyshev, A. M. Lyapunov, and A. A. Markov.

After graduating in 1929 from Leningrad University, Sergei Sobolev started to work in the Theoretical Department of Leningrad Seismological Institute under the direction of V. I. Smirnov. In that period, in close cooperation with V. I. Smirnov, they solved a number of fundamental mathematical problems in the wave transmission theory.

Since 1932, S. L. Sobolev worked in V. A. Steklov Mathematical Institute in Leningrad, and since 1934 – in Moscow. On February 1, 1933, when he was not yet 25 years old, he was elected a corresponding member of the USSR Academy of Sciences. He became a full member of the USSR Academy of Sciences on January 29, 1939. In 1941, for his works in mathematical theory of elasticity, S. L. Sobolev was awarded with the State Prize of the 1st Degree. During the Great Patriotic War, V. A. Steklov Mathematical Institute was evacuated in Kazan, and for a short period, from 1941 to 1943, S. L. Sobolev was the director of this institute. Since 1943, he worked in the institute headed by I. V. Kurchatov, which was then called the Laboratory of Measuring Instruments of the USSR Academy of Sciences (now I. V. Kurchatov Institute of Atomic Energy). He kept on working as research fellow at this institute before leaving for Novosibirsk.

S. L. Sobolev is known worldwide as a prominent mathematician and author of outstanding research works on the theory of differential equations, computational mathematics, and functional analysis. He gave rise to the wave transmission theory. He developed the theory of generalized functions as functionals on a set of smooth compactly-supported functions. On the basis of this theory, he defined the concept of a weak solution of a partial differential equation. S. L. Sobolev introduced new function spaces and proved embedding theorems for them (Sobolev spaces, Sobolev embedding theorems). He laid the foundations of the spectral theory for operators in spaces with indefinite metric in connection with studying solutions of hydrodynamic systems of rotating fluid. He made a significant contribution to the development of computational mathematics: he introduced the important concept of computational algorithm closure and constructed the theory of cubature formulas. He organized at Moscow University the country's first Chair of Computational Mathematics.

S. L. Sobolev was a forward-thinking man and a socially active person. For example, he vigorously supported cybernetics and mathematical economics when these schools of thought were victimized; he advocated protection of the unique ecosystem of the Baikal Lake. It is hard to enumerate all the important achievements that he attained.

S. L. Sobolev was involved in applied scientific projects that were highly important state matters – he developed mathematical support for the USSR nuclear project while working as deputy director for I. V. Kurchatov in the Measuring Instrument Laboratory.

S. L. Sobolev had great authority in world-wide science. He was elected an international member of the French Academy of Sciences, Accademia Nazionale dei Lincei in Roma, Berlin Academy of Sciences, Edinburgh Royal Society, honorary doctor of Charles University in Prague, honorary doctor of Humboldt University in Berlin, honorary doctor of Higher School of Architecture and Construction in Weimar, honorary member of Moscow and American Mathematical Societies.

The services S. L. Sobolev rendered to science and our country were highly valued and he was awarded with numerous orders and prizes even before his arrival in Novosibirsk – in the Siberian Division of the USSR Academy of Sciences. For the works done at the I. V. Kurchatov Institute of Nuclear Power, S. L. Sobolev was conferred the honorary title of Hero of Socialist Labor, decorated with several Lenin Orders and many other decorations of the Soviet government.

In 1957, Academician S. L. Sobolev together with Academicians M. A. Lavrentiev and S. A. Khristianovich became one of the three founders of the Siberian Division of the USSR Academy of Sciences.

S. L. Sobolev was the founder and director of the Institute of Mathematics of the USSR Academy of Sciences. He held the position of director from 1957 to 1983 when, after celebration of his 75th birthday, he left to go to Moscow to work at the Steklov Mathematical Institute. In 1988, he was put forward for a M. V. Lomonosov Gold Medal of the USSR Academy of Sciences.

In the last years of his life, S. L. Sobolev was seriously ill, and he passed away on January 3, 1989. The M. V. Lomonosov Gold Medal of the USSR Academy of Sciences was awarded to him posthumously in 1989.

One of the main achievements of S. L. Sobolev in mathematics was construction of the theory of generalized functions, one of the most important directions of modern functional analysis, and creation of the theory of functions with generalized derivatives. In the literature these spaces are called Sobolev spaces. These two directions in the scientific research of S. L. Sobolev appear as one whole.

As a separate direction of mathematics, functional analysis had been formed at the end of the 19th, beginning of the 20th centuries. The creation of set theory and based on it general (set theoretic) topology and the theory of functions of a real variable created favorable circumstances for functional analysis. The appearance of functional analysis was an answer to certain questions of theoretical mathematics, possibly even implicitly stated, and its applications. In applications it is often important to know the conditions not only in the particular example, but rather for all problems of a certain class.

The need for development of research methods, not for particular functions or equations, but for entire classes of functions and equations, had led to the creation of functional analysis. The role of functional analysis in modern mathematics is by no means complete with this description.

The applications of functional analysis to problems of the theory of partial differential equations were already known before works of S. L. Sobolev. In this connection, we can indicate, for example, the famous D. Hilbert's works devoted to the validation of the Dirichlet principle for the Laplace equation. By virtue of S. L. Sobolev's investigations, functional analysis become a universal method for solving problems of mathematical physics.

In the 1920–30s, many scientists working in the theory of partial differential equations concentrated their efforts in order to understand what is a weak solution of a differential equation, and, in particular, how to extend the notion of the derivative of a function, so it would satisfy all needs of the theory of partial differential equations.

The most effective and, I would say, the most spectacular way of solving this problem was indicated by S. L. Sobolev. He noticed that any locally summable function of n variables generates a certain functional on the space of smooth compactly-supported functions. If one identifies the function with this functional, then it becomes possible to extend on locally integrable functions various operations performed on smooth functions by means of an adjoint operator.

The basics of the theory of generalized functions were presented briefly by S. L. Sobolev in his note in the journal "Doklady Akademii Nauk SSSR" (1935). The complete presentation was given in the article of S. L. Sobolev "Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales" (A new method of solving the Cauchy problem for linear normal hyperbolic equations. Mat. Sb., **1**, 39–72 (1936)). The Russian translation of this article is also given in the last edition of the book by S. L. Sobolev "Some Applications of Functional Analysis in Mathematical Physics", edited by O. A. Oleinik and published in 1988, with comments by V. I. Burenkov and V. P. Palamodov.

The basic ideas and constructions of the theory of generalized functions contained in S. L. Sobolev's articles appear in the modern theory practically without any changes. Let us point out the most important ideas.

1. A generalized function is defined as a functional on the space of smooth compactly-supported functions.

2. Linear differential operators in the space of generalized functions are introduced in the form of adjoints to the corresponding linear differential operators on the space of smooth compactly-supported functions.

3. The generalized functions are classified in the order of their singularity (in terms of S. L. Sobolev, by a class).

4. The regularization of generalized functions by means of convolution and approximation of an arbitrary generalized function by infinitely differentiable functions.

5. The flexible manipulation of spaces of test and generalized functions, defined by various conditions imposed on supports of test and generalized functions.

6. Reducing the Cauchy problem to a problem with a nontrivial right-hand side without initial conditions by transforming the initial conditions into sources of delta function type.

Let Ω be a domain, i.e., a connected open set in the space R^n. The function φ defined in Ω is called *compactly-supported*, if there exists a compact set $S_\varphi \subset \Omega$ such that $\varphi(x) = 0$ for $x \notin S_\varphi$. There is the smallest set among compact sets satisfying this condition. It is called the *support of the function* φ. Further we assume that S_φ is the support of the function φ. We say that the function $\varphi : \Omega \to R$ belongs to the class $C_0^r(\Omega)$, if it is compactly-supported and has all partial derivatives of order r in Ω, and all these derivatives are continuous. The symbol $C_0^\infty(\Omega)$ denotes the set of all functions φ belonging to the class $C_0^r(\Omega)$ for any $r \geq 1$.

The class $C_0^r(\Omega)$ is a vector space. We will consider linear functionals on the spaces $C_0^r(\Omega)$. The value of a functional f on a function $\varphi \in C_0^r(\Omega)$ is denoted by the symbol $\langle f, \varphi\rangle$. In the space $C_0^r(\Omega)$, a certain topology is introduced (I do not describe it in detail, referring instead to the book by S. L. Sobolev "Some Applications of Functional Analysis in Mathematical Physics"). A generalized function is a functional continuous in this topology.

In the work of S. L. Sobolev mentioned above (Mat. Sb., **1**, 39–72 (1936)) the generalized functions are simply called functionals. The term "generalized function" appeared later. French mathematician Laurent Schwartz used the term "distribution" to denote this object.

Let us present certain examples. They are significant for the theory of generalized functions.

1. Suppose that $f : \Omega \to R$ is an arbitrary measurable function in $L_{1,loc}(\Omega)$. The function f for every r defines on the space $C_0^r(\Omega)$ the linear functional $\widetilde{f}$ by the formula

$$\langle \widetilde{f}, \varphi\rangle = \int\limits_\Omega f(x)\varphi(x)\,dx.$$

The functional $\widetilde{f}$ is continuous in $C_0^r(\Omega)$ in the sense of the definition given above, and, hence, it is a certain generalized function.

The functional $\widetilde{f}$ defines the function f uniquely up to values on the set of measure zero. (This statement is known from the calculus of variations under the name of the Du Bois–Reymond lemma.) After S. L. Sobolev, in what follows, we identify the function $f \in L_{1,loc}(\Omega)$ with the functional $\widetilde{f} \in \mathcal{D}(\Omega)$. Therefore, I simply write f instead of $\widetilde{f}$. Thus, we obtain an embedding of $L_{1,loc}(\Omega)$ to the space $\mathcal{D}^r(\Omega)$ of linear functionals over the vector space $C_0^r(\Omega)$ for each integer $r > 0$. Thus, any function from the class $L_{1,loc}(\Omega)$ is a generalized function.

Similarly to this example, the notation $f(x)$ is used in the literature for any generalized function. According to this, instead of $\langle f, \varphi\rangle$ one uses the expression

$$\int_{\Omega} f(x)\varphi(x)\,dx.$$

2. Let $\Omega = R^n$ and let a be an arbitrary point in R^n. By the symbol $\delta(x-a)$ we denote the generalized function such that for any function $\varphi \in C_0^r(R^n)$ the following equality holds:

$$\int_{\Omega} \delta(x-a)\varphi(x)\,dx = \varphi(a).$$

We say that $\delta(x-a)$ is a δ-function concentrated at the point a of the space R^n. The notion of δ-function was introduced by Dirac and used in theoretical physics before the work of S. L. Sobolev.

Dirac defined $\delta(x-a)$ as the usual function such that $\delta(x-a) = 0$ for $x \neq a$, $\delta(0) = \infty$ and

$$\int_{R^n} \delta(x-a)\,dx = 1.$$

From a mathematical standpoint, the definition of Dirac is nonsense, even though its physical content is absolutely clear. For example, the Dirac δ-function is the unit mass concentrated in an arbitrarily small domain.

3. Let Ω be a domain in R^n. The symbol $\mathcal{B}_0(\Omega)$ denotes the union of all Borel sets $A \subset \Omega$, whose closures are compact and also contained in Ω. Let $\mu : \mathcal{B}_0(\Omega) \to R$ be a countably additive set function defined on the union of the sets $\mathcal{B}_0(\Omega)$. Then for any function $\varphi \in C_0^r(\Omega)$, $r \geq 1$, the following integral is defined:

$$\langle d\mu, \varphi\rangle = \int_{\Omega} \varphi(x)d\mu(x).$$

The set function μ is defined uniquely by the functional $d\mu$. Obviously, the notion of the δ-function is a particular case of the given example.

The generalized function $f(x)$ is called nonnegative in the domain Ω if for any nonnegative function $\varphi \in C_0^r(\Omega)$ the following inequality holds:

$$\int_{\Omega} f(x)\varphi(x)\,dx \geq 0.$$

The following statement can be easily proved: if the generalized function $f(x)$ is nonnegative, then $f = d\mu$, where μ is a nonnegative countably additive set function defined in Ω.

Let us show how the operations on usual functions are extended onto generalized functions. We use the example of differentiation for this.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be an n-dimensional multiindex, i.e., the vector in R^n, whose components are nonnegative integers. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and denote by the symbol D^α the operator of differentiation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

First we consider the case $n = 1$. Let Ω be an interval $(a, b) \subset R$, and let $f(x)$ be a function defined in Ω from the class C^r, i.e., it has a continuous derivative of order r at every point of this interval. Applying the rule of integration by parts, we obtain that for any function $\varphi \in C_0^r(\Omega)$ the inequality holds,

$$\int_a^b f^{(r)}(x) \varphi(x)\, dx = (-1)^r \int_a^b f(x) \varphi^{(r)}(x)\, dx.$$

Hence, by applying the Fubini theorem, we conclude that if Ω is a domain in R^n, and the function f belongs to the class $C^m(\Omega)$, $m = |\alpha|$, then for any function φ from the class $C_0^r(\Omega)$, $r \geq m$, the following equality holds:

$$\int_\Omega D^\alpha f(x)\, \varphi(x)\, dx = (-1)^m \int_\Omega f(x) D^\alpha \varphi(x)\, dx.$$

This equality presents a scheme to show how one can define the notion of a generalized derivative for arbitrary generalized functions.

If $f(x)$ is a generalized function in a domain Ω of the space R^n, then its derivative $D^\alpha f(x)$ is a linear functional, whose action on smooth functions is defined by the rule

$$\int_\Omega D^\alpha f(x)\, \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \varphi(x)\, dx.$$

By this definition, any generalized function has any derivative of any order.

Let us consider the simplest wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Any solution of this equation can be represented in the form

$$u(x, t) = f(x - at) + g(x + at). \tag{2}$$

To substitute the function $u(x, t)$ defined by (2) in equation (1) the functions f and g must have second order derivatives.

Each term in (2) has certain physical meaning. By (2), the function $u(x, t)$ is represented as a sum of two waves, one wave moves in one direction, and the other one moves in the opposite direction. The requirement of second order differentiability of the functions f and g is not much justified physically. The question of how to understand the solution of the wave equation was the subject of discussions among mathematicians already in the 18th century. In particular, they suggested to take any function of form (2) as a solution of the equation for any functions f and g.

For any locally integrable functions f and g, the function $u(x,t)$ defined by (2) always satisfy the wave equation under the condition that the derivatives in this equation are understood in the sense of the theory of generalized functions.

The definition given by S. L. Sobolev allows one to correct also "transgressions" of physicists, related to the δ-function, namely, to give a rigorous definition of the derivative of the δ-function. According to the definition of S. L. Sobolev, the derivative $D^\alpha\delta(x-a)$ is the generalized function such that the equality

$$\langle D^\alpha\delta, \varphi\rangle = (-1)^{|\alpha|}D^\alpha\varphi(a)$$

holds for any compactly-supported function from the corresponding class of smoothness.

It is necessary also to note the ingenious construction invented by S. L. Sobolev in order to smooth functions and generalized functions. This method allows one to approximate an arbitrary generalized function by functions from the class C^∞.

To illustrate this, let us indicate certain simple applications of the notions introduced by S. L. Sobolev.

The criterion of the monotonicity of a function, defined on a certain interval of the real line, is usually formulated in courses of differential calculus in the following way. If the function $f:(a,b)\to R$ is differentiable at each point of the interval (a,b), then it is increasing if and only if its derivative is always nonnegative. The theory of generalized functions allows one to remove the requirement of differentiability, more precisely, to replace it by a significantly weaker requirement of local integrability.

A locally integrable function $f:(a,b)\to R$ is increasing if and only if its derivative, as a generalized function, is nonnegative.

Similarly, a function that is locally integrable on the interval (a,b) is convex if and only if its second derivative is a nonnegative generalized function.

Let us also indicate that the condition: the function $f:(a,b)\to R$ is absolutely continuous, is equivalent to the condition: the function f is locally integrable and its derivative, as a generalized function, is a locally integrable function.

S. L. Sobolev also constructed the theory of classes of functions with generalized derivatives, the so-called spaces $W_p^l(\Omega)$. In the literature these spaces are called Sobolev spaces. For applications of functional analysis to mathematical physics, besides the general principles, it is necessary to have large sets of Banach spaces that can be used in problems of mathematical physics. The spaces $W_p^l(\Omega)$ provide such sets.

Let Ω be a domain in R^n, let $l \geq 1$ and $p \geq 1$ be real numbers such that l is the integer, and let f be a generalized function defined in Ω. We say that f belongs to the class $W_p^l(\Omega)$, if all its derivatives D^α, $|\alpha| \leq l$, belong to the class $L_p(\Omega)$. Naturally, these derivatives are understood in the sense of the definition given above.

S. L. Sobolev built a theory of the classes $W_p^l(\Omega)$. These functional classes have become the object of careful attention of many researches. At the same time, the techniques of studying such functions and methods proposed by S. L. Sobolev were universally recognized; they continue to be applied in many various studies.

S. L. Sobolev had constructed integral representations of the functions from the classes $W_p^l(\Omega)$ and studied different norms of the classes $W_p^l(\Omega)$. He showed that these classes form Banach spaces. Here, the main result of S. L. Sobolev is embedding theorems establishing connections between these spaces.

Let us make some statements.

Theorem 1. *Let Ω be a bounded domain in the space R^n with a boundary satisfying certain conditions of geometrical nature. If $lp > n$, then any function $f \in W_p^l(\Omega)$ is continuous. Moreover, the following inequality holds:*

$$\|f\|_{C(\Omega)} \leq M\|f\|_{W_p^l(\Omega)},$$

where $M = M(l, p, n, \Omega)$ is a positive constant.

Theorem 2. *Let Ω be a bounded domain in the space R^n with a boundary satisfying certain conditions of geometrical nature. If $lp \leq n$, then any function $f \in W_p^l(\Omega)$ for any q such that $1 \leq q < \frac{np}{n-lp}$ belongs to the class $L_q(\Omega)$. Moreover, the following inequality holds:*

$$\|f\|_{L_q(\Omega)} \leq M\|f\|_{W_p^l(\Omega)},$$

where $M = M(l, p, q, n, \Omega)$ is a positive constant.

The conditions on the boundary of the domain Ω, indicated by S. L. Sobolev in these theorems, have quite general character. For example, they are satisfied for any domain with a smooth boundary.

S. L. Sobolev was one of the founders of Novosibirsk State University in Akademgorodok. He gave the first lecture during the opening of Novosibirsk State University. Working in the Siberian Division of the USSR Academy of Sciences for 25 years, he was the head of the Chair of Differential Equations in the Department of Mechanics and Mathematics, lectured the classical course on equations of mathematical physics and a special course on cubature formulas, the theory which he had developed. The result of this research is his book "Introduction to the Theory of Cubature Formulas" (Nauka, Moscow (1974)). The scientific school in the field of the theory of cubature formulas was formed under the lead of S. L. Sobolev.

In the 1960s, S. L. Sobolev was also engaged in the problem of construction of electronic computers with processing power at least 1 billion operations per second (in the terminology used now, supercomputers). In this connection a group was formed in the Institute of Mathematics of the Siberian Division of the USSR Academy of Sciences. Such a supercomputer had to be a cluster of separate computers (processors) performing in parallel different steps of the work. The main technical principle was micro computerization. The time allocated for the completion of this project was said to be 20–25 years. The journal titled "Computational Systems" was published in the institute. It published papers devoted to electronic computers of high productivity. An interinstitutional seminar was organized, where everybody who studied this subject could present. Unfortunately, for objective (and possibly, subjective) reasons this work was not finished, since it did not find proper understanding and support. We can now say that, during the work conducted in the Institute of Mathematics, there was given a prognosis of ways of development of computer science. This prognosis turned out to be precise. The ideas formulated in the process of that work were implemented in real devices later on. For example, the proposal to use for connecting processors the network of the faces of the n-dimensional cube first was formulated in one of the papers published in the journal "Computational Systems" in 1962. This idea was realized in many parallel supercomputers working now.

The fact that such a remarkable mathematician as Academician S. L. Sobolev arrived in Novosibirsk in 1957 had great significance for the Siberian Division and development of mathematics in Siberia. The Sobolev Institute of Mathematics has been one of the world centers of mathematical research already for more than 40 years.

1. Application of the Theory of Plane Waves to the Lamb Problem*

S. L. Sobolev

Chapter 1

1. Professor H. Lamb in his article [1] considered the problem on propagation of disturbances in an infinite half-space.

At a point on a boundary of the half-space there is a force normal to the surface of boundary between the medium and the vacuum. The problem is to compute components of displacements at some other point of the surface (the observation point). The results obtained by H. Lamb allow us to compute these displacements in the form of definite integrals.

Our problem is to find analogous integral expressions for displacements at an arbitrary point inside the medium. Our method, in spite of a certain formal dissimilarity, is actually close to H. Lamb's method.

The essence of our method is the consideration of a disturbance propagating in the half-space as a sum of disturbances of a certain special type: the complex plane waves. We obtain these complex waves directly from the equations of elasticity; however, they can be obtained by summing in a certain order the multiple Fourier integrals used by H. Lamb. We are not going to prove the existence and uniqueness theorems for our integral representation, since they are the formal corollary of the corresponding theorems for the Fourier integrals. Moreover, the obtained result does not need a strict proof, since the final formulas allow us to verify all initial and boundary conditions.

Let us briefly outline the statement of the problem.

First, we investigate the two-dimensional Lamb problem, and then move on to the three-dimensional problem.

As a starting point, we take H. Lamb's expressions of the displacements on the boundary, which we use as the boundary conditions.

* Tr. Seism. Inst., **18** (1932), 41 p.
Tr. Seism. Inst. is Transactions of the Seismological Institute of the USSR Academy of Sciences. – *Ed.*

As in any method of representing a solution as a definite integral, for solving a problem we need to define so-called density of spectrum in the representation. For this purpose, we identify integrals obtained by H. Lamb with those obtained by us.

After defining in such way the spectral function, we transform the obtained results to a form more convenient for calculation.

Now we move on to the presentation of our method.

2. In our note [2] we had already studied the reflection of longitudinal and transverse elastic plane waves falling at different angles on the plane. However, because of the great importance of the plane waves for the problem in question, and also since the presentation of this question can be significantly simplified by using the theory of functions of a complex variable, we review this question again.

Consider an infinite elastic half-space and direct the y-axis along the normal to the boundary plane inward to the elastic medium, and the x- and z-axis along its surface. Suppose that we deal with a plane problem, and that the disturbance picture does not depend on the coordinate z. In this case, as is known, the components of the displacements u and v have the form

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{1}$$

where φ and ψ are scalar and vector potentials satisfying the equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = a^2 \frac{\partial^2 \varphi}{\partial t^2}, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = b^2 \frac{\partial^2 \psi}{\partial t^2}. \tag{2}$$

Here

$$a = \sqrt{\frac{\rho}{\lambda + 2\mu}}, \qquad b = \sqrt{\frac{\rho}{\mu}}$$

are the values reciprocal to the velocity of propagation of the longitudinal and transverse waves.

Let us consider the coordinate system moving along the x-axis with the velocity $\frac{1}{\theta}$, and assume that in this moving system of coordinates the disturbance picture, i.e., both the displacements and potentials, remain constant. In what follows, this quantity $\frac{1}{\theta}$ is called *apparent velocity*, and the described motion is called the *plane wave*. The meaning of this name will be explained later.

If we denote $\xi = t - \theta x$, then the system of ξ and y coordinates is our moving system of coordinates with the rescaled abscissa axis.

Our assumption is equivalent to the fact that both φ and ψ depend only on ξ and y.

Substituting these expressions into equations (2), we obtain

$$\theta^2 \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial y^2} = a^2 \frac{\partial^2 \varphi}{\partial \xi^2} \quad \text{and} \quad \theta^2 \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial y^2} = b^2 \frac{\partial^2 \psi}{\partial \xi^2}$$

or

$$(\theta^2 - a^2)\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad \text{and} \quad (\theta^2 - b^2)\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{3}$$

It is interesting to discuss separately three possible cases. In the first case, when $|\theta| < a$, both equations (3) are the vibrating string equations. In the second case, when $a < |\theta| < b$, the equation on φ is elliptic, and the equation on ψ is the vibrating string equation. Finally, in the third case, when $|\theta| > b$, both equations are elliptic. We discuss all three cases separately.

In the first case, substituting $\sqrt{a^2 - \theta^2}y = \eta_1$ into the first equation, and $\sqrt{b^2 - \theta^2}y = \eta_2$ into the second equation, we reduce both equations to the form

$$\frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\eta_1^2} = 0, \qquad \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta_2^2} = 0.$$

The general solution of these equations has the form

$$\begin{gathered}\varphi = \varphi_1(\xi + \eta_1) + \varphi_2(\xi - \eta_1) \\ = \varphi_1(t - \theta x + \sqrt{a^2 - \theta^2}y) + \varphi_2(t - \theta x - \sqrt{a^2 - \theta^2}y), \\ \psi = \psi_1(\xi + \eta_2) + \psi_2(\xi - \eta_2) \\ = \psi_1(t - \theta x + \sqrt{b^2 - \theta^2}y) + \psi_2(t - \theta x - \sqrt{b^2 - \theta^2}y).\end{gathered} \tag{4}$$

There only remains to satisfy conditions on the surface $y = 0$. If we assume that this surface is free of strains, then this condition can be written in the form of the system of two equalities[1]

$$\begin{gathered}\mu\left(2\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\bigg|_{y=0} = 0, \\ \mu\left(\frac{b^2}{a^2}\frac{\partial^2\varphi}{\partial y^2} + \left(\frac{b^2}{a^2} - 2\right)\frac{\partial^2\varphi}{\partial x^2} - 2\frac{\partial^2\psi}{\partial x\partial y}\right)\bigg|_{y=0} = 0.\end{gathered} \tag{5}$$

Substituting for $\dfrac{\partial^2\varphi}{\partial y^2}$ and $\dfrac{\partial^2\psi}{\partial y^2}$ their expressions from (3), we obviously obtain

$$\begin{gathered}\left(2\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\theta^2 - b^2}{\theta^2}\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial x^2}\right)\bigg|_{y=0} = 0, \\ \left(-\frac{b^2}{a^2}\left(\frac{\theta^2 - a^2}{\theta^2}\right)\frac{\partial^2\varphi}{\partial x^2} + \left(\frac{b^2}{a^2} - 2\right)\frac{\partial^2\varphi}{\partial x^2} - 2\frac{\partial^2\psi}{\partial x\partial y}\right)\bigg|_{y=0} = 0.\end{gathered}$$

[1] For details see the paper by S. L. Sobolev "Some questions of the theory of propagation of vibrations" in the book: Frank, F., Mises, R.: Differential and Integral Equations of Mathematical Physics. Vol. 2. ONTI, Leningrad – Moscow (1937). – *Ed.*

Obviously, we can integrate these conditions once with respect to x and omit a constant which does not change the result, since it presents some constant term in the displacement. Thus, we obtain

$$\begin{gathered}\left.\left[2\theta^2\frac{\partial\varphi}{\partial y}-(2\theta^2-b^2)\frac{\partial\psi}{\partial x}\right]\right|_{y=0}=0,\\ \left.\left[-(2\theta^2-b^2)\frac{\partial\varphi}{\partial x}-2\theta^2\frac{\partial\psi}{\partial y}\right]\right|_{y=0}=0.\end{gathered}\tag{6}$$

Substituting (4) in (6), we obtain the system of equations[2]

$$\begin{gathered}2\theta\sqrt{a^2-\theta^2}(\varphi_1'-\varphi_2')+(2\theta^2-b^2)(\psi_1'+\psi_2')=0,\\ (2\theta^2-b^2)(\varphi_1'+\varphi_2')-2\theta\sqrt{b^2-\theta^2}(\psi_1'-\psi_2')=0.\end{gathered}\tag{7}$$

To simplify the computations, we put

$$\varphi_2=[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_1,$$

$$\psi_2=[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_2.$$

Then, we have

$$\begin{aligned}\varphi=&[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_1(t-\theta x-\sqrt{a^2-\theta^2}y)\\ &-[(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_1(t-\theta x+\sqrt{a^2-\theta^2}y)\\ &-4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)f_2(t-\theta x+\sqrt{a^2-\theta^2}y),\end{aligned}\tag{8.1}$$

$$\begin{aligned}\psi=&[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_2(t-\theta x-\sqrt{b^2-\theta^2}y)\\ &-[(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]f_2(t-\theta x+\sqrt{b^2-\theta^2}y)\\ &+4\theta\sqrt{a^2-\theta^2}(2\theta^2-b^2)f_1(t-\theta x+\sqrt{b^2-\theta^2}y).\end{aligned}\tag{8.2}$$

It is not difficult to reveal the physical meaning of these formulas.

Obviously, each potential contains two distinct sets of terms. One set remains constant in one system of moving parallel lines defined by equations

$$t-\theta x\mp\sqrt{a^2-\theta^2}y=\text{const}\quad\text{for}\quad\varphi$$

and

$$t-\theta x\mp\sqrt{b^2-\theta^2}y=\text{const}\quad\text{for}\quad\psi,$$

and another set remains constant in the second system.

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[2] The argument of functions in the presented equalities is the difference $t-\theta x$.

The first set of terms (actually, only one term) is the *incident plane wave*, and another set is the *reflected plane wave*. It is not difficult to derive the relation between "the incidence angle" (or reflection) of the wave and the apparent velocity.

Indeed, if we denote by ϑ_1 the incidence angle of the wave, i.e., the angle between the normal to the plane of identical value of the scalar potential and the normal to the surface of the boundary, then we obtain for this angle the formula (see Fig. 1),

$$\tan\vartheta_1 = \frac{\theta}{\sqrt{a^2-\theta^2}} \quad \text{or} \quad \sin\vartheta_1 = \frac{\theta}{a}. \tag{9.1}$$

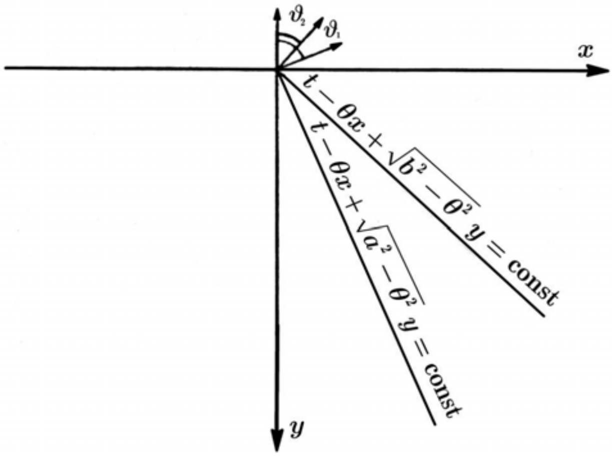

Fig. 1.

In the same way, we obtain the formula for the incident transverse wave

$$\sin\vartheta_2 = \frac{\theta}{b}. \tag{9.2}$$

Obviously, the reflection angles of both waves are equal to the corresponding incidence angles.

Hence we have the known law of reflection

$$\frac{\sin\vartheta_1}{\sin\vartheta_2} = \frac{b}{a}.$$

Solving the problem looks somewhat more difficult in the case when one of equations (2), namely, the equation on φ, is elliptic, i.e., when $a < |\theta| < b$.

In this case, putting $\eta_1 = \sqrt{\theta^2-a^2}y$ and $\eta_2 = \sqrt{b^2-\theta^2}y$, we obtain the system of two equations

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta_1^2} = 0, \qquad \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta_2^2} = 0$$

and the same boundary conditions (6).

It follows from the equation on φ that φ is the real part of an analytic function of a complex variable, regular in the upper half-plane.

Moreover, if we assume that it is bounded at infinity (this is exactly what we are interested in), then both φ and its derivatives are determined up to a constant by contour values of its real part. Henceforth, we assume that this function is regular up to the contour. Writing down the obtained result, we have

$$\varphi = \operatorname{Re}\,(\overline{\varphi}(\zeta)), \qquad \zeta = \xi + i\eta,$$

where $\overline{\varphi}$ denotes our function of a complex variable bounded at infinity and regular up to the contour.

For future reference, it is convenient to use also the imaginary part of the function $\overline{\varphi}$. Assuming $\overline{\varphi} = \varphi + i\varphi^*$, we have the known Cauchy–Riemann equations

$$\frac{\partial\varphi}{\partial\xi} = \frac{\partial\varphi^*}{\partial\eta}, \qquad \frac{\partial\varphi}{\partial\eta} = -\frac{\partial\varphi^*}{\partial\xi}.$$

Whence we obtain

$$-\frac{1}{\theta}\frac{\partial\varphi}{\partial x} = \frac{1}{\sqrt{\theta^2 - a^2}}\frac{\partial\varphi^*}{\partial y}, \qquad \frac{1}{\sqrt{\theta^2 - a^2}}\frac{\partial\varphi}{\partial y} = \frac{1}{\theta}\frac{\partial\varphi^*}{\partial x}.$$

Substituting this into the first equation in (6), we obtain

$$\left[2\theta\sqrt{\theta^2 - a^2}\frac{\partial\varphi^*}{\partial x} - (2\theta^2 - b^2)\frac{\partial\psi}{\partial x}\right]\Bigg|_{y=0} = 0. \tag{10.1}$$

If we now recall that

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial y} = \frac{\sqrt{b^2 - \theta^2}}{\theta}\left(-\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial x}\right),$$

then the second equation in (6) can be written in the form

$$\left[-(2\theta^2 - b^2)\frac{\partial\varphi}{\partial x} - 2\theta\sqrt{b^2 - \theta^2}\left(-\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial x}\right)\right]\Bigg|_{y=0} = 0. \tag{10.2}$$

Equalities (10) can be integrated with respect to x, omitting an arbitrary constant as unessential. Then we have

$$2\theta\sqrt{\theta^2 - a^2}\varphi^*|_{y=0} - (2\theta^2 - b^2)(\psi_1 + \psi_2) = 0,$$

$$-(2\theta^2 - b^2)\varphi|_{y=0} + 2\theta\sqrt{b^2 - \theta^2}(\psi_1 - \psi_2) = 0.$$

Solving these equations for ψ_2, we obtain

$$4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\psi_2 = -(2\theta^2-b^2)^2\varphi|_{y=0}$$
$$+4\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\varphi^*|_{y=0}$$

or, otherwise,

$$\psi_2 = -\mathrm{Re}\left[\frac{[(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{\varphi}|_{y=0}}{4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)}\right] = \mathrm{Re}\,[\overline{\psi}_2].$$

In the same way, we have

$$\psi_1 = \mathrm{Re}\left[\frac{[(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{\varphi}|_{y=0}}{4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)}\right] = \mathrm{Re}\,[\overline{\psi}_1].$$

Here we denote by $\overline{\psi}_2$ and $\overline{\psi}_1$ the complex expressions in square brackets. We also have the evident equalities

$$[(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{\varphi} = -4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\overline{\psi}_2,$$
$$[(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{\varphi} = 4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\overline{\psi}_1.$$

For the sake of simplicity we assume as before that

$$\overline{\psi}_2 = [(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{f}_2.$$

In this case we have

$$\overline{\varphi} = -4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\overline{f}_2,$$

and from the second equality we obtain

$$\overline{\psi}_1 = -\left[(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\right]\overline{f}_2.$$

Summing the obtained results, we immediately have

$$\varphi = \mathrm{Re}\,[-4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\overline{f}_2(t-\theta x+i\sqrt{\theta^2-a^2}y)],$$
$$\overline{\psi} = \mathrm{Re}\,\Big[[(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{f}_2(t-\theta x-\sqrt{b^2-\theta^2}y) \quad (11)$$
$$-\big[(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}]\overline{f}_2(t-\theta x+\sqrt{b^2-\theta^2}y)\big].$$

These formulas are absolutely analogous to formulas (8), which is expected. They admit the obvious physical interpretation:

$$\overline{f}_2(t-\theta x-\sqrt{b^2-\theta^2}y)$$

is, as before, the incident transverse wave, and

$$\overline{f}_2(t - \theta x + \sqrt{b^2 - \theta^2}y)$$

is the reflected transverse wave, while the longitudinal potential is neither incident nor reflected. Obviously, the longitudinal disturbance, being different from a harmonic function only by scaling, fills the entire half-space in this case. The reflected transverse wave differs from the incident wave in a shape as well.

It is not difficult to verify that our case corresponds to the one when the incidence angle of the transverse wave is larger than the limiting angle of the full inner reflection. From the law of sines, expressed by (9.1) and (9.2), in this case we see that $\sin\vartheta_1 > 1$, which obviously brings us to an imaginary angle.

As is known, this case is called the wave incidence with the angle greater than the limiting angle.

To conduct the study to the end, it is necessary to consider the last case, when $|\theta| > b$.

It is not difficult to see that in this case the most convenient approach is to perform a change of variables similar to the above one.

Putting

$$\eta_1 = \sqrt{\theta^2 - a^2}y, \qquad \eta_2 = \sqrt{\theta^2 - b^2}y,$$

we reduce both equations (2) to the Laplace equations

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta_1^2} = 0, \qquad \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta_2^2} = 0.$$

As before, we obtain

$$\varphi = \operatorname{Re}\overline{\varphi}, \quad \psi = \operatorname{Re}\overline{\psi}, \quad \overline{\varphi} = \varphi + i\varphi^*, \quad \overline{\psi} = \psi + i\psi^*$$

together with the Cauchy–Riemann equations

$$-\frac{1}{\theta}\frac{\partial\varphi}{\partial x} = \frac{1}{\sqrt{\theta^2 - a^2}}\frac{\partial\varphi^*}{\partial y}, \qquad \frac{1}{\sqrt{\theta^2 - a^2}}\frac{\partial\varphi}{\partial y} = \frac{1}{\theta}\frac{\partial\varphi^*}{\partial x},$$

$$-\frac{1}{\theta}\frac{\partial\psi}{\partial x} = \frac{1}{\sqrt{\theta^2 - b^2}}\frac{\partial\psi^*}{\partial y}, \qquad \frac{1}{\sqrt{\theta^2 - b^2}}\frac{\partial\psi}{\partial y} = \frac{1}{\theta}\frac{\partial\psi^*}{\partial x}.$$

As before, substituting our equalities into (6), we obtain

$$\begin{gathered} 2\theta\sqrt{\theta^2 - a^2}\frac{\partial\varphi^*}{\partial x}\bigg|_{y=0} - (2\theta^2 - b^2)\frac{\partial\psi}{\partial x}\bigg|_{y=0} = 0, \\ -(2\theta^2 - b^2)\frac{\partial\varphi}{\partial x}\bigg|_{y=0} - 2\theta\sqrt{\theta^2 - b^2}\frac{\partial\psi^*}{\partial x}\bigg|_{y=0} = 0. \end{gathered} \tag{12}$$

Integrating with respect to x and omitting an arbitrary constant, we have

$$
\begin{aligned}
&2\theta\sqrt{\theta^2-a^2}\varphi^*\Big|_{y=0} - (2\theta^2-b^2)\psi\big|_{y=0} = 0,\\
&-(2\theta^2-b^2)\varphi\big|_{y=0} - 2\theta\sqrt{\theta^2-b^2}\psi^*\Big|_{y=0} = 0.
\end{aligned} \tag{13}
$$

As before, we assume that $\overline{\varphi}$ and $\overline{\psi}$ are bounded at infinity. In this case, because of our assumption, the functions φ and ψ have to be determined by their values on the real axis. Hence their conjugate functions must be determined up to a constant summand.

From (13) we immediately obtain

$$
\mathrm{Re}\,[-2\theta i\sqrt{\theta^2-a^2}\overline{\varphi}]|_{y=0} = \mathrm{Re}\,[(2\theta^2-b^2)]\overline{\psi}|_{y=0},
$$

$$
\mathrm{Re}\,[-(2\theta^2-b^2)\overline{\varphi}]|_{y=0} = \mathrm{Re}\,[-2\theta i\sqrt{\theta^2-b^2}\overline{\psi}]|_{y=0}.
$$

In this case, from our assumptions it follows that

$$
-2\theta i\sqrt{\theta^2-a^2}\overline{\varphi} = (2\theta^2-b^2)\overline{\psi}, \quad (2\theta^2-b^2)\overline{\varphi} = 2\theta i\sqrt{\theta^2-b^2}\overline{\psi}. \tag{14}
$$

The system of these equations gives nonzero solutions for $\overline{\varphi}$ and $\overline{\psi}$ if and only if

$$
(2\theta^2-b^2)^2 - 4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2} = 0. \tag{15}
$$

Equation (15) is the known *Rayleigh equation*.

Thus, we see that the motion, that we call the plane wave, with an apparent velocity less than $\dfrac{1}{b}$ is possible only for a unique value of the apparent velocity equal to $\dfrac{1}{c}$, where c is a root of equation (15)[3].

To obtain the most convenient formulas, we put

$$
\overline{\psi} = -[(2\theta^2-b^2)^2 + 4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}]\overline{f}_2(t-\theta x + i\sqrt{\theta^2-b^2}y). \tag{16.1}
$$

Obviously, for $\overline{\varphi}$ we have

$$
\overline{\varphi} = -4i\theta\sqrt{\theta^2-b^2}(2\theta^2-b^2)\overline{f}_2(t-\theta x + i\sqrt{\theta^2-a^2}y). \tag{16.2}
$$

In this case the nature of both waves is completely analogous to the nature of the longitudinal wave in the previous case.

The disturbances fill the entire half-space, and by the maximum modulus principle, they attain maximum value on the boundary. Everywhere inside the medium they are continuous and all their derivatives are continuous as well. Possible discontinuations of the derivatives are located only on the contour. Such motion is called the *Rayleigh wave*.

Now it is not difficult to see that all three cases in question can be expressed by the same formulas, namely,

[3] The Rayleigh equation has a unique positive root $\theta = c$. – *Ed.*

$$\varphi = \mathrm{Re}\,\big\{[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]\overline{f}_1(t-\theta x-\sqrt{a^2-\theta^2}y)$$
$$-[(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]\overline{f}_1(t-\theta x+\sqrt{a^2-\theta^2}y)$$
$$-4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)\overline{f}_2(t-\theta x+\sqrt{a^2-\theta^2}y)\big\}, \tag{17.1}$$
$$\psi = \mathrm{Re}\,\big\{[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]\overline{f}_2(t-\theta x-\sqrt{b^2-\theta^2}y)$$
$$-[(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]\overline{f}_2(t-\theta x+\sqrt{b^2-\theta^2}y)$$
$$+4\theta\sqrt{a^2-\theta^2}(2\theta^2-b^2)\overline{f}_1(t-\theta x+\sqrt{b^2-\theta^2}y)\big\}, \tag{17.2}$$

where $\overline{f}_1$ and $\overline{f}_2$ are functions of complex variables, bounded and regular in the upper half-plane. For this purpose, it is necessary to make an agreement to assign either positive real or positive imaginary value to the radicals in this formula.

Obviously, the first case is obtained immediately, since the arguments of f_1 and f_2 are real, and their factors are real as well. As is known, the real part of a function of a complex variable on the real axis can take completely arbitrary value.

In the second case, the argument of $\overline{f}_1(t-\theta x-i\sqrt{\theta^2-a^2}y)$ lies in the lower half-plane, and, hence, we must assume that $\overline{f}_1$ is bounded and regular in the entire plane. As is known, such a function must be constant, and without loss of generality, we can assume that it is zero. Then we obtain exactly (11).

Finally, if $|\theta|>b$, then we have to assume that the functions $\overline{f}_1$ and $\overline{f}_2$ are zero, with the exception of the case when both coefficients at the terms with the argument in the lower half-plane are equal to zero. This again leads us to the Rayleigh equation (15). At the first glance, for $\theta=c$ our formulas contain two arbitrary functions; however, it is not difficult to see that equality holds

$$\frac{(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{4\theta\sqrt{b^2-\theta^2}(2\theta^2-b^2)} = -\frac{4\theta\sqrt{a^2-\theta^2}(2\theta^2-b^2)}{(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}.$$

Hence only one certain linear combination of $\overline{f}_1$ and $\overline{f}_2$ is contained in the expressions for both potentials. Without loss of generality, we can assume that $\overline{f}_1=0$. In such case, we immediately obtain (16).

As we have already noted, our assumption is reduced to the fact that the solution of the elasticity equations in the half-space is represented by the sum (integral) of the elementary solutions of form (17).

In other words, the solution of the elasticity equations in the half-space is composed from longitudinal and transverse waves reflecting at different angles (sometimes larger than the limiting angle), and, furthermore, from the Rayleigh wave, i.e., the solution of type (16).

From the point of view of the known principle of propagation of discontinuities it is interesting to point out the fact that in our representation, besides the surface discontinuities propagating inside the medium with the velocities

$\frac{1}{a}$ and $\frac{1}{b}$, there are also the linear discontinuities moving along the boundary with the normal velocity $\frac{1}{c}$.

Indeed, formulas (16) are particular solutions with such type of discontinuity. The function $\overline{f}_2$, appearing in these formulas, can have discontinuities on the boundary of its existence domain, i.e., on the real axis. These isolated linear discontinuities, sliding on the surface and not related to the inner surfaces of the discontinuities, have to move with the velocity $\frac{1}{c}$, as proved.

As we have already noted, our physical idea can be justified by summing the Fourier integrals used by H. Lamb in his memoir, and therefore, it is not new in principle. However, we think this idea was not explicitly presented yet.

3. For the sake of convenience of the further presentation, we need to give a somewhat different form of H. Lamb's formulas.

Therefore, let us transform them.

Due to H. Lamb, for $x > 0$ we have

$$u_0 = -\frac{H}{\mu}Q(t-cx) - \frac{2}{\pi\mu}\int\limits_a^b \frac{b^2\theta(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q(t-\theta x)\,d\theta,$$

$$v_0 = -\frac{1}{\pi\mu}\int\limits_a^b \frac{b^2(2\theta^2-b^2)^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q(t-\theta x)\,d\theta \qquad (18)$$

$$-\frac{1}{\pi\mu}\mathfrak{B}\int\limits_b^\infty \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}Q(t-\theta x)\,d\theta,$$

where H is a constant, $Q(t)$ is an acting force, and the symbol $\mathfrak{B}\int$ denotes the principal value of the divergent integral[4].

We use a certain artificial trick to transform these formulas such that they give a solution for any x.

Namely, we fix a certain moment of time t and construct the function $Q_t(t_1)$ as

$$Q_t(t_1) = \begin{cases} Q(t_1) & \text{for} \quad t_1 < t, \\ 0 & \text{for} \quad t_1 > t. \end{cases}$$

Finally, using the function Q_t, we construct the function of a complex variable

$$\overline{Q}_t(t_1) = Q_t(t_1) + iQ_t^*(t_1),$$

[4] Formula (18) was obtained in [1],

$$H = -\frac{c(2c^2-b^2-2\sqrt{c^2-a^2}\sqrt{c^2-b^2})}{F'(c)},$$

where $c > 0$ is a root of the Rayleigh equation $F(\theta) = 0$. – *Ed.*

regular in the upper half-plane of the argument t_1, whose real part equals $Q_t(t_1)$ on the real axis.

We assume that both the function $Q(t)$ and the function $Q'(t)$ tend to zero as $t \to -\infty$.

From above it follows that with a specific choice of $Q_t^*(t_1)$, our constructed function $\overline{Q}_t(t_1)$ vanishes at infinity.

Using the introduced function $\overline{Q}_t$, for u_0 we obtain the expression completely equivalent to the previous one; however, it possesses the symmetry property

$$u_0 = -\frac{H}{\mu} Q_t(t - cx) - \frac{2}{\pi\mu} \int_a^b \frac{b^2\theta(2\theta^2 - b^2)\sqrt{\theta^2 - a^2}\sqrt{b^2 - \theta^2}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} Q_t(t - \theta x)\, d\theta$$

$$+\frac{H}{\mu} Q_t(t + cx) + \frac{2}{\pi\mu} \int_a^b \frac{b^2\theta(2\theta^2 - b^2)\sqrt{\theta^2 - a^2}\sqrt{b^2 - \theta^2}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} Q_t(t + \theta x)\, d\theta.$$

For the sake of brevity, we denote

$$F(\theta) = (2\theta^2 - b^2)^2 - 4\theta^2\sqrt{\theta^2 - a^2}\sqrt{\theta^2 - b^2}.$$

Recalling the expression on H and expressing it through the corresponding residue, we can write our formula in the form

$$u_0 = \frac{c[(2c^2 - b^2) - 2\sqrt{c^2 - a^2}\sqrt{c^2 - b^2}]}{\mu} \mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right) Q_t(t + cx)$$

$$-\frac{1}{\pi\mu} \int_{-b}^{-a} \frac{2b^2\theta(2\theta^2 - b^2)\sqrt{\theta^2 - a^2}\sqrt{b^2 - \theta^2}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} Q_t(t - \theta x)\, d\theta$$

$$+\frac{c[2c^2 - b^2 - 2\sqrt{c^2 - a^2}\sqrt{c^2 - b^2}]}{\mu} \mathcal{E}_c\left(\frac{1}{F(\theta)}\right) Q_t(t - cx)$$

$$-\frac{1}{\pi\mu} \int_a^b \frac{2b^2\theta(2\theta^2 - b^2)\sqrt{\theta^2 - a^2}\sqrt{b^2 - \theta^2}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)} Q_t(t - \theta x)\, d\theta, \tag{19.1}$$

where symbols $\mathcal{E}_c(\ )$ and $\mathcal{E}_{-c}(\ )$ denote residues at the points c and $-c$ of the function in round brackets.

Let us study separately the transformation of the second term in the expression for v_0, substituting Q_t for Q. The value of this term remains the same, and it equals

$$-\frac{1}{\pi\mu} \mathfrak{B} \int_b^\infty \frac{b^2\sqrt{\theta^2 - a^2}}{(2\theta^2 - b^2)^2 - 4\theta^2\sqrt{\theta^2 - a^2}\sqrt{\theta^2 - b^2}} Q_t(t - \theta x)\, d\theta.$$

Let us consider the contour C pictured in Fig. 2 in the plane θ and study the integral

$$-\frac{1}{2\pi\mu}\int\limits_C \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}_t(t-\theta x)\,d\theta.$$

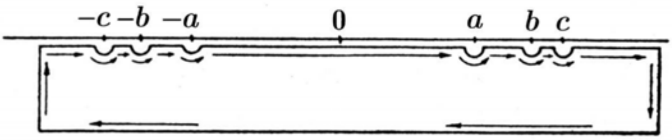

Fig. 2.

As we have proved [3], the denominator does not have roots on the given sheet of the Riemann surface. Therefore, this integral is equal to zero[5]. Furthermore, from evident estimates of the integrand we see that with infinite expansion of the contour only the integral over the real axis remains; it is equal to zero on the bases of classical theorems of the theory of functions. Indeed, the denominator of the integrand has order θ^2, and the numerator has order θ. Since $\overline{Q}$ is zero at infinity, we obtain our assertion.

Dividing the above integral into the terms corresponding to different pieces of the contour C, we obtain

$$\frac{1}{\pi\mu}\mathfrak{B}\int\limits_{-\infty}^{-b} \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}_t(t-\theta x)\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a} \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\overline{Q}_t(t-\theta x)\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{-a}^{a} \frac{ib^2\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\overline{Q}_t(t-\theta x)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b} \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\overline{Q}_t(t-\theta x)\,d\theta$$

---

[5] See the corresponding arguments in the paper [4] of Part I of this book (p. 148). – *Ed.*

$$-\frac{1}{\pi\mu}\mathfrak{B}\int\limits_b^\infty \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}_t(t-\theta x)\,d\theta$$
$$+\frac{\pi i}{\pi\mu}b^2\sqrt{c^2-a^2}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx)$$
$$-\frac{\pi i}{\pi\mu}b^2\sqrt{c^2-a^2}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx)=0.$$

Consider now the real part of this equality. Then, by the definition of the function $\overline{Q}_t$, we obtain

$$-\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{4\theta^2b^2(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^*(t-\theta x)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{-a}^{a}\frac{b^2\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}Q_t^*(t-\theta x)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{b^2(2\theta^2-b^2)^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t(t-\theta x)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{4\theta^2b^2(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^*(t-\theta x)\,d\theta$$

$$-\frac{1}{\pi\mu}\mathfrak{B}\int\limits_b^\infty \frac{b^2\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}Q_t(t-\theta x)\,d\theta$$
$$-\frac{b^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)Q_t^*(t+cx)$$
$$+\frac{b^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)Q_t^*(t-cx)=0. \tag{20}$$

Substituting into (18) for v_0 the expression of the improper integral over the interval $b<\theta<\infty$ obtained from the last formula, we have

$$v_0=\frac{b^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)Q_t^*(t+cx)-\frac{b^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)Q_t^*(t-cx)$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{4\theta^2b^2(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^*(t-\theta x)\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{-a}^{a}\frac{b^2\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}Q_t^*(t-\theta x)\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{4\theta^2b^2(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^*(t-\theta x)\,d\theta. \tag{19.2}$$

Formulas (19.1) and (19.2) give us the desired representation for u_0 and v_0.

4. If we would calculate u_0 and v_0 from equations (17) and integrate with respect to θ, then the form of the obtained solution would be completely analogous to the given one in (19). The simplest assumption, which will lead us to the goal, is simply identifying the integrands in both representations of u_0 and v_0. Hence we obtain equations which allow us to compute spectral functions for different values of θ. By the way, for the discrete spectral lines $\theta = \pm c$ and in the intervals $a < |\theta| < b$ we obtain two equations on one unknown function $\overline{f}_2$; however, these equations have solutions. For brevity, we denote by the symbol $\int\limits_E d\theta$ the integral taken over the entire set of the permissible spectral values of θ and added with two terms corresponding to $\pm c$. Then,

$$\begin{aligned} u_0 &= \int\limits_E \operatorname{Re}\,[-4b^2\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}\,\overline{f}_1'(t-\theta x) \\ &\quad +2b^2(2\theta^2-b^2)\sqrt{b^2-\theta^2}\,\overline{f}_2'(t-\theta x)]\,d\theta, \\ v_0 &= \int\limits_E \operatorname{Re}\,[2b^2(2\theta^2-b^2)\sqrt{a^2-\theta^2}\,\overline{f}_1'(t-\theta x) \\ &\quad +4\theta^2b^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}\,\overline{f}_2'(t-\theta x)]\,d\theta. \end{aligned} \tag{21}$$

Hence, for the interval $-a < \theta < a$, we have

$$-2\theta\sqrt{a^2-\theta^2}\overline{f}_1'(t-\theta x)+(2\theta^2-b^2)\overline{f}_2'(t-\theta x)=0,$$

$$2b^2\sqrt{a^2-\theta^2}\{(2\theta^2-b^2)\overline{f}_1'(t-\theta x)+2\theta\sqrt{b^2-\theta^2}\overline{f}_2'(t-\theta x)\}$$
$$=\frac{1}{\pi\mu i}\frac{b^2\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\overline{Q}(t-\theta x).$$

The determinant of this system of equations is nonzero, and we obtain

$$\begin{aligned} \overline{f}_1'(t-\theta x) &= \frac{1}{2\pi\mu i}\frac{(2\theta^2-b^2)}{[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]^2}\overline{Q}_t(t-\theta x), \\ \overline{f}_2'(t-\theta x) &= \frac{1}{2\pi\mu i}\frac{2\theta\sqrt{a^2-\theta^2}}{[(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}]^2}\overline{Q}_t(t-\theta x), \end{aligned} \tag{22}$$

$$-a < \theta < a.$$

For both intervals $a < |\theta| < b$ we obtain two identical equations

$$2b^2(2\theta^2 - b^2)\sqrt{b^2 - \theta^2}\overline{f}_2'(t - \theta x)$$

$$= -\frac{2}{\pi\mu}\frac{\theta b^2\sqrt{b^2 - \theta^2}\sqrt{\theta^2 - a^2}(2\theta^2 - b^2)}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)}\overline{Q}_t(t - \theta x),$$

$$4i\theta b^2\sqrt{b^2 - \theta^2}\sqrt{\theta^2 - a^2}\overline{f}_2'(t - \theta x)$$

$$= \frac{1}{\pi\mu i}\frac{4\theta^2 b^2(\theta^2 - a^2)\sqrt{b^2 - \theta^2}}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)}\overline{Q}_t(t - \theta x).$$

These two equations are reduced to the one, and give

$$\overline{f}_2'(t - \theta x) = \frac{1}{\pi\mu}\frac{\theta\sqrt{\theta^2 - a^2}\,\overline{Q}_t(t - \theta x)}{(2\theta^2 - b^2)^4 + 16\theta^4(\theta^2 - a^2)(b^2 - \theta^2)}, \quad a < |\theta| < b. \quad (23)$$

In the same way, we obtain also the values of the discrete spectral function for $\theta = +c$ and $\theta = -c$. Without presenting computations, we give the final result

$$\overline{f}_2'(t - cx) = \frac{-i}{4\mu c\sqrt{c^2 - b^2}}\mathcal{E}_c\left(\frac{1}{F(\theta)}\right)\overline{Q}(t - cx), \quad (\theta = c), \quad (24)$$

$$\overline{f}_2'(t + cx) = \frac{-i}{4\mu c\sqrt{c^2 - b^2}}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}(t + cx), \quad (\theta = -c). \quad (25)$$

From these formulas we immediately obtain the integral representation for u and v at any point inside the medium.

Thus, we have

$$u = \operatorname{Re}\left\{\left[\frac{c(2c^2 - b^2)}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t + cx + i\sqrt{c^2 - a^2}y)\right.\right.$$

$$+\frac{c(2c^2 - b^2)}{\mu}\mathcal{E}_c\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t - cx + i\sqrt{c^2 - a^2}y)$$

$$-\frac{1}{\pi\mu}\int_{-b}^{-a}\frac{4\theta^3\sqrt{b^2 - \theta^2}\sqrt{\theta^2 - a^2}(2\theta^2 - b^2)\overline{Q}_t(t - \theta x + i\sqrt{\theta^2 - a^2}y)\,d\theta}{(2\theta^2 - b^2)^4 - 16\theta^4(\theta^2 - a^2)(\theta^2 - b^2)}$$

$$+\frac{1}{2\pi\mu i}\int_{-a}^{a}\frac{\theta(2\theta^2 - b^2)}{(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}}$$

$$\times[\overline{Q}_t(t - \theta x + \sqrt{a^2 - \theta^2}y) - \overline{Q}_t(t - \theta x - \sqrt{a^2 - \theta^2}y)]\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{4\theta^3\sqrt{b^2-\theta^2}\sqrt{\theta^2-a^2}(2\theta^2-b^2)\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}\Bigg]$$

$$+\Bigg[\frac{-2c\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx+i\sqrt{c^2-b^2}y)$$

$$-\frac{2c\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{\mu}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx+i\sqrt{c^2-b^2}y)$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{\pi\mu i}\int\limits_{-a}^{a}\frac{\theta\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)-\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)]\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\Bigg]\Bigg\}, \tag{26.1}$$

$$v=\mathrm{Re}\Bigg\{\Bigg[\frac{i(2c^2-b^2)\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx+i\sqrt{c^2-a^2}y)$$

$$-\frac{i(2c^2-b^2)\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx+i\sqrt{c^2-a^2}y)$$

$$-\frac{1}{\pi\mu i}\int\limits_{-b}^{-a}\frac{(\theta^2-a^2)4\theta^2(2\theta^2-b^2)\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}$$

$$-\frac{1}{2\pi\mu i}\int\limits_{-a}^{a}\frac{(2\theta^2-b^2)\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}_t(t-\theta x-\sqrt{a^2-\theta^2}y)+\overline{Q}_t(t-\theta x+\sqrt{a^2-\theta^2}y)]\,d\theta$$

$$-\frac{1}{\pi\mu i}\int\limits_a^b \frac{(\theta^2-a^2)4\theta^2(2\theta^2-b^2)\sqrt{b^2-\theta^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}\Bigg]$$

$$+\Bigg[\frac{-2ic^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx+i\sqrt{c^2-b^2}y)$$

$$+\frac{2ic^2\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx+i\sqrt{c^2-b^2}y)$$

$$-\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta^2\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta^2\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{\pi\mu i}\int\limits_{-a}^{a}\frac{\theta^2\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)+\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)]\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{\theta^2\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{\theta^2\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x-\sqrt{b^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\Bigg]\Bigg\}. \tag{26.2}$$

It remains to transform these expressions to a simpler form.

5. For this purpose, we transform the terms containing the residues of $\left(\frac{1}{F(\theta)}\right)$. We replace these residues with the respectively chosen contour integrals. Obviously, the most convenient way to do it is to integrate in the plane of the variable of the function $\overline{Q}_t$. For this purpose, instead of θ we introduce the new variable H via the formula

$$H=\theta x+\sqrt{a^2-\theta^2}y. \tag{27}$$

For the sake of definiteness, we assume that $x>0$. Let us make a cut in the plane θ along the real axis between the points $\pm a$. First, we consider the transformation of the real axis of θ. We take as the first sheet of the Riemann surface the one such that

$$H=\theta x+\sqrt{a^2-\theta^2}y$$

on the lower lip of the cut and

$$H = \theta x - \sqrt{a^2 - \theta^2}\, y$$

on the upper lip. Obviously, on this sheet

$$H = \theta x + i\sqrt{\theta^2 - a^2}\, y$$

on the right side of a and

$$H = \theta x - i\sqrt{\theta^2 - a^2}\, y$$

on the left side of $-a$. From the formula

$$\frac{(\theta x)^2}{x^2} - \frac{\left(\sqrt{\theta^2 - a^2}\, y\right)^2}{y^2} = a^2$$

it follows that the intervals of the real axis right of a and left of $-a$ are transformed into two pieces of the same hyperbola. It is not difficult to see that on the upper lip of the cut $(-a, a)$ for $-a < \theta < \dfrac{-ax}{\sqrt{x^2 + y^2}}$ the variable H has a real value and is decreasing from $-ax$ to $-a\sqrt{x^2 + y^2}$. For

$$-\frac{ax}{\sqrt{x^2 + y^2}} < \theta < a$$

we obtain

$$-a\sqrt{x^2 + y^2} < H < ax.$$

On the lower lip for

$$-a < \theta < \frac{ax}{\sqrt{x^2 + y^2}}$$

we have

$$-ax < H < a\sqrt{x^2 + y^2},$$

and for

$$\frac{ax}{\sqrt{x^2 + y^2}} < \theta < a$$

we obtain

$$a\sqrt{x^2 + y^2} > H > ax.$$

Solving (27) for θ, we obtain on the upper lip of the cut of the first sheet

$$\theta = \frac{Hx + y\sqrt{a^2(x^2 + y^2) - H^2}}{x^2 + y^2},$$

and on the lower lip

$$\theta = \frac{Hx - y\sqrt{a^2(x^2+y^2) - H^2}}{x^2+y^2}.$$

Studying now what the real axis of H is transformed into, we come to the result pictured in Fig. 3[6]. For clarity, we use identical shading for parts of lines transformed one into another.

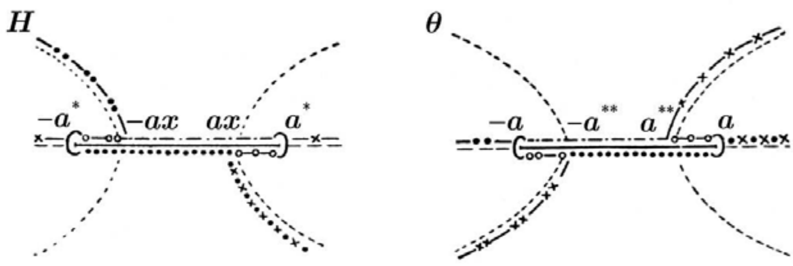

Fig. 3.

It is not difficult to see that the transformations of the plane of the first sheet are one-to-one.

The second sheet, where the roots are taken with the opposite signs, is studied in the same way (see Fig. 4).

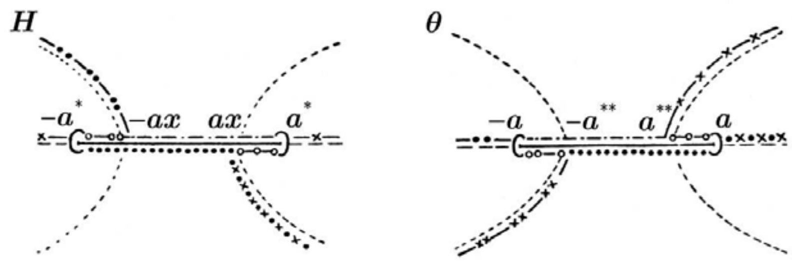

Fig. 4.

Besides the variable H we also need one more variable Ξ,

$$\Xi = \theta x + \sqrt{b^2 - \theta^2}y. \tag{28}$$

[6] In Figs. 3–6

$$a^* = a\sqrt{x^2+y^2}, \quad a^{**} = \frac{ax}{\sqrt{x^2+y^2}}. \text{ – } Ed.$$

Since the transformation is completely analogous to the previous one with the only difference that instead of a we have b here, we do not repeat our reasoning.

6. Let us consider the contour L_1 on the first sheet of the plane H and the corresponding contour L_1' on the first sheet of the plane θ (see Fig. 5).

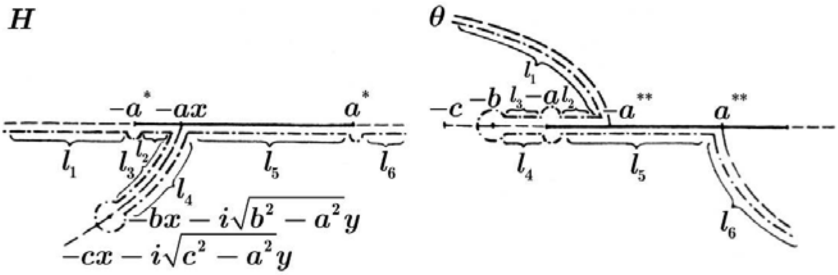

Fig. 5.

Let us compile a table of values of functions we need on separate parts of the contour L_1 and at a singular point. The names of the pieces of the contour are indicated in Fig. 5.

For the sake of brevity, let us introduce the positive quantities T_1 and T_2 by means of the formulas

$$T_1^2 = \Big([b^2(x^2+y^2)^2 - a^2y^2(x^2+y^2) - H^2(x^2-y^2)]^2 + 4H^2x^2y^2[H^2 - a^2(x^2+y^2)]\Big)^{1/2} + [b^2(x^2+y^2)^2 - a^2y^2(x^2+y^2) - H^2(x^2-y^2)], \tag{29.1}$$

$$T_2^2 = \Big([b^2(x^2+y^2)^2 - a^2y^2(x^2+y^2) - H^2(x^2-y^2)]^2 + 4H^2x^2y^2[H^2 - a^2(x^2+y^2)]\Big)^{1/2} - [b^2(x^2+y^2)^2 - a^2y^2(x^2+y^2) - H^2(x^2-y^2)]. \tag{29.2}$$

Part of the contour	$H(\theta)$	$\theta(H)$	$\alpha(\theta)=\alpha(H)=\pm\sqrt{a^2-\theta^2}$	$\beta(\theta)=\beta(H)=\pm\sqrt{b^2-\theta^2}$
l_1		$\frac{Hx+iy\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{Hy-ix\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{-1}{\sqrt{2}(x^2+y^2)}(T_1+iT_2)$
l_2	$\theta x-\sqrt{a^2-\theta^2}y$	$\frac{-ax}{\sqrt{x^2+y^2}}>\theta>-a$	$-\sqrt{a^2-\theta^2}$	$-\sqrt{b^2-\theta^2}$
l_3	$\theta x-i\sqrt{\theta^2-a^2}y$	$-a>\theta>-b$	$-i\sqrt{\theta^2-a^2}$	$-\sqrt{b^2-\theta^2}$
l_4	$\theta x-i\sqrt{\theta^2-a^2}y$	$-b<\theta<-a$	$-i\sqrt{\theta^2-a^2}$	$-\sqrt{b^2-\theta^2}$
l_5	$\theta x+\sqrt{a^2-\theta^2}y$	$-a<\theta<\frac{ax}{\sqrt{x^2+y^2}}$	$\sqrt{a^2-\theta^2}$	$\sqrt{b^2-\theta^2}$
l_6		$\frac{Hx-iy\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{Hy+ix\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{1}{\sqrt{2}(x^2+y^2)}(T_1+iT_2)$
The singular point	$-cx-i\sqrt{c^2-a^2}y$	$-c$	$-i\sqrt{c^2-a^2}$	$-i\sqrt{c^2-b^2}$

As we have indicated above, to compute the residues in the expressions for u and v, we use the contour integration. We obtain

$$\frac{c(2c^2-b^2)}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx+i\sqrt{c^2-a^2}y)$$

$$=\frac{1}{2\pi i\mu}\int_{L_1}\left\{\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}dH$$

$$=\frac{1}{2\pi i\mu}\int_{-\infty}^{-a\sqrt{x^2+y^2}}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-H)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int_{a\sqrt{x^2+y^2}}^{\infty}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-H)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int_{\frac{-ax}{\sqrt{x^2+y^2}}}^{-a}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int_{-a}^{-b}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int_{-b}^{-a}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int_{-a}^{\frac{ax}{\sqrt{x^2+y^2}}}\frac{\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x-\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}. \tag{30}$$

In the same way, we have

$$\frac{i(2c^2-b^2)\sqrt{c^2-a^2}}{\mu}\mathcal{E}_{-c}\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t+cx+i\sqrt{c^2-a^2}y)$$

$$=\frac{1}{2\pi i\mu}\int_{L_1}\left\{\frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}dH$$

$$=\frac{1}{2\pi i\mu}\int_{-\infty}^{-a\sqrt{x^2+y^2}}\frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{\frac{-ax}{\sqrt{x^2+y^2}}}^{-a}\frac{(2\theta^2-b^2)\sqrt{a^2-\theta^2}\,\overline{Q}_t(t-\theta x+\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{-a}^{-b}\frac{i(2\theta^2-b^2)\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{-b}^{-a}\frac{i(2\theta^2-b^2)\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{-a}^{\frac{ax}{\sqrt{x^2+y^2}}}\frac{-(2\theta^2-b^2)\sqrt{a^2-\theta^2}\,\overline{Q}_t(t-\theta x-\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}. \qquad (31)$$

Together with the contour $L_1(L_1')$ we also consider the contour $L_2(L_2')$ on the second sheet of the surface (see Fig. 6).

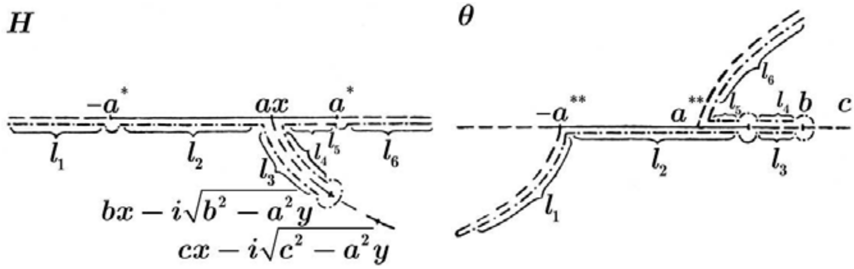

Fig. 6.

Below we present the table of the values we need of some functions, where T_1 and T_2 are the same as above.

Part of the contour	$H(\theta)$	$\theta(H)$	$\alpha(\theta)=\alpha(H)=\pm\sqrt{a^2-\theta^2}$	$\beta(\theta)=\beta(H)=\pm\sqrt{b^2-\theta^2}$
l_1		$\frac{Hx-iy\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{Hy+ix\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{-1}{\sqrt{2}(x^2+y^2)}(T_1-iT_2)$
l_2	$\theta x-\sqrt{a^2-\theta^2}y$	$\frac{-ax}{\sqrt{x^2+y^2}}<\theta<a$	$-\sqrt{a^2-\theta^2}$	$-\sqrt{b^2-\theta^2}$
l_3	$\theta x-i\sqrt{\theta^2-a^2}y$	$a<\theta<b$	$-i\sqrt{\theta^2-a^2}$	$-\sqrt{b^2-\theta^2}$
l_4	$\theta x-i\sqrt{\theta^2-a^2}y$	$b>\theta>a$	$-i\sqrt{\theta^2-a^2}$	$\sqrt{b^2-\theta^2}$
l_5	$\theta x+\sqrt{a^2-\theta^2}y$	$a>\theta>\frac{ax}{\sqrt{x^2+y^2}}$	$\sqrt{a^2-\theta^2}$	$\sqrt{b^2-\theta^2}$
l_6		$\frac{Hx+iy\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{Hy-ix\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2}$	$\frac{-1}{\sqrt{2}(x^2+y^2)}(T_1-iT_2)$
The singular point	$cx-i\sqrt{c^2-a^2}y$	c	$-i\sqrt{c^2-a^2}$	$-i\sqrt{c^2-b^2}$

Compiling the integrals over the contour, we obtain similarly as above

$$\frac{c(2c^2-b^2)}{\mu}\mathcal{E}_c\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx+i\sqrt{c^2-a^2}y)$$

$$=\frac{1}{2\pi i\mu}\int\limits_{L_2}\left\{\frac{-\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}dH$$

$$=\frac{1}{2\pi i\mu}\int\limits_{-\infty}^{-a\sqrt{x^2+y^2}}\frac{-\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\frac{-\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{\frac{-ax}{\sqrt{x^2+y^2}}}^{a}\frac{-\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a}^{b}\frac{-\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{b}^{a}\frac{-\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a}^{\frac{ax}{\sqrt{x^2+y^2}}}\frac{-\theta(2\theta^2-b^2)\overline{Q}_t(t-\theta x-\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}, \tag{32}$$

and also

$$\frac{i(2c^2-b^2)\sqrt{c^2-a^2}}{\mu}\mathcal{E}_c\left(\frac{1}{F(\theta)}\right)\overline{Q}_t(t-cx+i\sqrt{c^2-a^2}y)$$

$$=\frac{1}{2\pi i\mu}\int\limits_{L_2}\left\{\frac{-(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}dH$$

$$=\frac{1}{2\pi i\mu}\int\limits_{-\infty}^{-a\sqrt{x^2+y^2}}\frac{-(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\frac{-(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}\overline{Q}_t(t-H)}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,dH$$

$$+\frac{1}{2\pi i\mu}\int\limits_{\frac{-ax}{\sqrt{x^2+y^2}}}^{a}\frac{(2\theta^2-b^2)\sqrt{a^2-\theta^2}\,\overline{Q}_t(t-\theta x+\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a}^{b}\frac{i(2\theta^2-b^2)\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{b}^{a}\frac{i(2\theta^2-b^2)\sqrt{\theta^2-a^2}\,\overline{Q}_t(t-\theta x+i\sqrt{\theta^2-a^2}y)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi i\mu}\int\limits_{a}^{\frac{ax}{\sqrt{x^2+y^2}}}\frac{(2\theta^2-b^2)\sqrt{a^2-\theta^2}\,\overline{Q}_t(t-\theta x-\sqrt{a^2-\theta^2}y)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}. \tag{33}$$

These expressions make it possible to compute the value of the first square bracket in the expressions for u and v. The expression in the first bracket in (26.1) for u, after substitution of values of the residues, is written in the form

$$\frac{1}{2\pi i\mu}\int\limits_{-\infty}^{-a\sqrt{x^2+y^2}}\left\{\left.\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_1}-\left.\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_2}\right\}$$
$$\times\overline{Q}_t(t-H)\,dH$$
$$+\frac{1}{2\pi i\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\left\{\left.\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_1}-\left.\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_2}\right\}$$
$$\times\overline{Q}_t(t-H)\,dH.$$

It is not difficult to see that the coefficient at $\overline{Q}_t(t)$ is the difference of two conjugate functions, and it is pure imaginary. Focusing our attention on the imaginary coefficient and taking the real part, we bring this bracket to the form depending only on Q_t, and not on $\overline{Q}_t$. Taking into account the definition of the function Q_t, we finally obtain the result for the real part of the first bracket in expression (26.1) for u,

$$\frac{1}{\pi\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\mathrm{Im}\left\{\left.\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_1}\right\}Q_t(t-H)\,dH. \tag{34.1}$$

Similarly, for the real part of the first bracket in expression (26_2) for v we obtain

$$\frac{1}{\pi\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\operatorname{Im}\left\{\left.\frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right|_{L_1}\right\}Q_t(t-H)\,dH. \tag{34.2}$$

7. The transformation of the second square bracket in (26) depends on location of critical points on the Riemann surface, i.e., on values of x and y. Here we need to consider two cases:

$$\text{I.}\qquad a\sqrt{x^2+y^2}\geq bx,$$

$$\text{II.}\qquad a\sqrt{x^2+y^2}\leq bx.$$

Without making all calculations, which are quite similar to the previous ones, we present only the final result for the displacements in both cases by collecting all said above. For $a\sqrt{x^2+y^2}\geq bx$,

$$u=\frac{1}{\pi\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\operatorname{Im}\left\{\frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}Q_t(t-H)\,dH$$

$$+\frac{1}{\pi\mu}\int\limits_{b\sqrt{x^2+y^2}}^{\infty}\operatorname{Im}\left\{\frac{2\theta\alpha\beta\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}Q_t(t-\Xi)\,d\Xi, \tag{35.1}$$

$$v=\frac{1}{\pi\mu}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\operatorname{Im}\left\{\frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}Q_t(t-H)\,dH$$

$$-\frac{1}{\pi\mu}\int\limits_{b\sqrt{x^2+y^2}}^{\infty}\operatorname{Im}\left\{\frac{2\theta^2\alpha\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\right\}Q_t(t-\Xi)\,d\Xi, \tag{35.2}$$

where[7]

$$\theta(H)=\frac{Hx-iy\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2},\qquad \alpha(H)=\frac{Hy+ix\sqrt{H^2-a^2(x^2+y^2)}}{x^2+y^2},$$

$$\beta(H)=\frac{1}{\sqrt{2}(x^2+y^2)}(T_1+T_2),$$

$$\theta(\Xi)=\frac{\Xi x-iy\sqrt{\Xi^2-b^2(x^2+y^2)}}{x^2+y^2},\qquad \alpha(\Xi)=\frac{1}{\sqrt{2}(x^2+y^2)}(S_1+iS_2),$$

[7] Ξ, T_1, T_2 are defined in (28), (29). – *Ed.*

$$\beta(\Xi) = \frac{\Xi y + ix\sqrt{\Xi^2 - b^2(x^2+y^2)}}{x^2+y^2},$$

$$\begin{aligned} S_1^2 = \Big(&[a^2(x^2+y^2)^2 - b^2y^2(x^2+y^2) - \Xi^2(x^2-y^2)]^2 \\ &+4\Xi^2x^2y^2[\Xi^2 - b^2(x^2+y^2)]\Big)^{1/2} \\ &+[a^2(x^2+y^2)^2 - b^2y^2(x^2+y^2) - \Xi^2(x^2-y^2)], \\ S_2^2 = \Big(&[a^2(x^2+y^2)^2 - b^2y^2(x^2+y^2) - \Xi^2(x^2-y^2)]^2 \\ &+4\Xi^2x^2y^2[\Xi^2 - b^2(x^2+y^2)]\Big)^{1/2} \\ &-[a^2(x^2+y^2)^2 - b^2y^2(x^2+y^2) - \Xi^2(x^2-y^2)]. \end{aligned} \tag{36}$$

In the same way, for $a\sqrt{x^2+y^2} \le bx$ we have

$$\begin{aligned} u = \frac{1}{\pi\mu} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \operatorname{Im}\left\{ \frac{\theta(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2 + 4\theta^2\alpha\beta} \right\} Q_t(t-H)\,dH \\ +\frac{1}{\pi\mu} \int\limits_{ax+\sqrt{b^2-a^2}y}^{\infty} \operatorname{Im}\left\{ \frac{2\theta\alpha\beta\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2 + 4\theta^2\alpha\beta} \right\} Q_t(t-\Xi)\,d\Xi, \end{aligned} \tag{37.1}$$

$$\begin{aligned} v = \frac{1}{\pi\mu} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \operatorname{Im}\left\{ \frac{(2\theta^2-b^2)\alpha\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2 + 4\theta^2\alpha\beta} \right\} Q_t(t-H)\,dH \\ -\frac{1}{\pi\mu} \int\limits_{ax+\sqrt{b^2-a^2}y}^{\infty} \operatorname{Im}\left\{ \frac{2\theta^2\alpha\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2 + 4\theta^2\alpha\beta} \right\} Q_t(t-\Xi)\,d\Xi, \end{aligned} \tag{37.2}$$

where the functions θ, α, and β have the previous values on the intervals $a\sqrt{x^2+y^2} < H < \infty$ and $b\sqrt{x^2+y^2} < \Xi < \infty$, and

$$\theta = \frac{\Xi x - y\sqrt{b^2(x^2+y^2) - \Xi^2}}{x^2+y^2}, \quad \beta = \sqrt{b^2-\theta^2}, \quad \alpha = i\sqrt{\theta^2-a^2} \tag{38}$$

on the interval $ax + \sqrt{b^2-a^2}y < \Xi < b\sqrt{x^2+y^2}$. Formulas (35)–(38) present the final answer to the question.

8. In conclusion, let us point out several simple physical consequences following from our formulas.

It is interesting to trace moments of appearance of different phases of vibrations inside the medium in the case when initial conditions are zero for $t < t_0$, and the force begins to act for $t \geq t_0$.

For simplification, we assume that the force acting on the surface of the medium is the short and very intensive force of impulse type.

This is reduced to the assumption that we have to take the function $Q(t)$ under the integral sign, extremely large in a small interval of time and vanishing outside this interval.

In this case, we can replace the integral by a finite expression as H. Lamb does in the cited memoir.

From the method of derivation of our formulas it is obvious that the integrals (or the functions obtained from the integrals) containing H are the longitudinal waves, and the integrals containing Ξ are the transverse waves.

Moving on to the investigation of the force of impulse type, we see that in this case the components of the displacement are expressed by the factor in front of the function Q under the integral sign, if we substitute $t - t_0$ for H or Ξ, respectively, where t_0 is the moment of the impulse activation. For simplicity, putting $t_0 = 0$, we obtain that we again need to consider two different cases for our motion:

$$a\sqrt{x^2 + y^2} > bx, \tag{39}$$

$$a\sqrt{x^2 + y^2} < bx. \tag{40}$$

In the first case, both components of the displacement are equal to zero for $t < a\sqrt{x^2 + y^2}$. For $t = a\sqrt{x^2 + y^2}$ the longitudinal wave appears; moreover, u and v are defined by the formulas

$$\begin{aligned} u_1 &= \frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{\theta_1(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial t}}{(2\theta_1^2 - b^2)^2 + 4\theta_1^2\alpha\beta}\right\}, \\ v_1 &= \frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{(2\theta_1^2 - b^2)\alpha_1\frac{\partial\theta_1}{\partial t}}{(2\theta_1^2 - b^2)^2 + 4\theta_1^2\alpha_1\beta_1}\right\}, \end{aligned} \tag{41}$$

where θ_1 denotes the result of the substitution of t for H in the function θ.

Starting at $t = b\sqrt{x^2 + y^2}$, the transverse wave is superimposed on this longitudinal wave. The transverse wave gives the components

$$\begin{aligned} u_2 &= \frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{2\theta_2\alpha_2\beta_2\frac{\partial\theta_2}{\partial t}}{(2\theta_2^2 - b^2)^2 + 4\theta_2^2\alpha_2\beta_2}\right\}, \\ v_2 &= -\frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{2\theta_2^2\alpha_2\frac{\partial\theta_2}{\partial t}}{(2\theta_2^2 - b^2)^2 + 4\theta_2^2\alpha_2\beta_2}\right\}. \end{aligned} \tag{42}$$

It is easy to verify that

$$\left.\frac{\partial\theta_1}{\partial t}\right|_{t=a\sqrt{x^2+y^2}} = \infty$$

and

$$\left.\frac{\partial\theta_2}{\partial t}\right|_{t=b\sqrt{x^2+y^2}} = \infty.$$

It leads to the fact that at any point (x, y) (for $y > 0$) the infinite displacement takes place at the moments of time $t = a\sqrt{x^2 + y^2}$ and $t = b\sqrt{x^2 + y^2}$.

In the second case, the arriving longitudinal wave is the same as in the first case. However, the moment of arrival of the transverse wave is somewhat different; namely, the transverse wave appears at the moment

$$t = ax \pm \sqrt{b^2 - a^2}y$$

and is defined by the formulas

$$\begin{aligned} u_2 &= \frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{2\theta_2\alpha_2\beta_2\frac{\partial\theta_2}{\partial t}}{(2\theta_2^2 - b^2)^2 + 4\theta_2^2\alpha_2\beta_2}\right\}, \\ v_2 &= -\frac{1}{\pi\mu}\mathrm{Im}\left\{\frac{2\theta_2^2\alpha_2\frac{\partial\theta_2}{\partial t}}{(2\theta_2^2 - b^2)^2 + 4\theta_2^2\alpha_2\beta_2}\right\}. \end{aligned} \tag{43}$$

However, the integrand form essentially depends on the sign of the expression $t - b\sqrt{x^2 + y^2}$. At the moment $t = b\sqrt{x^2 + y^2}$ we again have infinite displacements giving a new phase of the same transverse wave. Also, note that at the beginning moment of the first transverse phase in the second case the radical α in the analytic expressions for u and v gets a critical point, and we can verify that the derivatives of the displacements with respect to the time become infinite. This gives us the basis to talk about a push at the moment of the appearance of this new phase.

The denominator of all our functions vanishes at the surface points for $t = \pm cx$. At points located close to these two points the displacements attain very large values.

It is obvious that the moments $t = a\sqrt{x^2 + y^2}$ and $t = b\sqrt{x^2 + y^2}$ correspond to the shortest transit time of the disturbance along the straight line from the source to the observation point with the velocities of the longitudinal wave $\frac{1}{a}$ and the transverse wave $\frac{1}{b}$. If we picture the fronts of the different phases at a certain moment of time, we obtain Fig. 7.

Two circles with radii $\frac{t}{a}$ and $\frac{t}{b}$ give the obvious fronts of the wave inside $|a\sqrt{x^2 + y^2}| > bx$, two tangents $ax \pm \sqrt{b^2 - a^2}y = t$ to the inner circle give the fronts of the transverse wave inside the angle between the lines

$$\pm a\sqrt{x^2 + y^2} - bx = 0.$$

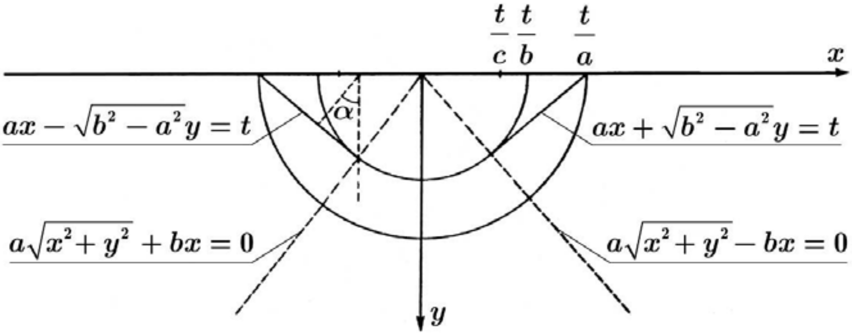

Fig. 7.

Let us note that the moments $t = ax \pm \sqrt{b^2 - a^2}y$ also correspond to the shortest transit time from the source to the observation point, but along the piece-wise line. One of its pieces is the boundary of the medium, and another piece consists of the line angled at a limiting angle of the full inner reflection; moreover, along the boundary of the medium the disturbance moves with the velocity $\dfrac{1}{a}$, and inside with the velocity $\dfrac{1}{b}$. Indeed, denoting by α the limiting angle of the full inner reflection and denoting by h the distance from the source to the point, where the ray leaves the surface, we obtain for the transition along such path

$$y = \frac{x - h}{\tan\alpha}, \qquad t = ah + \frac{by}{\cos\alpha}.$$

Hence,

$$h = x - y\tan\alpha$$

and

$$t = ax - y\left(a\tan\alpha - \frac{b}{\cos\alpha}\right).$$

Recalling now that

$$\sin\alpha = \pm\frac{a}{b}, \qquad \cos\alpha = \frac{\sqrt{b^2 - a^2}}{b},$$

and

$$\tan\alpha = \pm\frac{a}{\sqrt{b^2 - a^2}},$$

we obtain

$$t = ax - y\left(\pm\frac{a^2}{\sqrt{b^2 - a^2}} \mp \frac{b^2}{\sqrt{b^2 - a^2}}\right) = ax \pm \sqrt{b^2 - a^2}y,$$

which is required.

Thus, the results of our study completely confirm the known Fermat principle frequently used by seismologists.

We can also physically explain this result by using relatively simple descriptive ideas.

The longitudinal wave, propagating along the disk and remaining near the surface, generates behind the fissure of the transverse wave similar to the one which remains behind a boat moving faster than a wave. Therefore, for a certain mutual arrangement of the source and an observer, we have clearly expressed the phenomenon at the moment when this “forced” transverse wave approaches the observer.

Furthermore, we should point out that the discontinuities moving with the velocity of the Rayleigh wave must be concentrated actually on the surface in the form of two points running over the surface with the Rayleigh velocity.

Moreover, in the entire phenomenon we should pay attention to the fact that the vibrations beginning with a sharp push come to naught gradually with the infinite increase of t.

We could compute curves given by (41), (42), and (43); however, since it is quite simple, we leave this to the reader.

Chapter 2

In the same way, one can also solve the second problem stated by H. Lamb on an action of a given concentrated force in the three-dimensional space. As in the plane problem, first, we need to rewrite the formulas given by H. Lamb in a form more convenient for us. H. Lamb's formulas, derived by him in the second part of the work cited earlier, have the form

$$q_0 = \frac{H}{\pi\mu}\frac{\partial}{\partial\overline{\omega}}\int_0^\infty R(t - c\overline{\omega}\cosh u)\,du$$

$$-\frac{2}{\pi^2 b\mu}\int_a^b U(\theta)\frac{\partial}{\partial\overline{\omega}}\int_0^\infty R(t - \theta\overline{\omega}\cosh u)\,du\,d\theta, \tag{44}$$

$$w_0 = \frac{1}{\pi^2 b\mu}\mathfrak{B}\int_a^\infty \theta V(\theta)\frac{\partial}{\partial t}\int_0^\infty R(t - \theta\overline{\omega}\cosh u)\,du\,d\theta. \tag{45}$$

In this case, the surface of the half-space is assumed to be the (x, y)-plane, the concentrated force acts vertically at the origin, $\overline{\omega}$ is the distance between the point and the z-axis, q_0 and w_0 are the projections of the displacement of a certain point of the surface on the direction of $\overline{\omega}$ and z-axis. Here $U(\theta)$ and $V(\theta)$ are the same functions, which were the factors of the integrand in the plane problem, and $R(t)$ denotes the acting force.

Let us first transform the integral

$$\int_0^\infty R(t-\theta\overline{\omega}\cosh u)\,du.$$

Let us change somewhat the notation. We denote by $Q(t)$ the function $R(t)$, and instead of $\overline{\omega}$ we introduce ϱ.

The integral can be rewritten as

$$\int_0^\infty Q(t-\theta\varrho\cosh u)\,du.$$

Let us construct the function $\overline{Q}_t(t_1) = Q_t(t_1) + iQ_t^*(t_1)$ regular in the upper half-plane of the complex variable and possessing on the real axis the property

$$Q_t(t_1) = \begin{cases} Q(t), & t_1 > t, \\ Q(t_1), & t_1 < t. \end{cases}$$

Then, the function $Q_t'(t_1)$ is single-valued, and on the real axis it has infinity at most of logarithmic order.

In this case, for positive θ (in H. Lamb's formulas we deal only with such ones),

$$\int_0^\infty Q_t'(t-\theta\varrho\cosh u)\,du = \operatorname{Re}\int_1^\infty \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{\xi^2-1}}\,d\xi.$$

The function $\dfrac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{\xi^2-1}}$ is regular in the lower half-plane of the variable ξ. Thus, if only $Q'(\infty)=0$ and the integral written above has meaning, then

$$-\int_\infty^{-1} \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{\xi^2-1}}\,d\xi + i\int_{-1}^{1} \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{1-\xi^2}}\,d\xi + \int_1^\infty \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{\xi^2-1}}\,d\xi = 0.$$

By the definition of the function $\overline{Q}_t$, the real part of the first term is zero. Then,

$$\operatorname{Re}\int_1^\infty \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{\xi^2-1}}\,d\xi = \operatorname{Im}\int_{-1}^{1} \frac{\overline{Q}_t'(t-\theta\varrho\xi)}{\sqrt{1-\xi^2}}\,d\xi. \tag{46}$$

Putting $\xi=\cos\vartheta$, we have

$$\int_0^\infty Q'(t-\varrho\theta\cosh u)\,du = \int_0^\pi Q_t^{*\prime}(t-\varrho\theta\cos\vartheta)\,d\vartheta.$$

Using similar transformations, we can obtain the formula

$$\int_0^\infty Q'(t-\varrho\theta\cosh u)\cosh u\,du = \int_0^\pi Q_t^{*'}(t-\varrho\theta\cos\vartheta)\cos\vartheta\,d\vartheta. \tag{47}$$

If $Q(t)$ decreases rapidly enough as $t\to\infty$, we can differentiate with respect to the parameter in H. Lamb's formula for q_0. Then the formula can be written in the form

$$q_0 = -\frac{cH}{\pi\mu}\int_0^\infty Q'(t-c\varrho\cosh u)\cosh u\,du$$

$$+\frac{2}{\pi^2 b\mu}\int_a^b U(\theta)\theta\int_0^\infty Q'(t-\theta\varrho\cosh u)\cosh u\,du\,d\theta. \tag{48}$$

Substituting into H. Lamb's formula for q_0 the value of the integral from (47), we obtain

$$q_0 = -\frac{cH}{\pi\mu}\int_0^\pi Q_t^{*'}(t-c\varrho\cos\vartheta)\cos\vartheta\,d\vartheta$$

$$-\frac{2}{\pi^2\mu}\int_a^b\left[\frac{b^2\theta^2(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(\theta^2-b^2)}\right.$$

$$\left.\times\int_0^\pi Q_t^{*'}(t-\theta\varrho\cos\vartheta)\cos\vartheta\,d\vartheta\right]d\theta.$$

Finally, for convenience, we replace ϑ by $\pi-\vartheta_1$ and θ by $-\theta_1$. In this formula, denoting again ϑ_1 and θ_1 by ϑ and θ, respectively, and taking for q_0 the half-sum of the obtained results, we have

$$q_0 = \frac{1}{2\pi}\int_0^\pi \cos\vartheta\,d\vartheta\left\{\frac{-cH}{\mu}Q_t^{*'}(t-\varrho c\cos\vartheta)+\frac{cH}{\mu}Q_t^{*'}(t+\varrho c\cos\vartheta)\right.$$

$$-\frac{2}{\pi\mu}\int_a^b\frac{b^2\theta^2(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}Q_t^{*'}(t-\theta\varrho\cos\vartheta)\,d\theta$$

$$\left.+\frac{2}{\pi\mu}\int_{-b}^{-a}\frac{b^2\theta^2(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^{*'}(t-\theta\varrho\cos\vartheta)\,d\theta\right\}. \tag{49}$$

Let us transform the expression for w_0. We differentiate the formula with respect to the parameter, substitute our expression for H. Lamb's integral, and replace ϑ by $\pi - \vartheta_1$ and θ by $-\theta_1$. Renaming and taking the half-sum of the results, we obtain

$$
\begin{aligned}
w_0 = \frac{1}{2\pi}\int\limits_0^{\pi/2} d\vartheta\Bigg\{ &\frac{-1}{\pi\mu}\int\limits_a^b \frac{b^2\theta(2\theta^2-b^2)\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\\
&-\frac{1}{\pi\mu}\mathfrak{B}\int\limits_b^\infty \frac{b^2\theta\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\\
&+\frac{1}{\pi\mu}\int\limits_{-b}^{-a} \frac{b^2\theta(2\theta^2-b^2)^2\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\\
&+\frac{1}{\pi\mu}\mathfrak{B}\int\limits_{-\infty}^{-b} \frac{b^2\theta\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\Bigg\}\\
+\frac{1}{2\pi}\int\limits_{\pi/2}^{\pi} d\vartheta\Bigg\{ &\frac{-1}{\pi\mu}\int\limits_a^b \frac{b^2\theta(2\theta^2-b^2)\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\\
&-\frac{1}{\pi\mu}\mathfrak{B}\int\limits_b^\infty \frac{b^2\theta\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\\
&+\frac{1}{\pi\mu}\int\limits_{-b}^{-a} \frac{b^2\theta(2\theta^2-b^2)^2\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\\
&+\frac{1}{\pi\mu}\mathfrak{B}\int\limits_{-\infty}^{-b} \frac{b^2\theta\sqrt{\theta^2-a^2}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\Bigg\}.
\end{aligned} \tag{50}
$$

Let us transform now both integrals written by using the theory of functions of complex variable.

In the first integral $\cos\vartheta > 0$, hence, the function $\overline{Q}_t'(t-\varrho\theta\cos\vartheta)$ is regular in the lower half-plane of the complex variable θ. To transform it, we consider the integral of the following function of the complex variable over the contour I (see Fig. 8)

$$\frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}_t'(t-\varrho\theta\cos\vartheta).$$

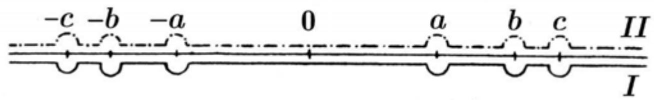

Fig. 8.

If we assume that $\overline{Q}'(\infty)$ vanishes as θ^{-1-a}, then the integral over this contour vanishes, and we can write

$$\mathfrak{B}\int_{-\infty}^{-b} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\pi i\frac{cb^2\sqrt{c^2-a^2}}{F'(-c)}\overline{Q}'_t(t+\varrho c\cos\vartheta)$$

$$+\int_{-b}^{-a} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$+i\int_{-a}^{a} \frac{b^2\theta\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\int_{a}^{b} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\mathfrak{B}\int_{b}^{\infty} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\vartheta$$

$$-\pi i\frac{cb^2\sqrt{c^2-a^2}}{F'(c)}\overline{Q}'_t(t-\varrho c\cos\vartheta)=0.$$

Taking the imaginary part of this equality, we obtain

$$\mathfrak{B}\int_{b}^{\infty} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$+\int_{a}^{b} \frac{b^2\theta(2\theta^2-b^2)\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\mathfrak{B}\int_{-\infty}^{-b} \frac{b^2\theta\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{\theta^2-a^2}\sqrt{\theta^2-b^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\int\limits_{-b}^{-a} \frac{b^2\theta(2\theta^2-b^2)\sqrt{\theta^2-a^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$= -\frac{\pi cb^2\sqrt{c^2-a^2}}{F'(-c)}\overline{Q}'_t(t+\varrho c\cos\vartheta) - \frac{\pi cb^2\sqrt{c^2-a^2}}{F'(c)}\overline{Q}'_t(t-\varrho c\cos\vartheta)$$

$$+\int\limits_{-b}^{-a} \frac{b^2\theta\sqrt{b^2-\theta^2}(\theta^2-a^2)}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$+\int\limits_{-a}^{a} \frac{b^2\theta\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2-4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$+\int\limits_{a}^{b} \frac{b^2(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}\overline{Q}'_t(t-\varrho\theta\cos\vartheta)\,d\theta. \qquad (51)$$

If we now consider the part of the integral, where $0 > \cos\vartheta > -1$, i.e., $\frac{\pi}{2} < \vartheta < \pi$, then the function $\overline{Q}'_t(t-\varrho\theta\cos\vartheta)$ is regular in the upper half-plane of the complex variable θ. If we again perform the transformation by using the integration over the contour II, then, because of the fact that all values of the radicals are replaced by their conjugates, we obtain a formula that differs from (51) only by the sign of its right side. However, it is not difficult to present these two representations by using one formula. Indeed, by the definition of the function Q_t, all terms containing negative values of θ vanish. Therefore, we can change their sign, at the same time the formula remains valid. By the same reason, we can change the sign in all terms with positive θ in the formula corresponding to the interval $\frac{\pi}{2} < \vartheta < \pi$.

Thus, we bring both formulas to the same form. Substituting the result into the expression for w_0, we obtain the final expressions for q_0 and w_0 in the form

$$q_0 = \frac{1}{2\pi}\int\limits_0^{\pi} d\vartheta\, \cos\vartheta$$

$$\times\left\{\frac{c|-c|(2c^2-b^2-2\sqrt{c^2-a^2}\sqrt{c^2-b^2})}{\mu F'(-c)}Q_t^{*'}(t+\varrho c\cos\vartheta)\right.$$

$$+\frac{c|c|(2c^2-b^2-2\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{\mu F'(c)}Q_t^{*'}(t-\varrho c\cos\vartheta)$$

$$-\frac{2}{\pi\mu}\int\limits_{-b}^{-a} \frac{b^2\theta|\theta|(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t^{*'}(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\frac{2}{\pi\mu}\int\limits_a^b \frac{b^2\theta|\theta|(2\theta^2-b^2)\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)} Q_t^{*\prime}(t-\varrho\theta\cos\vartheta)\,d\theta\Bigg\}, \qquad (52.1)$$

$$w_0 = \frac{1}{2\pi}\int\limits_0^\pi d\vartheta\Bigg\{\frac{-|-c|b^2\sqrt{c^2-a^2}}{\mu F'(-c)}Q_t'(t+\varrho c\cos\vartheta)$$

$$+\frac{|c|b^2\sqrt{c^2-a^2}}{\mu F'(c)}Q_t'(t-\varrho c\cos\vartheta)$$

$$-\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{b^2|\theta|(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t'(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{b^2|\theta|(\theta^2-a^2)\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^4+16\theta^4(\theta^2-a^2)(b^2-\theta^2)}Q_t'(t-\varrho\theta\cos\vartheta)\,d\theta$$

$$-\frac{1}{\pi\mu}\int\limits_{-a}^{a}\frac{b^2|\theta|\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}Q_t'(t-\varrho\theta\cos\vartheta)\,d\theta\Bigg\}. \qquad (52.2)$$

Comparing these formulas with (19.1) and (19.2) in Chap. 1, we easily observe that they are completely analogous. Formulas (52) are obtained from (19), if in the expressions for u_0 and v_0 we replace the function $\overline{Q}$ by $-i|\theta|\overline{Q}$ and, putting $x' = \varrho\cos\vartheta'$, integrate $u_0\cos\vartheta'$ and v_0 with respect to the parameter ϑ'. Obviously, the physical meaning of these operations is that in our space we consider a plane problem depending on the coordinates $x' = \varrho\cos\vartheta'$ and $y' = z$. Then, rotating this coordinate system, we sum the obtained results. Obviously, this plane problem is presented as a sum of plane waves analogous to the ones that had been studied in solving the Lamb plane problem. It is only necessary to substitute $-i|\theta|\overline{Q}'$ for $\overline{Q}$. Hence the simplest assumption is reduced to the one that in the depth of the medium the solution is also represented by the same sum of plane waves. Thus, we can immediately write the final formula for the depth

$$q = \mathrm{Re}\Bigg\{\frac{1}{2\pi}\int\limits_0^\pi d\vartheta\,\cos\vartheta$$

$$\times\Bigg\{\Bigg[-i\frac{c|-c|(2c^2-b^2)}{\mu F'(-c)}\overline{Q}_t'(t+c\varrho\cos\vartheta+i\sqrt{c^2-a^2}z)$$

$$+\frac{i}{\pi\mu}\int\limits_{-b}^{-a}\frac{4\theta^3|\theta|\sqrt{b^2-\theta^2}\sqrt{\theta^2-a^2}(2\theta^2-b^2)\overline{Q}_t'(t-\varrho\theta\cos\vartheta+i\sqrt{\theta^2-a^2}z)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}$$

$$-\frac{1}{2\pi\mu}\int\limits_{-a}^{a}\frac{\theta|\theta|(2\theta^2-b^2)}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}'_t(t-\varrho\theta\cos\vartheta+\sqrt{a^2-\theta^2}z)-\overline{Q}'_t(t-\varrho\theta\cos\vartheta-\sqrt{a^2-\theta^2}z)]\,d\theta$$

$$+\frac{i}{\pi\mu}\int\limits_{a}^{b}\frac{4\theta^3|\theta|\sqrt{b^2-\theta^2}\sqrt{\theta^2-a^2}(2\theta^2-b^2)\overline{Q}'_t(t-\varrho\theta\cos\vartheta+i\sqrt{\theta^2-a^2}z)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}$$

$$-i\frac{c|c|(2c^2-b^2)}{\mu F'(c)}\overline{Q}'_t(t-c\varrho\cos\vartheta+i\sqrt{c^2-a^2}z)\Big]$$

$$+\Big[\frac{i2c|-c|\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{\mu F'(-c)}\overline{Q}'_t(t+c\varrho\cos\vartheta+i\sqrt{c^2-b^2}z)$$

$$-\frac{i}{\pi\mu}\int\limits_{-b}^{-a}\frac{|\theta|\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{i}{\pi\mu}\int\limits_{-b}^{-a}\frac{|\theta|\theta\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{1}{\pi\mu}\int\limits_{-a}^{a}\frac{\theta|\theta|\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)-\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)]\,d\theta$$

$$-\frac{i}{\pi\mu}\int\limits_{a}^{b}\frac{\theta|\theta|\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{i}{\pi\mu}\int\limits_{a}^{b}\frac{\theta|\theta|\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+i\frac{2c|c|\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{\mu F'(c)}\overline{Q}'_t(t-c\varrho\cos\vartheta+i\sqrt{c^2-b^2}z)\Big]\Big\}\Big\}. \tag{53.1}$$

In the same way,

$$w=\mathrm{Re}\Big\{\frac{1}{2\pi}\int\limits_{0}^{\pi}d\vartheta\Big\{\Big[\frac{(2c^2-b^2)|-c|\sqrt{c^2-b^2}}{\mu F'(-c)}\overline{Q}'_t(t+c\varrho\cos\vartheta+i\sqrt{c^2-a^2}z)$$

$$+\frac{1}{\pi\mu}\int\limits_{-b}^{-a}\frac{4\theta^2|\theta|(2\theta^2-b^2)(\theta^2-a^2)\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\varrho\theta\cos\vartheta+i\sqrt{\theta^2-a^2}z)\,d\theta}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{1}{2\pi\mu}\int\limits_{-a}^{a}\frac{|\theta|(2\theta^2-b^2)\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}'_t(t-\varrho\theta\cos\vartheta-\sqrt{a^2-\theta^2}z)+\overline{Q}'_t(t-\varrho\theta\cos\vartheta+\sqrt{a^2-\theta^2}z)]\,d\theta$$

$$+\frac{1}{\pi\mu}\int\limits_{a}^{b}\frac{4\theta^2|\theta|(2\theta^2-b^2)(\theta^2-a^2)\sqrt{b^2-\theta^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta+i\sqrt{\theta^2-a^2}z)\,d\theta}{(2\theta^2-b^2)^4-16\theta^4(\theta^2-a^2)(\theta^2-b^2)}$$

$$-\frac{|c|(2c^2-b^2)\sqrt{c^2-a^2}}{\mu F'(c)}\overline{Q}'_t(t-c\varrho\cos\vartheta+i\sqrt{c^2-a^2}z)\Bigg]$$

$$+\Bigg[\frac{2c^2|c|\sqrt{c^2-a^2}}{\mu F'(-c)}\overline{Q}'_t(t+c\varrho\cos\vartheta+i\sqrt{c^2-b^2}z)$$

$$+\frac{i}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta^2|\theta|\sqrt{\theta^2-a^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{i}{\pi\mu}\int\limits_{-b}^{-a}\frac{\theta^2|\theta|\sqrt{\theta^2-a^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{1}{\pi\mu}\int\limits_{-a}^{a}\frac{\theta^2|\theta|\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}$$

$$\times[\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)+\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)]\,d\theta$$

$$-\frac{i}{\pi\mu}\int\limits_{a}^{b}\frac{\theta^2|\theta|\sqrt{\theta^2-a^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta+\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2+4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$+\frac{i}{\pi\mu}\int\limits_{a}^{b}\frac{\theta^2|\theta|\sqrt{\theta^2-a^2}\,\overline{Q}'_t(t-\theta\varrho\cos\vartheta-\sqrt{b^2-\theta^2}z)\,d\theta}{(2\theta^2-b^2)^2-4i\theta^2\sqrt{\theta^2-a^2}\sqrt{b^2-\theta^2}}$$

$$-\frac{2c^2|c|}{\mu F'(c)}\overline{Q}'_t(t-c\varrho\cos\vartheta+i\sqrt{c^2-b^2}z)\Bigg]\Bigg\}\Bigg\}. \qquad (53.2)$$

In the same way as in the plane problem, one can rewrite these expressions in a form more compact and more convenient for computations.

The details of computations are completely similar to the ones already carried out, and we do not repeat them here. Let us point out only the final result. If the function $Q'(t)$ rapidly enough vanishes as $t\to\infty$, then q and w have the form

$$q = \int\limits_{a\sqrt{\varrho^2+z^2}}^{\infty} Q'(t-H)\left\{\frac{1}{2\pi^2\mu}\mathrm{Re}\int\limits_0^{\pi}\frac{\theta^2(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\cos\vartheta\,d\vartheta\right\}dH$$

$$+ \int\limits_{\substack{b\sqrt{\varrho^2+z^2} \quad (a\sqrt{\varrho^2+z^2}>b\varrho)\\ a\varrho+\sqrt{b^2-a^2}z \quad (a\sqrt{\varrho^2+z^2}<b\varrho)}}^{\infty} Q'(t-\Xi)$$

$$\times\left\{\frac{1}{2\pi^2\mu}\mathrm{Re}\int\limits_0^{\pi}\frac{2\theta^2\alpha\beta\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\cos\vartheta\,d\vartheta\right\}d\Xi \tag{54.1}$$

and

$$w = \int\limits_{a\sqrt{\varrho^2+z^2}}^{\infty} Q'(t-H)\left\{\frac{1}{2\pi^2\mu}\mathrm{Re}\int\limits_0^{\pi}\frac{\theta\alpha(2\theta^2-b^2)\frac{\partial\theta}{\partial H}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,d\vartheta\right\}dH$$

$$- \int\limits_{\substack{b\sqrt{\varrho^2+z^2} \quad (a\sqrt{\varrho^2+z^2}>b\varrho)\\ a\varrho+\sqrt{b^2-a^2}z \quad (a\sqrt{\varrho^2+z^2}<b\varrho)}}^{\infty} Q'(t-\Xi)$$

$$\times\left\{\frac{1}{2\pi^2\mu}\mathrm{Re}\int\limits_0^{\pi}\frac{2\theta^3\alpha\frac{\partial\theta}{\partial\Xi}}{(2\theta^2-b^2)^2+4\theta^2\alpha\beta}\,d\vartheta\right\}d\Xi. \tag{54.2}$$

In these formulas values of the functions $\theta(H)$, $\alpha(H)$, $\beta(H)$, $\theta(\Xi)$, $\alpha(\Xi)$, and $\beta(\Xi)$ are obtained from (36) and (38), if we replace x by $\varrho\cos\vartheta$. Formulas (54) give the final answer to the stated problem. It is not difficult to see that the picture of the wave front motion in the case of the three-dimensional problem completely coincides with the picture obtained by the front rotation for the solution of the plane problem. For the solution obtained by us one can repeat all the same that was said for the solution of the plane problem related to the nature of the Rayleigh waves, etc.

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2. On a New Method in the Plane Problem on Elastic Vibrations*

V. I. Smirnov and S. L. Sobolev

1. The problem on vibrations of an elastic half-space bounded by the vacuum was posed by H. Lamb in his well-known article [1]. He considered a series of problems on vibrations under the action of different forces. In some cases he solved these problems completely, and in other cases he only presented formulas containing divergent Fourier integrals. First, H. Lamb considered the force periodic in time and spatial coordinates, and then he applied the Fourier integral to arrive at the general case.

In the present work we propose a new method, which allows us to solve some of H. Lamb's problems by means of simple calculations. Our method gives tools to determine displacements not only on the surface (as H. Lamb), but also inside the half-space.

The essential feature of our method is the reduction of a problem with three independent variables to one with one or two independent variables.

Two real variables can be reduced to one complex variable, and we can use the theory of functions of a complex variable to find the solution.

First, we consider the problem discussed by H. Lamb on vibrations of the half-space under the action of a vertical impact on the surface. Then, we discuss problems when the source of the force is located inside the elastic medium. Under some fundamental assumptions, we find a solution by reducing a number of independent variables. Obtained solutions satisfy initial and boundary conditions.

Our general reasoning allows us to study the reflection of elastic waves of special types on the plane.

For instance, we can solve the problem on vibration of an elastic layer.

2. Let us state the first problem on vibrations of the half-space under the action of a vertical impact on the surface.

Assume that the surface of the medium is the (x, z)-plane and suppose that the motion does not depend on the coordinate z. Then, our problem is reduced to the two-dimensional problem, which is very important later.

* Tr. Seism. Inst., **20** (1932), 37 p.

For the components of the displacement u and v we have

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{1}$$

and the functions φ and ψ must satisfy the equations

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{a^2}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right), \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{b^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right), \tag{2}$$

where

$$a = \sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad b = \sqrt{\frac{\rho}{\mu}}. \tag{3}$$

Denote by ρ the density of the medium, λ and μ are the Lame elastic constants.

Suppose that $R(x,t)$ is the vertical force acting along the x-axis and normal to the surface $y = 0$. Then we have the boundary conditions

$$\left[2\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right]\bigg|_{y=0} = 0, \tag{4}$$

$$\left[\left(\frac{b^2}{a^2} - 2\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + 2\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x \partial y}\right]\bigg|_{y=0} = \frac{R(x,t)}{\mu}. \tag{5}$$

To consider the case of the impact concentrated at the point $x = 0$ at the moment $t = 0$, we pass to the limit.

Let

$$P_\varepsilon(x,t) = \frac{1}{\varepsilon^2} P\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right),$$

where $P(x,t)$ is a function continuous in the rectangle

$$-1 \le x \le 1, \quad 0 \le t \le 1,$$

$$P(x,t) \equiv 0 \quad \text{for} \quad |x| \ge 1 \quad \text{or} \quad \left|t - \frac{1}{2}\right| \ge \frac{1}{2}.$$

Let $\varphi_\varepsilon(x,y,t)$ and $\psi_\varepsilon(x,y,t)$ be solutions of equations (2) with conditions (4) and (5), where we replace $R(x,t)$ by $P_\varepsilon(x,t)$.

We consider the problem on vibrations under the action of the impact as the limiting case of the stated problem as $\varepsilon \to 0$.

Thus, we have

$$\varphi(x,y,t) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x,y,t), \quad \psi(x,y,t) = \lim_{\varepsilon \to 0} \psi_\varepsilon(x,y,t).$$

The value of the impact is defined as

$$Q = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dx \int_0^{\varepsilon} P_\varepsilon(x,t)\, dt = \int_{-1}^{1} dx \int_0^1 P(x,t)\, dt.$$

After defining the functions φ_ε and ψ_ε, we have

$$\varphi_\varepsilon(kx, ky, kt) = \varphi_{\varepsilon/k}(x, y, t) \quad \text{and} \quad \psi_\varepsilon(kx, ky, kt) = \psi_{\varepsilon/k}(x, y, t).$$

This property of the functions φ_ε and ψ_ε is stipulated by the form of equations (2), conditions (4) and (5), and by the definition of $P_\varepsilon(x, t)$. Passing to the limit, we have

$$\varphi(kx, ky, kt) = \varphi(x, y, t) \quad \text{and} \quad \psi(kx, ky, kt) = \psi(x, y, t),$$

i.e., the functions φ and ψ are homogeneous of degree 0. Hence they depend on two variables

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}. \tag{6}$$

Also, note the case when the potentials φ and ψ are homogeneous functions. Let $P(x)$ be an odd function for $-1 \leq x \leq 1$. In (5) we put

$$R(x, t) = 0 \quad \text{for} \quad t < 0 \quad \text{and} \quad R(x, t) = \frac{1}{\varepsilon^2} P\left(\frac{x}{\varepsilon}\right) \quad \text{for} \quad t > 0.$$

In this case, we have

$$\int\limits_{-\varepsilon}^{\varepsilon} R(x, t)\, dx = \frac{1}{\varepsilon} \int\limits_{-1}^{1} P(x)\, dx = 0,$$

and the moment with respect to $x = 0$ is equal to

$$\frac{2}{\varepsilon^2} \int\limits_0^{\varepsilon} x P\left(\frac{x}{\varepsilon}\right) dx = 2 \int\limits_0^1 x P(x)\, dx = q.$$

As $\varepsilon \to 0$, we have the focused moment q applied at $t = 0$.

Therefore, we see that the case of homogeneous potentials can arise under different mechanical circumstances. In this connection, later we will see that a solution of the problem contains several arbitrary constants, defined by mechanical conditions of the problem. It should be noted that we again deal with nonuniqueness of the solution. Later we will have an equation on the boundary of the existence domain of an analytic function. This equation will express the fact that the real part of a linear operator must vanish on this function. Assuming that the mentioned operator vanishes everywhere, we will select the simplest solution of this equation. We will also be able to obtain other solutions of the problem. For this, we equate this operator to a regular function, whose real part has zero boundary value on the entire contour with the exception of a unique singular point of this function. We will not study the family of all solutions, but we hope to do it in a future paper.

Moving on to consideration of the functions φ and ψ, let us note a fact, which we will encounter later.

Using homogeneity of the functions φ and ψ, we reduce equations (2) to two equations with two independent variables. Furthermore, by suitable choice of these variables, we reduce these equations to the Laplace equation or the vibrating string equation. Indeed, if the functions φ and ψ depend only on quantities (6), then equations (2) take the form

$$\begin{aligned}(a^2\xi^2-1)\frac{\partial^2\varphi}{\partial\xi^2}+2a^2\xi\eta\frac{\partial^2\varphi}{\partial\xi\partial\eta}+(a^2\eta^2-1)\frac{\partial^2\varphi}{\partial\eta^2}+2a^2\xi\frac{\partial\varphi}{\partial\xi}+2a^2\eta\frac{\partial\varphi}{\partial\eta}&=0,\\(b^2\xi^2-1)\frac{\partial^2\psi}{\partial\xi^2}+2b^2\xi\eta\frac{\partial^2\psi}{\partial\xi\partial\eta}+(b^2\eta^2-1)\frac{\partial^2\psi}{\partial\eta^2}+2b^2\xi\frac{\partial\psi}{\partial\xi}+2b^2\eta\frac{\partial\psi}{\partial\eta}&=0.\end{aligned}\tag{7}$$

Characteristics for the first equation in (7) are determined by the ordinary differential equation

$$(a^2\xi^2-1)d\eta^2-2a^2\xi\eta d\xi d\eta+(a^2\eta^2-1)d\xi^2=0$$

and by a similar equation for the second equation.

The last equation can be written in the form

$$a^2(\xi d\eta-\eta d\xi)^2-(d\xi^2+d\eta^2)=0.$$

Let ds be an element of the characteristic arc. Then we can write our equation in the form

$$\xi\frac{d\eta}{ds}-\eta\frac{d\xi}{ds}=\pm\frac{1}{a},$$

hence we see that the characteristics touch the circle

$$\xi^2+\eta^2=\frac{1}{a^2}.$$

The first equation in (7) is elliptic, if

$$\xi^2+\eta^2<\frac{1}{a^2},\tag{8.1}$$

and hyperbolic, if

$$\xi^2+\eta^2>\frac{1}{a^2}.\tag{8.2}$$

In the last case, two families of characteristics are expressed by the equation

$$-C\xi\pm\sqrt{a^2-C^2}\eta+1=0,$$

where C is an arbitrary constant. This equation gives for C two complex conjugate values under condition (8.1). Let us begin our analysis with this case. Then, we have the imaginary characteristics

$$\frac{\xi}{\xi^2+\eta^2}\pm i\frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}=C.$$

Putting

$$\sigma = \frac{\xi}{\xi^2+\eta^2}, \quad \tau = \frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}, \tag{9.1}$$

we reduce the first equation in (7) to the Laplace equation

$$\frac{\partial^2\varphi}{\partial\sigma^2} + \frac{\partial^2\varphi}{\partial\tau^2} = 0. \tag{10.1}$$

Similarly, under condition (8.2), by the real transform

$$\sigma = \frac{\xi}{\xi^2+\eta^2}, \quad \tau = \frac{\eta\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}, \tag{9.2}$$

we bring the first equation in (7) to the vibrating string equation

$$\frac{\partial^2\varphi}{\partial\sigma^2} - \frac{\partial^2\varphi}{\partial\tau^2} = 0. \tag{10.2}$$

In the second part of our work we discuss a more general and simple way of the reduction of equations (2) to canonical form (10.1) or (10.2).

3. Taking into account that the initial moment $t = 0$ of the action of our force corresponds to the rest of the half-space and that vibrations cannot propagate with a velocity more than the velocity of longitudinal vibrations, we can assert that a required solution will vanish outside the circle

$$\xi^2+\eta^2 = \frac{1}{a^2}. \tag{11.1}$$

Thus, to find the potential φ, we have to integrate equation (10.1).

As regards the search for the potential ψ, a should be replaced by b in all previous formulas. The characteristics of the second equation in (7) will be tangent to the circle

$$\xi^2+\eta^2 = \frac{1}{b^2}, \tag{11.2}$$

and this equation will be reduced to (10.2) outside this circle. If the point (ξ, η) is located not only outside circle (11.2), but also outside circle (11.1), then the value of ψ must also vanish at this point.

Note that at each point outside circle (11.2) ψ is a sum of two terms[1], each of which is constant along one of two characteristics passing through this point. Then we can assert that ψ can differ from zero outside circle (11.2) only on the intervals of tangents between the point of tangency and the axis

[1] The function ψ has the form

$$f_1\left(\frac{\xi+\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}\right) + f_2\left(\frac{\xi-\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}\right). \text{ – } \textit{Ed.}$$

$\eta = 0$, and on such tangents, which have a projection on this axis less than $\dfrac{1}{a}$ by counting from the origin of coordinates.

Therefore, for the transverse wave, the front in the (ξ, η)-plane consists of the arc AB of circle (11.2) and two segments of tangents AA_1 and BB_1 such that $\overline{OA_1} = \overline{OB_1} = \dfrac{1}{a}$ (see Fig. 1). For the longitudinal wave, i.e., for the potential φ, the front consists only of semicircle (11.1). The shape of the front of the transverse wave (see Fig. 1) can be immediately obtained from the Fermat principle. It should be noted that vibrations propagate over the surface with the velocity $\dfrac{1}{a}$, and each point of this surface is a source of not only longitudinal, but also transverse vibrations. At the same time these transverse vibrations propagate inside with the velocity $\dfrac{1}{b}$.

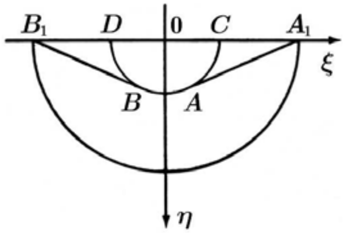

Fig. 1.

The equation of the straight line AA_1 in the (ξ, η)-plane is

$$a\xi + \sqrt{b^2 - a^2}\,\eta - 1 = 0. \tag{12.1}$$

Returning to the variables x, y, t, we obtain the rectilinear front

$$ax + \sqrt{b^2 - a^2}\,y - t = 0. \tag{12.2}$$

To study equation (10.1), we introduce the complex variable

$$\theta_1 = \sigma + i\tau = \frac{\xi}{\xi^2 + \eta^2} + i\frac{\eta\sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}.$$

This transform maps the semidisk

$$\xi^2 + \eta^2 < \frac{1}{a^2}, \quad \eta > 0,$$

onto the half-plane $\tau > 0$ of the complex variable θ_1, the diameter B_1A_1 onto the intervals $(-\infty, -a)$ and $(+a, +\infty)$ of the axis $\tau = 0$, and the semicircle B_1A_1 onto the interval $(-a, +a)$ of this axis (see Fig. 2). In the half-plane

$\tau > 0$ the potential φ is a harmonic function and can be expressed as the real part of an analytic function $\Phi(\theta_1) = \varphi + i\varphi^*$:

$$\varphi = \operatorname{Re}[\Phi(\theta_1)].$$

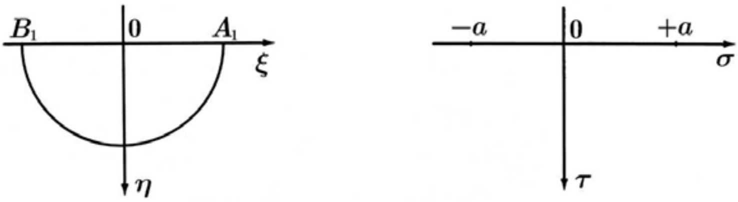

Fig. 2.

Similarly, introducing the complex variable

$$\theta_2 = \sigma + i\tau = \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-b^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},$$

in the semidisk

$$\xi^2+\eta^2 < \frac{1}{b^2}, \quad \eta > 0,$$

we can express the potential ψ as the real part of a function $\Psi(\theta_2) = \psi + i\psi^*$ analytic in the half-plane $\tau > 0$:

$$\psi = \operatorname{Re}[\Psi(\theta_2)].$$

The formulas

$$\begin{aligned}\theta_1 &= \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},\\ \theta_2 &= \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-b^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}\end{aligned} \tag{13}$$

prove that the values of θ_1 and θ_2 coincide at the points of the diameter CD (see Fig. 1), which will be essential later.

It is easy to prove that on the plane θ_2 the points D and C correspond to the points $+b$ and $-b$ of the axis $\tau = 0$, and the points B and A correspond to the points $+a$ and $-a$ of this axis.

Let us now introduce the boundary conditions with respect to the new variables. For any $t > 0$, there are no stresses on the surface of the half-space.

Therefore, for φ and ψ we should take conditions (4) and (5) with $P(x,t) = 0$. We obtain

$$\begin{aligned} D_1(\varphi,\psi) &= \left[2\frac{\partial^2\varphi}{\partial\xi\partial\eta} - \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\right]\Big|_{y=0} = 0, \\ D_2(\varphi,\psi) &= \left[\left(\frac{b^2}{a^2} - 2\right)\left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2}\right) + 2\frac{\partial^2\varphi}{\partial\eta^2} - 2\frac{\partial^2\psi}{\partial\xi\partial\eta}\right]\Big|_{y=0} = 0, \end{aligned} \tag{14}$$

where we denote by D_1 and D_2 the linear operators on the left side of our conditions. Differentiation with respect to ξ and η can be replaced by differentiation with respect to θ_1 and θ_2. It is easy to see that for $\eta = 0$ we have

$$\frac{\partial\theta_1}{\partial\xi} = -\theta_1^2, \quad \frac{\partial^2\theta_1}{\partial\xi^2} = 2\theta_1^3, \quad \frac{\partial\theta_1}{\partial\eta} = -\theta_1\sqrt{a^2-\theta_1^2}, \quad \frac{\partial^2\theta_1}{\partial\eta^2} = -2\theta_1^3,$$

$$\frac{\partial^2\theta_1}{\partial\xi\partial\eta} = -\frac{2\theta_1^4 - a^2\theta_1^2}{\sqrt{a^2-\theta_1^2}},$$

where the square root has the negative imaginary value for $\theta_1 > a$.

We have similar expressions for θ_2. Conditions (14) take the form

$$\begin{aligned} &\mathrm{Re}\left[2\theta\sqrt{a^2-\theta^2}\Phi''(\theta) + 2\frac{a^2-2\theta^2}{\sqrt{a^2-\theta^2}}\Phi'(\theta)\right. \\ &\qquad\qquad \left. -(2\theta^2-b^2)\Psi''(\theta) - 4\theta\Psi'(\theta)\right]\Big|_{\tau=0} = 0, \\ &\mathrm{Re}\left[(b^2-2\theta^2)\Phi''(\theta) - 4\theta\Phi'(\theta)\right. \\ &\qquad\qquad \left. -2\theta\sqrt{b^2-\theta^2}\Psi''(\theta) - 2\frac{b^2-2\theta^2}{\sqrt{b^2-\theta^2}}\Psi'(\theta)\right]\Big|_{\tau=0} = 0. \end{aligned} \tag{15}$$

Since θ_1 and θ_2 coincide on the axis $\eta = 0$, we denote the variables by θ without index.

Conditions (15) must be satisfied on the part that corresponds to the diameters of the semicircles.

Taking into account what we said about the correspondence between θ_1, θ_2, ξ and η, we see that conditions (15) must be satisfied on the intervals $\sigma \le -b$ and $\sigma \ge +b$. Note once again that the interval $-a \le \sigma \le +a$ of the variables θ_1 and θ_2 corresponds to the arcs of the semicircles, forming the front of propagation of longitudinal and transverse vibrations. Consequently, the functions φ and ψ, i.e., the real parts of Φ and Ψ, must vanish on this interval. Taking into account that all coefficients on the left sides of (15) are real for $-a \le \theta \le +a$, we can assert that conditions (15) must be also satisfied on the interval $-a \le \theta \le +a$.

Later, we show that these conditions must hold also on two intervals $-b \leq \theta \leq -a$ and $a \leq \theta \leq b$. For this purpose we consider an equation of hyperbolic type for ψ in the curvilinear triangles AA_1C and BB_1D (see Fig. 1). It is enough to consider the triangle BB_1D. Introducing the variables

$$\sigma = \frac{\xi}{\xi^2+\eta^2}, \qquad \tau = \frac{\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}, \tag{16}$$

for ψ we have the vibrating string equation

$$\frac{\partial^2\psi}{\partial\sigma^2} - \frac{\partial^2\psi}{\partial\tau^2} = 0,$$

whose solution is

$$\psi = f_1(\sigma+\tau) + f_2(\sigma-\tau).$$

Since ψ is equal to zero outside circle (11.1), as above, we can assert that the last expression for ψ contains at most one term different from zero on the pieces of the characteristics, made of segments of tangents between the arc BD and the axis $\eta = 0$.

The mentioned segments can be defined by the values of the real parameter θ_3,

$$\theta_3 = \frac{\xi}{\xi^2+\eta^2} - \frac{\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}, \quad a \leq \theta_3 \leq b, \tag{17}$$

and the function ψ depends only on θ_3 inside the triangle BB_1D. It is easy to see that the value of θ_3 coincides on each tangent with the corresponding value of θ_2 on the arc BD. Hence, in view of continuity of ψ, in the triangle BB_1D we should take

$$\psi = \operatorname{Re}\,[\Psi(\theta_3)].$$

On the interval B_1D of the axis $\eta = 0$ the values of θ_3 coincide with the values of θ_1.

Returning to conditions (14), we can express the derivatives with respect to ξ and η by the derivatives with respect to θ_1 and θ_3. These variables can be denoted by the same letter θ, and $a \leq \theta \leq b$.

Conditions (14) take the form

$$\operatorname{Re}\left\{2\theta\sqrt{a^2-\theta^2}\Phi''(\theta) + 2\frac{a^2-2\theta^2}{\sqrt{a^2-\theta^2}}\Phi'(\theta)\right\}$$
$$-(2\theta^2-b^2)\operatorname{Re}\,[\Psi''(\theta)] - 4\theta\operatorname{Re}\,[\Psi'(\theta)] = 0,$$
$$\operatorname{Re}\,\{(b^2-2\theta^2)\Phi''(\theta) - 4\theta\Phi'(\theta)\} - 2\theta\sqrt{b^2-\theta^2}\operatorname{Re}\,[\Psi''(\theta)]$$
$$-2\frac{b^2-2\theta^2}{\sqrt{b^2-\theta^2}}\operatorname{Re}\,[\Psi'(\theta)] = 0,$$
$$a \leq \theta \leq b.$$

Hence conditions (15) must hold on the interval $a \le \theta \le b$.

Considering the triangle AA_1C, we can similarly show that conditions (15) must hold also on the interval $-b \le \theta \le -a$. Thus, conditions (15) are established on the entire real axis of the plane θ.

The simplest conclusion from this fact is that the analytic functions on the left sides of conditions (15) are equal to imaginary constants. This conclusion is necessary, if we assume that the passage to the limit on the axis $\tau = 0$ is continuous everywhere. Thus, we obtain

$$-2\theta\sqrt{a^2-\theta^2}\Phi''(\theta) - 2\frac{a^2-2\theta^2}{\sqrt{a^2-\theta^2}}\Phi'(\theta) + (2\theta^2-b^2)\Psi''(\theta) + 4\theta\Psi'(\theta) = \alpha i,$$

$$(b^2-2\theta^2)\Phi''(\theta) - 4\theta\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi''(\theta) - 2\frac{b^2-2\theta^2}{\sqrt{b^2-\theta^2}}\Psi'(\theta) = \beta i,$$

where α and β are real constants.

Integrating the equations with respect to θ, we have

$$\begin{aligned} &-2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (2\theta^2-b^2)\Psi'(\theta) = \alpha i\theta + C_1, \\ &(b^2-2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi'(\theta) = \beta i\theta + C_2, \end{aligned} \tag{18}$$

hence,

$$\begin{aligned} \Phi'(\theta) &= \frac{-(\alpha i\theta + C_1)2\theta\sqrt{b^2-\theta^2} - (\beta i\theta + C_2)(2\theta^2-b^2)}{(2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}, \\ \Psi'(\theta) &= \frac{(\alpha i\theta + C_1)(2\theta^2-b^2) - (\beta i\theta + C_2)2\theta\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}, \end{aligned} \tag{19}$$

where C_1 and C_2 are complex constants. Consider real values of θ on the interval $-a \le \theta \le +a$. As above, this interval corresponds to the front of the longitudinal wave and to a part of the front of the transverse wave. Consequently, the real parts of $\Phi'(\theta)$ and $\Psi'(\theta)$ must be equal to zero on the interval $-a \le \theta \le +a$. Hence C_1 and C_2 are pure imaginary.

To find the constants, we express the projections of the displacements u, v by the functions Φ and Ψ by using (1). We have

$$u = \operatorname{Re}\left[\Phi'(\theta_1)\frac{\partial\theta_1}{\partial x} + \Psi'(\theta_2)\frac{\partial\theta_2}{\partial y}\right], \qquad v = \operatorname{Re}\left[\Phi'(\theta_1)\frac{\partial\theta_1}{\partial y} - \Psi'(\theta_2)\frac{\partial\theta_2}{\partial x}\right]. \tag{20}$$

The expressions for θ_1 and θ_2 give

$$\begin{aligned} \frac{\partial\theta_1}{\partial x} &= -\theta_1\frac{\partial\theta_1}{\partial t}, \qquad & \frac{\partial\theta_1}{\partial y} &= -\sqrt{a^2-\theta_1^2}\frac{\partial\theta_1}{\partial t}, \\ \frac{\partial\theta_2}{\partial x} &= -\theta_2\frac{\partial\theta_2}{\partial t}, \qquad & \frac{\partial\theta_2}{\partial y} &= -\sqrt{b^2-\theta_2^2}\frac{\partial\theta_2}{\partial t}, \end{aligned} \tag{21}$$

where the square roots are negative imaginary for θ_1 and $\theta_2 > b$. Indeed, for the variables θ_1 and θ_2 we have

$$\begin{aligned} \theta_1 &= \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-a^2(x^2+y^2)}}{x^2+y^2}, \\ \theta_2 &= \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-b^2(x^2+y^2)}}{x^2+y^2}. \end{aligned} \tag{22}$$

Consider now values of u and v on the axis $x = 0$. We assume that the impact is concentrated at the point $x = 0$ and acts along the axis $x = 0$. Hence $u = 0$ on this axis. Obviously, θ_1, θ_2, $\frac{\partial \theta_1}{\partial t}$ and $\frac{\partial \theta_2}{\partial t}$ are pure imaginary on this axis. Consequently, $\frac{\partial \theta_1}{\partial x}$ is real, and $\frac{\partial \theta_2}{\partial y}$ is pure imaginary. From the first of equations (20) we can conclude that $C_1 = \beta = 0$. Denote C_2 by $-Ci$, where C is a real constant. Then, we can write

$$\begin{aligned} \varPhi'(\theta) &= i\frac{-2\alpha\theta^2\sqrt{b^2-\theta^2} + C(2\theta^2-b^2)}{F(\theta)}, \\ \varPsi'(\theta) &= i\frac{\alpha\theta(2\theta^2-b^2) + C2\theta\sqrt{a^2-\theta^2}}{F(\theta)}, \end{aligned} \tag{23}$$

where

$$F(\theta) = (2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}. \tag{24}$$

Formulas (23) contain two real constants α and C. Consider the displacements u and v at a point of the axis $y = 0$ and assume that the time t tends to infinity. Under these assumptions, the variables θ_1 and θ_2 equal $\frac{t}{x}$ and tend to infinity. The expression

$$F(\theta) = (2\theta^2-b^2)^2 - 4\theta^4\left(1-\frac{a^2}{\theta^2}\right)^{1/2}\left(1-\frac{b^2}{\theta^2}\right)^{1/2} = (2a^2-2b^2)\theta^2 + \cdots$$

has order θ^2.

Using the expression for θ,

$$\theta = \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-c^2(x^2+y^2)}}{x^2+y^2}, \quad c^2 = a^2 \text{ or } b^2, \tag{25}$$

it easy to expand u and v in power series with respect to $\frac{1}{t}$. If $\alpha \neq 0$, then these series begin with a constant term, and we have the displacements different from zero as $t \to \infty$. This term is equal to zero for $\alpha = 0$. This fact forces us to put $\alpha = 0$. Then formulas (23) give us

$$\varPhi'(\theta) = iC\frac{2\theta^2-b^2}{F(\theta)}, \quad \varPsi'(\theta) = iC\frac{2\theta\sqrt{a^2-\theta^2}}{F(\theta)}. \tag{26}$$

The elementary potential ψ will be defined by the real part of the analytic function $\Psi(\theta)$ not only inside the semidisk

$$\xi^2 + \eta^2 < \frac{1}{b^2},$$

but also in two triangles, if we replace θ_2 by the variable θ_3 defined above.

4. The constant C in (26) depends on the concentrated impact Q. Assume that this constant is determined by the condition that Q is equal to 1. Also, assume that the force $Q(t)$ acts at the point $x = 0$ of the axis $y = 0$, where $Q(t)$ is a continuous function of t. Let $\varphi_0(x, y, t)$ and $\psi_0(x, y, t)$ be elementary potentials at the given point (x, y) at the moment t. We can construct these potentials by means of superposition of the effects of the action of the elementary impulses $Q(t - H)dH$ concentrated at the moment $t - H$, where the variable H belongs to the interval (H_0, ∞). We denote by H_0 the time interval necessary for the impulse to propagate to the point (x, y). For the longitudinal wave, H_0 is equal to $a\sqrt{x^2 + y^2}$. In the case of the transverse wave, the expression for H_0 depends on the position of the point (x, y). If this point is located inside the angle AOB (see Fig. 1), where the front of the transverse wave has the shape of a circular arc, then $H_0 = b\sqrt{x^2 + y^2}$. If, on the contrary, this point is located outside this angle, then we have $H_0 = ax + \sqrt{b^2 - a^2}\, y$. These expressions for H_0 follow immediately from equation (12.1) (in this case we assume that $x > 0$). Finally, using equations (20) and (26), we obtain two expressions for the components of the displacement:

$$u = C\,\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial \theta_1}{\partial x}}{F(\theta_1)} Q(t - H)\, dH$$

$$+ C\,\mathrm{Im} \int\limits_{b\sqrt{x^2+y^2}}^{\infty} \frac{2\theta_2\sqrt{a^2 - \theta_2^2}\,\frac{\partial \theta_2}{\partial y}}{F(\theta_2)} Q(t - H)\, dH, \tag{27.1}$$

$$v = C\,\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial \theta_1}{\partial y}}{F(\theta_1)} Q(t - H) dH$$

$$- C\,\mathrm{Im} \int\limits_{b\sqrt{x^2+y^2}}^{\infty} \frac{2\theta_2\sqrt{a^2 - \theta_2^2}\,\frac{\partial \theta_2}{\partial x}}{F(\theta_2)} Q(t - H)\, dH. \tag{27.2}$$

Expressions (27) are related to the case when (x, y) are located inside AOB, i.e., if $b^2x^2 \leq a^2(x^2 + y^2)$. In the case $b^2x^2 \geq a^2(x^2 + y^2)$, we have

$$u = C\,\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial \theta_1}{\partial x}}{F(\theta_1)} Q(t - H)\, dH$$

$$+C\mathrm{Im}\int\limits_{ax+\sqrt{b^2-a^2}\,y}^{\infty}\frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial y}}{F(\theta_2)}Q(t-H)\,dH, \tag{28.1}$$

$$v=C\mathrm{Im}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\frac{(2\theta_1^2-b^2)\frac{\partial\theta_1}{\partial y}}{F(\theta_1)}Q(t-H)\,dH$$

$$-C\mathrm{Im}\int\limits_{ax+\sqrt{b^2-a^2}\,y}^{\infty}\frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial x}}{F(\theta_2)}Q(t-H)\,dH. \tag{28.2}$$

In these formulas we should take

$$\theta_2=\frac{Hx}{x^2+y^2}+i\frac{y\sqrt{H^2-b^2(x^2+y^2)}}{x^2+y^2}\quad\text{for}\quad H^2\geq b^2(x^2+y^2),$$

$$\theta_2=\frac{Hx}{x^2+y^2}-\frac{y\sqrt{b^2(x^2+y^2)-H^2}}{x^2+y^2}\quad\text{for}\quad H^2\leq b^2(x^2+y^2),$$

with the arithmetical square root. To determine the derivatives of θ with respect to x and y, one can use formulas (21). Obviously, we should assume that the behavior of the function $Q(t)$ as $t\to-\infty$ is such that the integrals mentioned above converge.

Formulas (27) and (28) coincide with the formulas derived in the work of S. L. Sobolev [2], but the method described here is simpler and allows to solve many other questions without any application of the Fourier integral. It is known that such application frequently leads to essential complexities in solving the problem.

The analysis of formulas (26), (27) and (28) was carried out in the mentioned work of S. L. Sobolev, nevertheless, we repeat some moments of this analysis here.

First of all, note that in the case of the concentrated impact, the components of u and v are infinite on circles (11.1) and (11.2). This fact follows from the expressions for the derivatives

$$\frac{\partial\theta}{\partial x}\quad\text{and}\quad\frac{\partial\theta}{\partial y}.$$

A unique exception are points on the axis $\eta=0$, where the displacement is equal to zero. The mentioned circumstance also take place on the parts AC and BD of circle (11.2), which does not compose the front of the disturbance propagation. At the moments corresponding to such parts, we have the beginning of a new phase of vibrations. On the lines

$$\pm a\xi+\sqrt{b^2-a^2}\eta=1,$$

which compose the front of the transverse wave, the derivatives $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ are infinite. This follows from the fact that $\Psi'(\theta)$ contains the factor $\sqrt{a^2-\theta^2}$, and the mentioned lines correspond to the case $\theta^2 = a^2$.

The regular functions $\Phi(\theta)$ and $\Psi(\theta)$ defined by (26) have two poles $\theta = \pm c$ on the real axis. These poles are roots of the equation

$$F(\theta) = 0. \tag{29}$$

It is easy to see that $\theta = c$ is a number reciprocal to the velocity of the surface waves, which were first studied by Lord Rayleigh. Taking into account that $\theta = \dfrac{ht}{x}$ on the real axis, we can assert that such poles give an infinite displacement propagating on the surface in two directions with the velocity $\dfrac{1}{c}$. With the exception of these poles, the functions $\Phi(\theta)$ and $\Psi(\theta)$ do not have any singular point.

The proof of this fact is contained, for example, in the work of V. D. Kupradze and S. L. Sobolev [3][2].

5. It is now easy to obtain formulas for the displacement also in the case when the force is distributed continuously along the axis $y = 0$. Let $f(x)$ be a density of this distribution. If the impact happens at the moment $t = 0$, then the formulas have the form

$$u(x,y,t) = C\mathrm{Im}\int_{-\infty}^{+\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial x}}{F(\theta_1)} f(\xi)\,d\xi$$

$$+ C\mathrm{Im}\int_{-\infty}^{+\infty} \frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial y}}{F(\theta_2)} f(\xi)\,d\xi, \tag{30.1}$$

$$v(x,y,t) = C\mathrm{Im}\int_{-\infty}^{+\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial y}}{F(\theta_1)} f(\xi)\,d\xi$$

$$- C\mathrm{Im}\int_{-\infty}^{+\infty} \frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial x}}{F(\theta_2)} f(\xi)\,d\xi, \tag{30.2}$$

where

$$\theta_1 = \frac{(x-\xi)t}{(x-\xi)^2+y^2} + i\frac{y\sqrt{t^2 - a^2(x-\xi)^2 - a^2y^2}}{(x-\xi)^2+y^2},$$

[2] See corresponding reasoning in the paper [4] of Part I of this book (p. 148). – *Ed.*

$$\theta_2 = \begin{cases} \dfrac{(x-\xi)t}{(x-\xi)^2+y^2} + i\dfrac{y\sqrt{t^2-b^2(x-\xi)^2-b^2y^2}}{(x-\xi)^2+y^2} \\ \qquad\qquad\qquad\qquad\qquad \text{for} \quad b^2(x-\xi)^2+b^2y^2<t^2, \\ \dfrac{(x-\xi)t}{(x-\xi)^2+y^2} - \dfrac{y\sqrt{b^2(x-\xi)^2+b^2y^2-t^2}}{(x-\xi)^2+y^2} \\ \qquad\qquad\qquad\qquad\qquad \text{for} \quad b^2(x-\xi)^2+b^2y^2>t^2. \end{cases}$$

Note that the imaginary parts of all integrands in formulas (30) are equal to zero outside the fronts of the corresponding waves. Assume that the force is distributed not only along the axis $y=0$, but the image of its action in time is of unconcentrated nature. Then, multiplying the elementary potentials by $Q(\xi, t-H)$, we have to integrate with respect to H as in (27), (28), and with respect to ξ as in (30). The lower limit of integration with respect to H in the first integral is

$$a\sqrt{(x-\xi)^2+y^2}.$$

In the second integral the lower limit is

$$b\sqrt{(x-\xi)^2+y^2}$$

for

$$b^2(x-\xi)^2 \le a^2[(x-\xi)^2+y^2],$$

and

$$a|x-\xi| + \sqrt{b^2-a^2}\,y$$

for

$$b^2(x-\xi)^2 \ge a^2[(x-\xi)^2+y^2].$$

6. All previous conclusions up to formulas (19) remain valid also in the case of a focused force acting along the axis $y=0$. In this case, we need only to determine the constants in (19) somewhat differently. It is easy to see that in this case the component v must vanish at the points on the axis $x=0$. Indeed, if we change the direction of the force acting along $y=0$, then, by the symmetry principle, the component v must remain unchanged on the axis $x=0$, at the same time u must change sign. On the other hand, the displacement vector can only change its direction. Whence $v=0$. Arguing in the same way as above, by (26) we obtain the formulas

$$\Phi'(\theta) = -iC\frac{2\theta\sqrt{b^2-\theta^2}}{F(\theta)}, \qquad \Psi'(\theta) = iC\frac{2\theta^2-b^2}{F(\theta)}. \tag{31}$$

7. Before moving on to solving other problems, we present some general considerations, which were essential in the preceding discussion and will be even more important in the future. The essential moment in solving the problem is reducing the wave equation (2) for the potential

$$c^2 \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \qquad c^2 = a^2 \text{ or } b^2,$$

to the Laplace equation in new independent variables σ and τ. In the case $c^2 = b^2$, we obtained the solution of (2) with an arbitrary function of one variable, which we denoted above by $(\sigma - \tau)$. In the first case, the dependence of the complex variable $\theta = \sigma + i\tau$ on the original variables (x, y, t) is expressed by the formula

$$-\theta x - \sqrt{c^2 - \theta^2}\, y + t = 0. \tag{32}$$

If we consider the three-dimensional space S with the coordinates (x, y, t), then from the preceding computations it follows that equation (32) has complex roots inside the cone

$$c^2(x^2 + y^2) - t^2 = 0. \tag{33}$$

If we take a root θ of this equation with the positive imaginary part, then for the root $\sqrt{c^2 - \theta^2}$ in (32) we have to choose the negative imaginary value for $\theta > c$. Outside cone (33), i.e., for

$$c^2(x^2 + y^2) - t^2 > 0,$$

equation (32) has two real roots, and an arbitrary function of each of these roots satisfies equation (2).

We point out a more general class of solutions of equation (2), which is obtained by the reduction of this equation to the Laplace equation.

For the dependence of the new variable $\theta = \sigma + i\tau$ on the variables (x, y, t) we use a linear function of x, y, and t with coefficients, which are analytic functions of θ. Obviously, the coefficient at t may be taken equal to 1. This leads us to the relation

$$t + \chi_1(\theta)x + \chi_2(\theta)y = \chi(\theta). \tag{34}$$

Assume that in a domain of the space S this equation has a complex root $\theta = \sigma + i\tau$, which is a function of (x, y, t). Consider a solution of (2), depending only on σ and τ.

In this case, one can verify that equation (2) can be reduced to the form

$$\frac{\partial^2 \varphi}{\partial \sigma^2} + \frac{\partial^2 \varphi}{\partial \tau^2} = 0$$

under the condition

$$\chi_1^2(\theta) + \chi_2^2(\theta) = c^2.$$

This circumstance is a consequence of the geometric nature of the lines $\sigma = \text{const}$, $\tau = \text{const}$, which are the straight lines in our three-dimensional space S. However, since we do not use this fact, we will not discuss it in detail. Taking into account that a harmonic function is mapped to a harmonic function under the action of the conformal mapping, we can take $\chi_1(\theta)$ as a new complex variable. Then, in view of the condition mentioned above, we have

$$\chi_2(\theta) = \pm\sqrt{c^2 - \theta^2},$$

and we can reduce relation (34) to the form

$$t - \theta x \pm \sqrt{c^2 - \theta^2}\, y - \chi(\theta) = 0. \tag{35}$$

If this equation has a real root in a domain of the space S, then an arbitrary function of this root satisfies equation (2).

All these assertions can be verified by simple calculation.

We present the corresponding formulas, since they will be useful later.

Denote by δ the left side of equation (35) and by δ' the partial derivative $\dfrac{\partial\delta}{\partial\theta}$. We have

$$\frac{\partial\theta}{\partial x} = \frac{\theta}{\delta'}, \quad \frac{\partial\theta}{\partial y} = \mp\frac{\sqrt{c^2-\theta^2}}{\delta'}, \quad \frac{\partial\theta}{\partial t} = -\frac{1}{\delta'}. \tag{36}$$

The second-order derivatives are

$$\begin{aligned} \frac{\partial^2\theta}{\partial x^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{\theta^2}{\delta'}\right), \quad & \frac{\partial^2\theta}{\partial y^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{c^2-\theta^2}{\delta'}\right), \\ \frac{\partial^2\theta}{\partial t^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{1}{\delta'}\right), \quad & \frac{\partial^2\theta}{\partial x\partial y} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{\mp\theta\sqrt{c^2-\theta^2}}{\delta'}\right). \end{aligned} \tag{37}$$

By (36), if equation (35) has a real root θ in a domain of the space S, then this root satisfies the inequality $-c \le \theta \le +c$, and the function $\chi(\theta)$ must have real values.

Let us note also some formulas used later. Let θ be a complex root of (35), let $f(\theta)$ be an analytic function. Using (36) and (37), we obtain the following expressions for the derivatives of $f(\theta)$ with respect to (x, y, t):

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left[f'(\theta)\frac{\theta^2}{\delta'}\right], \quad & \frac{\partial^2 f}{\partial y^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left[f'(\theta)\frac{c^2-\theta^2}{\delta'}\right], \\ \frac{\partial^2 f}{\partial t^2} &= \frac{1}{\delta'}\frac{\partial}{\partial\theta}\left[f'(\theta)\frac{1}{\delta'}\right], \quad & \frac{\partial^2 f}{\partial x\partial y} &= \mp\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left[f'(\theta)\frac{\theta\sqrt{c^2-\theta^2}}{\delta'}\right]. \end{aligned} \tag{38}$$

The same formulas remain valid for the function $f(\theta)$ of the real argument θ, if θ is a real root of equation (35).

8. Let us now discuss the two-dimensional problem on vibrations of the half-space under the action of a source of force F, located inside the half-space. As before, assume that the elastic half-plane is $y \ge 0$. Let $x = 0$, $y = f$ be the coordinates of the force source. We assume that the force action is concentrated at some moment. As above, we denote by t the time passed from this moment.

Introduce two functions $X(\alpha, t)$ and $Y(\alpha, t)$ defined on

$$0 \leq \alpha \leq 2\pi, \qquad 0 \leq t \leq 1.$$

Consider vibrations of the half-plane, being at rest at the moment $t = 0$, under the action of stresses

$$\frac{1}{\varepsilon^2} X\left(\alpha, \frac{t}{\varepsilon}\right) \quad \text{and} \quad \frac{1}{\varepsilon^2} Y\left(\alpha, \frac{t}{\varepsilon}\right)$$

applied at the points of a circle of radius ε with center $F(0, f)$, where the interval of the action of stresses is $0 \leq t \leq \varepsilon$. As $\varepsilon \to 0$, we have vibrations of the half-plane with a singularity at the point $F(0, f)$ and with potentials φ and ψ homogeneous in x, $(y - f)$, and t. A similar result is obtained if the moment is at $t = 0$. Note that the singularity of this type, generally speaking, is homogeneous. We assume that our source has such singularity.

In another work we hope to conduct a mechanical analysis of this concept of homogeneous singularity.

On the interval $0 \leq t \leq af$ there is no wave reflected from the plane $y = 0$ of the space S, and, as discussed above, the elementary potentials φ and ψ depend only on the ratios $\dfrac{x}{t}$ and $\dfrac{y-f}{t}$, i.e., they must remain constant on the straight lines of the space S, passing through the point $x = 0$, $y = f$, $t = 0$. Subsequently, these lines will be called the rays of the space S. First of all, we consider the case when the source F is the source of longitudinal waves, i.e., we assume that the potential ψ is equal to zero on the interval $0 \leq t \leq af$. The potential φ is not equal to zero only for

$$t^2 > a^2[x^2 + (y - f)^2],$$

i.e., inside the cone T_0 of the space S with apex F. The equation of the cone is

$$t^2 - a^2[x^2 + (y - f)^2] = 0. \tag{39}$$

We consider only the inner part of this cone, where $y \geq 0$ and $t > 0$.

Introduce the complex variable θ_1 determined, as in (35), by the equality

$$t - \theta_1 x + \sqrt{a^2 - \theta_1^2}(y - f) = 0,$$

i.e.,

$$\delta_1 = t - \theta_1 x + \sqrt{a^2 - \theta_1^2}\, y - \sqrt{a^2 - \theta_1^2}\, f = 0. \tag{40}$$

Then φ must be the real part of an analytic function of the complex variable θ_1

$$\varphi_1 = \operatorname{Re}[\Phi_1(\theta_1)]. \tag{41}$$

Expression (40) sets in the correspondence to each ray inside the cone T_0 a value of θ_1, and φ_1 remains constant along each ray. Consider this correspondence in detail. Solving equation (40) with respect to θ_1, we obtain

$$\theta_1 = \frac{xt - i(y-f)\sqrt{t^2 - a^2[x^2 + (y-f)^2]}}{x^2 + (y-f)^2}, \tag{42}$$

where the radical is taken with "+" sign. The rays, located in the half-space $t > 0$ and crossing the plane $y = 0$, correspond to the complex values of θ_1 from the upper plane, i.e., with the positive imaginary part. Formula (42) establishes the law of the correspondence between the rays and the values of θ_1. The family of rays, forming the part of the cone where $t > 0$, corresponds to the entire complex plane with the cut $(-a, +a)$ along the real axis. However, the points of this cut correspond to the generators of the cone. The intervals $(-\infty, -a)$ and $(+a, +\infty)$ of the real axis of θ_1 correspond to the rays located on the plane $y = f$, the imaginary axis corresponds to the rays of the plane $x = 0$, and the upper half $(0, +i\infty)$ of this axis corresponds to the rays for which $y < f$, and which further intersect the plane $y = 0$. From the last fact and equation (40) it follows that in this equation the radical $\sqrt{a^2 - \theta_1^2}$ is positive for the values of θ_1 on the imaginary semiaxis $(0, +i\infty)$. This is equivalent to the assumption that the value of the radical $\sqrt{a^2 - \theta_1^2}$ is negative imaginary for $\theta_1 > a$.

The generators of the cone T_0 correspond to the front of propagation of vibrations. Consequently, φ_1 must vanish in the corresponding points, i.e., the function $\Phi_1(\theta_1)$ in (41) must be purely imaginary on the cut $(-a, +a)$. The points of the axis of the cone T_0 correspond to the source of different moments, and this axis corresponds to the point of the plane θ_1 at infinity. Since we know the source, we do the singularity of $\Phi_1(\theta)$ at infinity.

Thus, the function $\Phi_1(\theta)$ is determined. A more detailed analysis of different sources will be conducted later. Our assumption, that the potential φ remains constant along each ray emanating from the point $x = 0$, $y = f$, $t = 0$, leads us to the fact that the singularity of φ_1 in the force source takes place at all moments $t > 0$.

9. The given elementary potential φ_1 determines the motion when $t < af$. For $t > af$ we have to add two more potentials: one φ_2 for the longitudinal wave, and another ψ_1 for the transverse wave. We select these potentials in the same way as above, i.e., we assume that these potentials must remain constant along some rays of the space S. These rays are called the *reflected rays*. Beginning with the construction of φ_2, first of all, we note that φ_2 must be the real part of an analytic function:

$$\varphi_2 = \mathrm{Re}\,[\Phi_2(\theta_2)], \tag{43}$$

where θ_2 is defined by equation (35) for $c = a$. We choose the function $\chi(\theta)$ in this equation such that the values of θ_1 and θ_2 coincide for $y = 0$, i.e., we select $\chi(\theta)$ as in equation (40).

Then, for θ_2 we have the equation

$$\delta_2 = t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y - \sqrt{a^2 - \theta_2^2}\, f = 0. \tag{44}$$

It is easy to verify that these reflected rays generate the cone

$$t^2 - a^2[x^2 + (y+f)^2)] \geq 0$$

with apex $(0, -f, 0)$. We select in equation (44) the opposite sign of the radical than in equation (40), so the rays going to the domain $t > 0$, $y > 0$, correspond to the complex values of θ_2 with the positive imaginary parts.

Constructing the potential ψ_1, we should put $c = b$ in equation (35). The term $\chi(\theta)$ is chosen in the same way as in equation (40). The sign of the radical in the coefficient at y should be taken such that the rays, along which y and t increase, correspond to the values of θ with the positive imaginary parts. It is easy to show that we should take "$-$" sign.

Then, for θ_3 we have the equation

$$\delta_3 = t - \theta_3 x - \sqrt{b^2 - \theta_3^2}\, y - \sqrt{a^2 - \theta_3^2}\, f = 0. \tag{45}$$

For $y = 0$ the values of θ_3 coincide with the values of θ_1 and θ_2.

The potential ψ_1 is the real part of an analytic function

$$\psi_1 = \mathrm{Re}\,[\Psi(\theta_3)]. \tag{46}$$

Before we construct the functions $\Phi_2(\theta_2)$ and $\Psi(\theta_3)$, let us point out the connection between the variables θ. For this purpose, consider the section of the main cone T_0 by the plane $y = 0$, where we have the reflection. In the section we have the hyperbola

$$t^2 - a^2(x^2 + f^2) = 0.$$

Each point (x, t) of the plane $y = 0$, located inside this hyperbola, for which

$$t^2 - a^2(x^2 + f^2) \geq 0 \quad \text{and} \quad t > 0,$$

corresponds to a complex value of θ_1 from the upper half-plane or the real axis. By the construction of equations (44) and (45), the values of θ_2 and θ_3 coinciding with the values of θ_1 correspond to the point (x, t). Thus, choosing the point (x, t), we define the complex values of θ_2 and θ_3 from the upper half-plane. Substituting these values into (44) and (45), we obtain two reflected rays in the space S. The potential φ_2 remains constant along one of these rays, and ψ_1 remains constant along another one. The values of y and t increase along these reflected rays. Hence the addition of the potentials φ_2 and ψ_1 does not influence the motion for $t < af$ and does not change the initial data. If we fix a point (x, y) and a moment t, then the corresponding values of θ_2

and θ_3 from the upper half-plane are obtained from equations (44) and (45). Obviously, it is impossible to define complex θ_2 and θ_3 for some points (x, y, t).

This corresponds to the fact that the reflected rays do not fill the entire domain $t > 0$, $y > 0$.

If, for example, the reflected ray of the potential φ_2 does not pass through the point (x, y, t), then we should not add the potential φ_2 at this point in order to construct the solution.

It is easy to verify that, by (35), the complex value of θ characterizes some direction in the space S without any dependence on the term $\chi(\theta)$. Thus, the above reasoning gives us the law of the correspondence between the directions of the incident and reflected rays. We will not discuss this anymore, since our goal is only the effective construction of the solution.

For the displacement components we have

$$\begin{aligned} u &= \operatorname{Re}\left[\Phi_1'(\theta_1)\frac{\partial\theta_1}{\partial x} + \Phi_2'(\theta_2)\frac{\partial\theta_2}{\partial x} + \Psi'(\theta_3)\frac{\partial\theta_3}{\partial y}\right], \\ v &= \operatorname{Re}\left[\Phi_1'(\theta_1)\frac{\partial\theta_1}{\partial y} + \Phi_2'(\theta_2)\frac{\partial\theta_2}{\partial y} - \Psi'(\theta_3)\frac{\partial\theta_3}{\partial x}\right]. \end{aligned} \tag{47}$$

Inside the hyperbola $t^2 - a^2(x^2 + f^2) = 0$ on the plane $y = 0$ the boundary conditions expressing the absence of stresses must hold. However, note that the variables θ_1, θ_2 and θ_3 coincide for $y = 0$. This allows us to omit index. Furthermore, let δ' without index denote the general value of the variables δ_1', δ_2' and δ_3' for $y = 0$. Using (38), we can write the boundary conditions in the form

$$\begin{aligned} &\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{-2\theta\sqrt{a^2-\theta^2}[\Phi_1'(\theta) - \Phi_2'(\theta)] + (b^2 - 2\theta^2)\Psi'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0, \\ &\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2 - 2\theta^2)[\Phi_1'(\theta) + \Phi_2'(\theta)] - 2\theta\sqrt{b^2-\theta^2}\Psi'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0. \end{aligned} \tag{48}$$

The expressions under the sign of the real part Re contain the complex variable θ, which can take arbitrary values from the upper half-plane, and the real variable x, which appears in the formula for δ.

First, note that θ can be expressed in terms of x and t. This follows from formula (42) for $y = 0$.

Thus, in the expression

$$\delta' = -x + \frac{\theta}{\sqrt{a^2 - \theta^2}}\, f \tag{49}$$

we can replace x by means of the formula

$$x = \frac{t}{\theta} - \frac{\sqrt{a^2-\theta^2}\, f}{\theta}.$$

Hence we have one complex variable θ and one real parameter t under the sign of the real part on the left sides of (48).

Consider the interval $(-a, +a)$ of the real axis of the plane θ, corresponding to the generators of the cone T_0, i.e., to the front of propagation of vibrations on the plane $y = 0$. All three potentials φ_1, φ_2 and ψ must vanish on this front, i.e., the real parts of the functions Φ_1, Φ_2 and Ψ must be equal to zero on this interval. Obviously, we can make the same conclusion about the derivatives Φ_1', Φ_2', and Ψ'. Taking into account that the radical $\sqrt{a^2-\theta^2}$ is real on this interval, we can assert that conditions (48) are satisfied for each positive real value of t on the interval $-a < \theta < +a$. Fix now a value of t and prove that conditions (48) will be satisfied for this value of t and for all θ from the upper half-plane. If t is fixed, and x is changing from

$$-\frac{\sqrt{t^2-a^2f^2}}{a}$$

to

$$+\frac{\sqrt{t^2-a^2f^2}}{a},$$

then the complex variable

$$\theta = \frac{xt}{x^2+f^2} \pm i\frac{f\sqrt{t^2-a^2(x^2+f^2)}}{x^2+f^2}$$

describes a curve l, issuing from a point A on the interval $(-a, +a)$ and arriving at another point B on the same interval. The curve l together with the interval AB of the real axis form a closed contour. By (48) for fixed t the expressions for θ and t along this contour have zero real parts. Then, these real parts must vanish on the entire upper half-plane of θ. Making the change of variables

$$t = \theta x + \sqrt{a^2-\theta^2}\, f,$$

we can conclude that conditions (48), where δ' is defined by (49), must hold for an arbitrary value of x on the entire upper half-plane of θ. Let us prove that we then have

$$\begin{aligned} &-2\theta\sqrt{a^2-\theta^2}[\Phi_1'(\theta)-\Phi_2'(\theta)] + (b^2-2\theta^2)\Psi'(\theta) = 0,\\ &(b^2-2\theta^2)[\Phi_1'(\theta)+\Phi_2'(\theta)] - 2\theta\sqrt{b^2-\theta^2}\Psi'(\theta) = 0. \end{aligned} \tag{50}$$

Denoting by $\sigma_1(\theta)$ the left side of the first of these equalities and putting

$$\sigma_2(\theta) = \frac{\theta f}{\sqrt{a^2-\theta^2}},$$

we can express the first condition in (48) in the form

$$\frac{\sigma_1'(\theta)[-x+\sigma_2(\theta)] - \sigma_2'(\theta)\sigma_1(\theta)}{[-x+\sigma_2(\theta)]^2} = Ci,$$

where C is a real constant depending only on x, and θ can take arbitrary values in the upper half-plane. From the last equality it follows that the coefficient at x and the term independent of x in the numerator of this fraction must vanish. Hence $\sigma_1(\theta) = 0$, i.e., the first equality in (50) holds. Similarly, we can prove the second equality.

Solving equations (50) with respect to $\Phi_2'(\theta)$ and $\Psi'(\theta)$, we obtain

$$\begin{aligned} \Phi_2'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Phi_1'(\theta), \\ \Psi'(\theta) &= -\frac{4\theta(2\theta^2-b^2)\sqrt{a^2-\theta^2}}{F(\theta)}\Phi_1'(\theta), \end{aligned} \tag{51}$$

where

$$F(\theta) = (2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}. \tag{52}$$

The fractions in (51) are real on the interval $-a < \theta < +a$ of the real axis. On the other hand, the real part of the function $\Phi_1(\theta)$ vanishes on this interval. Whence, by condition, the real parts of $\Phi_2'(\theta)$ and $\Psi'(\theta)$ also vanish on this interval. Integrating, we can choose additive constants in the expressions for the potentials $\Phi_2(\theta)$ and $\Psi(\theta)$ such that the real parts of $\Phi_2(\theta)$ and $\Psi(\theta)$ will also be equal to zero. By formulas (51) and (47), we can determine the displacement components.

10. Let us point out some consequences of the obtained formulas. Consider equation (40) having complex roots inside the cone T_0 and real roots from the interval $-a < \theta < +a$ on the generators of this cone. As is known, these generators correspond to the front of propagation of the longitudinal wave in the domain $t > 0$, $y > 0$ of the space S. Let $\theta_1 = \theta_0$ be a value from the interval $-a < \theta_1 < +a$, let λ_0 be a corresponding generator. If we put $\theta_1 = \theta_0$ in equation (40), then we have the equation of the plane tangent to the cone T_0 along λ_0. Therefore points (x, y, t) in the exterior of the cone T_0 correspond to real values of θ_1 from the interval $-a < \theta_1 < +a$. Then, $\delta_1'' = 0$ along each generator λ_0, i.e., the derivative of the left side of equation (40) is equal to zero. Hence the derivatives $\dfrac{\partial\theta_1}{\partial x}$ and $\dfrac{\partial\theta_1}{\partial y}$ are infinite along these generators, and we have infinite displacements on the front of propagation of the longitudinal wave. The study of equations (44) and (45) leads to a similar conclusion, and we have infinite displacements on the fronts of reflected waves.

Expanding the left side of equation (40) in powers of $(\theta_1 - \theta_0)$, we can conclude that $\dfrac{\partial\theta_1}{\partial x}$ and $\dfrac{\partial\theta_1}{\partial y}$ are infinite of the orders

$$\frac{1}{\sqrt{x - x_0}} \quad \text{and} \quad \frac{1}{\sqrt{y - y_0}},$$

respectively.

Let us now move on to finding asymptotic estimates of the obtained solution as $t \to \infty$. This will give us the phenomenon of surface waves in the clear form.

Let $\xi = x - \frac{t}{c}$, $\eta = y$, where c is a positive root of the equation $F(\theta) = 0$, i.e., $\dfrac{1}{c}$ is the known velocity of the Rayleigh wave [2]. Assuming that ξ and η remain bounded, let us construct the asymptotic expansions for θ_1 and θ_2 up to the terms of order $\dfrac{1}{t^2}$. It is easy to see that we have

$$\begin{aligned} \theta_1 &= c - \frac{c^2\xi}{t} - i\frac{c\sqrt{c^2-a^2}(\eta - f)}{t} + O\left(\frac{1}{t^2}\right), \\ \theta_2 &= c - \frac{c^2\xi}{t} + i\frac{c\sqrt{c^2-a^2}(\eta + f)}{t} + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{53}$$

Hence,

$$\begin{aligned} \frac{\partial\theta_1}{\partial x} &= -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_1}{\partial y} = -i\frac{c\sqrt{c^2-a^2}}{t} + O\left(\frac{1}{t^2}\right), \\ \frac{\partial\theta_2}{\partial x} &= -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_2}{\partial y} = i\frac{c\sqrt{c^2-a^2}}{t} + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{54}$$

By $F(c) = 0$, one can also verify that

$$F(\theta_1) = F'(c)\frac{-c^2\xi - ic\sqrt{c^2-a^2}(\eta - f)}{t} + O\left(\frac{1}{t^2}\right), \tag{55}$$

$$F(\theta_2) = F'(c)\frac{-c^2\xi + ic\sqrt{c^2-a^2}(\eta + f)}{t} + O\left(\frac{1}{t^2}\right). \tag{56}$$

Similarly, for θ_3 we obtain

$$\theta_3 = c - \frac{c^2\xi}{t} + i\frac{c\sqrt{c^2-b^2}\eta}{t} + i\frac{c\sqrt{c^2-a^2}\, f}{t} + O\left(\frac{1}{t^2}\right), \tag{57}$$

$$\frac{\partial\theta_3}{\partial x} = -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_3}{\partial y} = i\frac{c\sqrt{c^2-b^2}}{t} + O\left(\frac{1}{t^2}\right). \tag{58}$$

This allows us to write the asymptotic expansions for the displacement components up to the term of order $\dfrac{1}{t}$. Taking into account (47) and (51)[3], we have

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[3] The authors use also the formula

$$F(\theta_3) = F'(c)\frac{-c^2\xi + ic\sqrt{c^2-b^2}\eta + ic\sqrt{c^2-a^2}f}{t} + O\left(\frac{1}{t^2}\right). \text{ – } \textit{Ed.}$$

$$
\begin{aligned}
u = \operatorname{Re}\Bigg\{ & \frac{-(2c^2-b^2)^2 - 4c^2\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{F'(c)} \\
& \times \frac{-c}{-c\xi + i\sqrt{c^2-a^2}(\eta+f)} \Phi_1'(c) - \frac{i4c(2c^2-b^2)\sqrt{c^2-a^2}}{F'(c)} \\
& \times \frac{i\sqrt{c^2-b^2}}{-c\xi + i\sqrt{c^2-b^2}\eta + i\sqrt{c^2-a^2}\, f} \Phi_1'(c) \Bigg\} + O\left(\frac{1}{t}\right),
\end{aligned} \tag{59.1}
$$

$$
\begin{aligned}
v = \operatorname{Re}\Bigg\{ & \frac{-(2c^2-b^2)^2 - 4c^2\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{F'(c)} \\
& \times \frac{i\sqrt{c^2-a^2}}{-c\xi + i\sqrt{c^2-a^2}(\eta+f)} \Phi_1'(c) + \frac{i4c(2c^2-b^2)\sqrt{c^2-a^2}}{F'(c)} \\
& \times \frac{-c}{-c\xi + i\sqrt{c^2-b^2}\eta + i\sqrt{c^2-a^2}\, f} \Phi_1'(c) \Bigg\} + O\left(\frac{1}{t}\right).
\end{aligned} \tag{59.2}
$$

Our analysis allows us to note that at infinity vibrations produce the wave propagating with the velocity $\frac{1}{c}$ with bounded amplitude. It is easy to see that this wave is a natural generalization of the Rayleigh wave[4].

In the case of the concentrated source of the force inside the medium, we see that the surface wave has nonperiodic nature. We should also mention that the exponential law of damping in the depth is not valid anymore. Obviously, the concept of wave length does not make sense.

11. Let us now move on to the source of transverse waves. As in the previous problem, we assume that this source is regular, i.e., the given elementary potential of the transverse waves ψ_1 is the real part of a regular analytic function

$$\psi_1 = \operatorname{Re}[\Psi_1(\theta_1)], \tag{60}$$

where the complex variable θ_1 is defined by an equation similar to (40),

$$\delta_1 = t - \theta_1 x + \sqrt{b^2-\theta_1^2}\, y - \sqrt{b^2-\theta_1^2}\, f = 0. \tag{61}$$

In this case, the cone T_0 is defined by the equation

$$t^2 - b^2[x^2 + (y-f)^2] = 0, \tag{62}$$

and the rays located inside this cone correspond to the plane of the complex variable θ_1 with the cut $(-b, +b)$ along the real axis. The values of θ_1 on this cut correspond to the generators of the cone. We look for the potential of longitudinal reflected waves in the form of the real part of a function analytic in the upper half-plane

$$\varphi = \operatorname{Re}[\Phi(\theta_2)], \tag{63}$$

[4] These waves were studied by S. L. Sobolev in his work cited above.

where θ_2 is defined by the equation

$$\delta_2 = t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y - \sqrt{b^2 - \theta_2^2}\, f = 0. \tag{64}$$

As before, we chose an equation such that it coincides with equation (61) for $y = 0$. In the section of cone (62), by the plane $y = 0$, we have the hyperbola

$$t^2 - b^2(x^2 + f^2) = 0, \quad t > 0. \tag{65}$$

Each point P from the interior of this hyperbola corresponds to a complex value of θ_1 from the upper half-plane, and points of the hyperbola correspond to values of θ_1 on the interval $(-b, +b)$ of the real axis. To obtain a reflected ray $l_{x,t}$ of the potential φ of the longitudinal wave passing through a point P with coordinates (x, t) of the plane $y = 0$, we should take the corresponding value of θ_1 and substitute it for θ_2 into equation (64). This ray $l_{x,t}$ passes through the point P, and equation (64) defines its direction.

As we have already noted, the direction of the straight lines, obtained from equation (64), is completely defined by the first three terms on the left side of this equation. Hence the direction is the same as one obtained from equation (44) with the same value of θ. The straight lines of equation (44) form the already known cone with apex $x = 0$, $y = -f$, $t = 0$ and the apex angle equal to $\arctan\dfrac{1}{a}$. For this cone as well as for the cone T_0 from our problem, the values of θ from the upper half-plane correspond to the rays along which y and t increase simultaneously. When the value of θ tends to a point of the real interval $(-a, +a)$, the direction of the corresponding ray coincides with the direction of the corresponding generator of the cone. When θ tends to a point of the real axis outside the interval $(-a, +a)$, the ray direction is parallel to the plane $y = 0$ in the limit. In the present case, the points of hyperbola (62) correspond to the values of θ_1 on the interval $(-b, +b)$. Let A and B be the points of this hyperbola for $\theta_1 = \pm a$ (see Fig. 3).

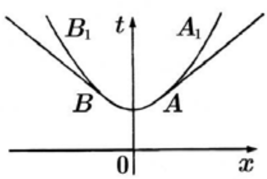

Fig. 3.

The arc AB of the hyperbola corresponds to the values of θ_1 from $-a < \theta_1 < +a$.

The infinite branches AA_1 and BB_1 correspond to the values of θ_1 from the intervals $a \le \theta_1 < b$ and $-b \le \theta_1 < -a$. The above reasoning leads us to the following conclusion: if a point $P(x,t)$ tends to a point on the arc AA_1 or BB_1, then the angle between the corresponding ray $l_{x,t}$ of the reflected longitudinal wave and the plane $y = 0$ tends to zero. The limit for the points located on these arcs is on the plane $y = 0$.

Substituting into (64) instead of θ_2 some value from the interval (a,b) or $(-b,-a)$, we obtain the equation of the ray $l_{x,t}$ passing through a point of the arc AA_1 or BB_1 and located on the plane $y = 0$:

$$t - \theta_2 x - \sqrt{b^2 - \theta_2^2}\, f = 0.$$

It is easy to show that the last equation defines the tangents to hyperbola (65). Hence, for each point of the arcs AA_1 and BB_1 of the hyperbola, the corresponding ray of the reflected longitudinal potential is tangent to hyperbola (65) at this point.

Later we will see that the potential of the reflected longitudinal waves will be equal to zero only on the interval $(-a,+a)$ of the real axis, as in the case of the longitudinal source, but it will not be equal to zero on the intervals (a,b) and $(-b,-a)$. Also, it will not vanish in two domains of the plane bounded by the arcs AA_1 and BB_1 of the hyperbola and two tangents to the hyperbola at the points A and B. We denote these domains by (I) and (II). There is no incident transverse wave in these domains. To satisfy the boundary conditions, we have to define the potential ψ_2 of the reflected transverse wave not only inside hyperbola (65), but also outside this hyperbola in the domains (I) and (II).

We will see later how to do it. We now move on to the definition of ψ_2 inside the hyperbola, i.e., for complex values of θ from the upper half-plane. Here, ψ_2 is the real part of an analytic function

$$\psi_2 = \mathrm{Re}\,[\Psi_2(\theta_3)], \tag{66}$$

where θ_3 is defined by the equation

$$\delta_3 = t - \theta_3 x - \sqrt{b^2 - \theta_3^2}\, y - \sqrt{b^2 - \theta_3^2}\, f = 0, \tag{67}$$

which defines the conical beam T_1 of rays with apex

$$F_1(x = 0,\ y = -f,\ t = 0)$$

and angle $\arctan \dfrac{1}{b}$ at the apex. We consider only those rays of this beam which pass through the domain $y > 0$, $t > 0$ of the space S.

Let us now write the boundary conditions for the mentioned points, i.e., for the values of θ from the upper half-plane. The values of θ_1, θ_2 and θ_3 coincide for $y = 0$. Denoting by θ this common value, we have

$$\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{2\theta\sqrt{a^2-\theta^2}\Phi'(\theta)+(b^2-2\theta^2)[\Psi_1'(\theta)+\Psi_2'(\theta)]}{\delta'}\right\}\bigg|_{y=0}=0,$$
$$\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2-2\theta^2)\Phi'(\theta)+2\theta\sqrt{b^2-\theta^2}[\Psi_1'(\theta)-\Psi_2'(\theta)]}{\delta'}\right\}\bigg|_{y=0}=0, \tag{68}$$

where δ' is the derivative of the expression $(t-\theta x-\sqrt{b^2-\theta^2}\,f)$ with respect to θ.

As in the case of the source of longitudinal waves, from above we obtain

$$2\theta\sqrt{a^2-\theta^2}\Phi'(\theta)+(b^2-2\theta^2)[\Psi_1'(\theta)+\Psi_2'(\theta)]=0,$$
$$(b^2-2\theta^2)\Phi'(\theta)+2\theta\sqrt{b^2-\theta^2}[\Psi_1'(\theta)-\Psi_2'(\theta)]=0. \tag{69}$$

Then, we can define the functions $\Phi'(\theta)$ and $\Psi_2'(\theta)$

$$\Phi'(\theta)=\frac{4\theta(2\theta^2-b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta),$$
$$\Psi_2'(\theta)=\frac{-(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta). \tag{70}$$

The potential ψ_1 of the transverse waves propagating from the source must vanish on the wave front. It means that the real parts of the functions $\Psi_1(\theta)$ and $\Psi_1'(\theta)$ must be equal to zero for $-b\le\theta\le+b$. Taking into account that the fractions in (70) are real for $-a\le\theta\le+a$, we can assert that $\Phi'(\theta)$ and $\Psi_2'(\theta)$ have zero real part for $-a\le\theta\le+a$. This fact is not valid anymore on the intervals $a<\theta<b$ and $-b<\theta<-a$, since the indicated fractions contain the radical $\sqrt{a^2-\theta^2}$. Therefore, in the domains (I) and (II) of the plane $y=0$ the potential φ equal to the real part of $\Phi(\theta)$ is not equal to zero. These domains are generated by the rays $l_{x,t}$ or the l_θ, corresponding to the real values of θ from the intervals (a,b) and $(-b,-a)$. If we substitute such value of θ for θ_3 into equation (67), we have the equation of some plane in the space S. The section of this plane by the plane $y=0$ is the ray l_θ. It is easy to see that this plane is tangent to the cone T_1 of the reflected transverse wave. Thus, we have the family of planes tangent to the cone T_1 along the generators passing through the points P of the arcs AA_1 and BB_1 of the hyperbola. Consider one of the planes tangent to the cone along the generator F_1P. Let θ be the real value of the parameter θ, corresponding to this generator F_1P. Denote by U_θ the domain of this tangent plane, bounded by the generator F_1P and the ray l_θ of the plane $y=0$, and located in the half-space $y>0$. The values of θ belong to the intervals (a,b) or $(-b,-a)$.

The domains U_θ fill a domain R in the space S. In this domain we define the potential ψ_2 as a function of the real variable θ. This function is constant in each U_θ. As already mentioned in Sect. 7, an arbitrary function of a real root θ of equation (67) in the domain R satisfies the wave equation (2) for $c = b$.

Our choice of U_θ allows us to assert that we did not break the initial conditions, since we have $t > af$ for U_θ. Similar circumstances will be valid for the future problems, and we will not discuss it anymore. As we will see later, our procedure always determines the potential continuously. Moving on to the effective computation of this potential, we have to choose a function of θ, which defines the potential ψ_2 in the domain R such that the boundary conditions are always satisfied in the domains (I) and (II) of the plane $y = 0$. The second formula in (70) gives us $\Psi_2'(\theta)$ on the intervals (a, b) and $(-b, -a)$. Integrating along the real axis, we obtain $\Psi_2(\theta)$. Obviously, one can put $\Psi_2(\pm a) = 0$. It is easy to prove that if the potential ψ_2 is equal to the real part of the indicated function $\Psi_2(\theta)$ on the planes U_θ, then the boundary conditions will be satisfied also in the domains (I) and (II) of the plane $y = 0$. Indeed, returning to equalities (69), we can assert that they hold also on the intervals $a \le \theta \le b$ and $-b \le \theta \le -a$. However, the real part of $\Psi_1'(\theta)$ is equal to zero on these intervals, and the coefficients of this function in equation (69) do not contain the radical $\sqrt{a^2 - \theta^2}$. Hence these coefficients are real. Taking into account once again the fact that δ' is also real, in the discussed case we have the condition in form (68) with $\Psi_1'(\theta) = 0$, i.e.,

$$\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\Psi_2'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0,$$

$$\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi_2'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0.$$

These equations show that the boundary conditions hold in the domains (I) and (II) of the plane $y = 0$. Thus, the problem is solved.

Let us note again that the value of ψ_2 on U_θ is equal to the value of this function along the generator F_1P, through which U_θ passes.

12. Let us now derive some consequences of the obtained results. As in Sect. 10, we can prove that the displacements are infinite on the fronts of the waves. We will not return to this point anymore.

If we cross the constructions made in the space S by the plane $t = \text{const}$, we obtain the fronts of the waves at the time moment t (see Fig. 4). Let us take sufficiently large t such that the plane $t = \text{const}$ to pass through the domain R of the space S. In this case, the front of the transverse waves consists of three parts. The first part is the arc AHB of the circle that is the section of the cone T_0 by the plane $t = \text{const}$. This is a wave propagating from the source. The second part is the arc $AEFB$ of the circle that is the section of the cone T_1 by the plane $t = \text{const}$. The third part consists of two lines CE and DF that are the sections U_{+a} and U_{-a} by the plane $t = \text{const}$. This last part is generated by the longitudinal waves propagating along the plane $y = 0$ with the velocity $\frac{1}{a}$. The points E and F are the points of the intersection of the plane $t = \text{const}$ with the generators of the cone T_0, corresponding to the values $\theta = \pm a$. The front of the longitudinal waves is the curve CGD enveloping the lines

$$\theta x + \sqrt{a^2 - \theta^2}\, y + \sqrt{b^2 - \theta^2}\, f = t, \qquad -a \le \theta \le +a, \quad t = \text{const}.$$

All these fronts propagate according to the Fermat principle. As in the previous case, one can give asymptotic representations of the displacements and to reveal the surface wave. The explanation is completely analogous to the above one.

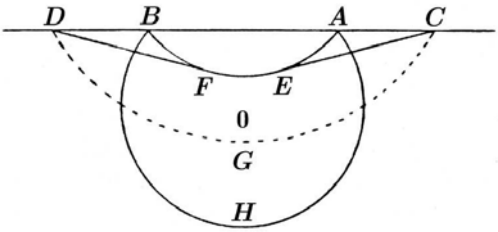

Fig. 4.

13. The presented approach can be applied not only to the two-dimensional problem on vibrations of the half-space, but it also gives the general law of reflection of a beam of rays of special type from a plane in the space S.

For this special type, the potential (longitudinal or transverse) is the real part of an analytic function of θ in the upper half-plane, where θ is a root of the equation

$$t - \theta x \pm \sqrt{c^2 - \theta^2}\, y - \chi(\theta) = 0, \quad c = a \text{ or } b.$$

As mentioned above, this form is equivalent to form (34). We will say that in this case vibrations have imaginary potentials.

The indicated analytic function satisfies also some boundary conditions. In the last cases it is necessary to consider real values of θ which correspond to planes in the space S. The potential must remain constant on each of these planes. We do not consider the entire plane, but rather only its part concluded between the reflective plane and the terminal position of the ray obtained when θ from the upper half-plane tends to the discussed real value corresponding to the plane. The presented method gives, for example, a solution of the problem on vibrations of a layer.

Let $2f$ be the thickness of the plane layer bounded by the lines $y = 0$ and $y = 2f$. Suppose that we have a source of longitudinal type at the point $x = 0$, $y = f$ with the singularity of the type described above. Let the potential of this source be given by the formula

$$\varphi = \operatorname{Re}[\Phi(\theta)], \tag{71}$$

where the analytic function $\Phi(\theta)$ is defined on the entire plane with the cut $(-a, +a)$ along the real axis; let the real part of $\Phi(\theta)$ be equal to zero on this cut. Consider the part Ω of the space S, bounded by the planes $y = 0$ and $y = 2f$. Denote by S_0 the first of these planes, and the second by S_1. If we are at rest for $t < 0$, then we have longitudinal vibrations with the given potential φ for $0 \le t < af$. The rays corresponding to this wave form the cone T_0 with apex $(x = 0,\ y = f,\ t = 0)$ and angle $\arctan \dfrac{1}{a}$ at the apex. At the moment $t = af$ we have reflected rays of longitudinal and transverse waves with respect to the planes S_0 and S_1. All these rays follow the direction of growth of t. Hence, in the domain Ω bounded by the planes $t = 0$ and $t = af$, the displacement is defined by the fundamental cone T_0. In expression (71), θ is defined by the equality

$$t - \theta x + \sqrt{a^2 - \theta^2}\, y - \sqrt{a^2 - \theta^2}\, f = 0.$$

Let φ_1 and ψ_1 be the potentials of the longitudinal and transverse waves reflected from the plane S_0, let φ_2 and ψ_2 be the analogous potentials for the reflection from S_1.

We have

$$\varphi_1 = \operatorname{Re}[\Phi_1(\theta_1)] \quad \text{and} \quad \psi_1 = \operatorname{Re}[\Psi_1(\theta_1')], \tag{72}$$

where θ_1 and θ_1' are complex values from the upper half-plane, defined by the equations

$$\begin{gathered} t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\, y - \sqrt{a^2 - \theta_1^2}\, f = 0, \\ t - \theta_1' x - \sqrt{b^2 - {\theta_1'}^2}\, y - \sqrt{a^2 - {\theta_1'}^2}\, f = 0. \end{gathered} \tag{73}$$

Equations (51) allow us to obtain the functions $\Phi_1(\theta_1)$ and $\Psi_1(\theta_1)$ for values of the argument from the upper half-plane

$$\begin{aligned} \Phi_1'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Phi'(\theta), \\ \Psi_1'(\theta) &= \frac{-4\theta(2\theta^2 - b^2)\sqrt{a^2-\theta^2}}{F(\theta)}\Phi'(\theta). \end{aligned} \tag{74}$$

In this case, the real parts of $\Phi_1'(\theta)$ and $\Psi_1'(\theta)$ are equal to zero on the interval $-a \le \theta \le +a$. These reflected rays pass through points of the plane S_0, located inside the hyperbola $t^2 - a^2(x^2 + f^2) = 0$. The rays fall on the plane S_1 above the line $t = 3af$.

Values of θ from the lower half-plane correspond to rays of the cone T_0, falling on the plane S_1. Let

$$\varphi_2 = \operatorname{Re}[\Phi_2(\theta_2)], \quad \psi_2 = \operatorname{Re}[\Psi_2(\theta_2')] \tag{75}$$

be the potentials of the longitudinal and transverse waves reflected from the plane S_1. Complex values of θ_2 and θ_2' from the lower half-plane must coincide with θ for $y = 2f$. It is easy to see that θ_2 and θ_2' are defined by the equations

$$\begin{aligned} &t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y + 3\sqrt{a^2 - \theta_2^2}\, f = 0, \\ &t - \theta_2' x - \sqrt{b^2 - {\theta_2'}^2}\, y + 2\sqrt{b^2 - {\theta_2'}^2}\, f - \sqrt{a^2 - {\theta_2'}^2}\, f = 0. \end{aligned} \tag{76}$$

The derivatives of the functions Φ_2 and Ψ_2 are determined by the formulas obtained from (74) by the sign change in front of the radical $\sqrt{a^2 - \theta^2}$ in the second formula. The displacement of the layer, bounded by the planes $t = f$ and $t = 3f$ in the domain Ω, is determined by the potentials φ, φ_1, φ_2, ψ_1, and ψ_2. Further, we have to consider the reflection of the rays corresponding to the potentials φ_1, φ_2, ψ_1, and ψ_2. First, consider the potential φ_1. The corresponding rays reflected from the plane S_0 fall on the plane S_1 and create reflected rays of longitudinal and transverse vibrations. Let us introduce the corresponding potentials

$$\varphi_3 = \mathrm{Re}\,[\Phi_3(\theta_3)], \quad \psi_3 = \mathrm{Re}\,[\Psi_3(\theta_3')], \tag{77}$$

where the variables θ_3 and θ_3' from the upper half-plane must coincide for $y = 2f$ with θ_1 defined by the first equation in (73). It is easy to see that the equations on these variables have the form

$$\begin{aligned} &t - \theta_3 x + \sqrt{a^2 - \theta_3^2}\, y - 5\sqrt{a^2 - \theta_3^2}\, f = 0, \\ &t - \theta_3' x + \sqrt{b^2 - {\theta_3'}^2}\, y - 2\sqrt{b^2 - {\theta_3'}^2}\, f - 3\sqrt{a^2 - {\theta_3'}^2}\, f = 0. \end{aligned} \tag{78}$$

The functions Φ_3' and Ψ_3' are determined through Φ_1' by the formulas obtained from (74) by the sign change in front of the radical $\sqrt{a^2 - \theta^2}$ in the second formula.

Introduce the potentials φ_4 and ψ_4 for rays corresponding to the reflection of the beam of rays of transverse vibrations with the potential ψ_1 from the plane S_1,

$$\varphi_4 = \mathrm{Re}\,[\Phi_4(\theta_4)], \quad \psi_4 = \mathrm{Re}\,[\Psi_4(\theta_4')], \tag{79}$$

where θ_4 and θ_4' from the upper half-plane satisfy the equations

$$\begin{aligned} &t - \theta_4 x + \sqrt{a^2 - \theta_4^2}\, y - 2\sqrt{b^2 - \theta_4^2}\, f - 3\sqrt{a^2 - \theta_4^2}\, f = 0, \\ &t - \theta_4' x + \sqrt{b^2 - {\theta_4'}^2}\, y - 4\sqrt{b^2 - {\theta_4'}^2}\, f - \sqrt{a^2 - {\theta_4'}^2}\, f = 0. \end{aligned} \tag{80}$$

The functions Φ_4' and Ψ_4' are determined through Ψ_1' by the formulas obtained from (70) by the sign change in front of the radical in the first formula

$$\begin{aligned} \Phi_4'(\theta) &= \frac{-4\theta(2\theta^2 - b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta), \\ \Psi_4'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta). \end{aligned} \tag{81}$$

Note that the range of the argument of the function $\Psi_1'(\theta)$ consists of the upper half-plane and the interval $(-a, +a)$, and the real part of $\Psi_1'(\theta)$ vanish on this interval. Analogous results are valid for all remaining functions obtained by the reflection from the planes S_0 and S_1. It is completely clear how we should continue the calculations.

In the case of a source of transverse vibrations we have somewhat different circumstances.

14. Assume that the formula $\psi = \mathrm{Re}\,[\Psi(\theta)]$ gives us the potential of a source of transverse vibrations, where the variable θ is defined by the equation

$$t - \theta x + \sqrt{b^2-\theta^2}\, y - \sqrt{b^2-\theta^2}\, f = 0, \tag{82}$$

and the range of change of this variable is the entire complex plane with the cut $(-b, +b)$ along the real axis. We construct the potentials φ_1 and ψ_1 reflected from the plane S_0,

$$\varphi_1 = \mathrm{Re}\,[\Phi_1(\theta_1)], \quad \psi_1 = \mathrm{Re}\,[\Psi_1(\theta_1')], \tag{83}$$

where

$$\begin{aligned} \Phi_1'(\theta) &= \frac{4\theta(2\theta^2 - b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi'(\theta), \\ \Psi_1'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi'(\theta). \end{aligned} \tag{84}$$

Real values of θ from $a \le |\theta| \le b$ correspond to rays of longitudinal vibrations in the plane S_0 and to the plane, where $\Psi_1(\theta)$ is equal to a constant defined by (84).

For θ_1 and θ_1' we have the equations

$$\begin{aligned} t - \theta_1 x - \sqrt{a^2-\theta_1^2}\, y - \sqrt{b^2-\theta_1^2}\, f &= 0, \\ t - \theta_1' x - \sqrt{b^2-{\theta_1'}^2}\, y - \sqrt{b^2-{\theta_1'}^2}\, f &= 0. \end{aligned} \tag{85}$$

Further, consider the reflection of the obtained rays of longitudinal vibrations from the plane S_1. We have the potentials of reflected longitudinal and transverse vibrations φ_2 and ψ_2,

$$\varphi_2 = \mathrm{Re}\,[\Phi_2(\theta_2)], \quad \psi_2 = \mathrm{Re}\,[\Psi_2(\theta_2')]. \tag{86}$$

For the variables θ_2 and θ_2' we have the equations

$$
\begin{aligned}
&t-\theta_2 x+\sqrt{a^2-\theta_2^2}\,y-\sqrt{b^2-\theta_2^2}\,f-3\sqrt{a^2-\theta_2^2}\,f=0,\\
&t-\theta_2' x+\sqrt{b^2-{\theta_2'}^2}\,y-3\sqrt{b^2-{\theta_2'}^2}\,f-2\sqrt{a^2-{\theta_2'}^2}\,f=0,
\end{aligned}
\tag{87}
$$

and the functions Φ_2' and Ψ_2' are defined by the formulas

$$
\begin{aligned}
\Phi_2'(\theta)&=\frac{-(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Phi_1'(\theta),\\
\Psi_2'(\theta)&=\frac{4\theta\sqrt{a^2-\theta^2}(2\theta^2-b^2)}{F(\theta)}\Phi_1'(\theta).
\end{aligned}
\tag{88}
$$

The rays of longitudinal vibrations of the potential φ_1, corresponding to the real values of θ from $a \le |\theta| \le b$, remain in the plane S_0. Hence the range of θ_2 and θ_2' is the upper half-plane and the interval $(-a,+a)$. This fact is valid for the variable θ in (88), and the real parts of $\Phi_2'(\theta)$ and $\Psi_2'(\theta)$ are equal to zero on the interval $(-a,+a)$.

Consider now the reflection of rays of transverse vibrations with the potential ψ_1 from the plane S_1. Introduce the potentials for the reflected rays

$$
\varphi_3=\operatorname{Re}[\Phi_3(\theta_3)],\quad \psi_3=\operatorname{Re}[\Psi_3(\theta_3')],
\tag{89}
$$

where

$$
\begin{aligned}
&t-\theta_3 x+\sqrt{a^2-\theta_3^2}\,y-3\sqrt{b^2-\theta_3^2}\,f-2\sqrt{a^2-\theta_3^2}\,f=0,\\
&t-\theta_3' x+\sqrt{b^2-{\theta_3'}^2}\,y-5\sqrt{b^2-{\theta_3'}^2}\,f=0
\end{aligned}
\tag{90}
$$

and

$$
\begin{aligned}
\Phi_3'(\theta)&=\frac{-4\theta(2\theta^2-b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta),\\
\Psi_3'(\theta)&=\frac{-(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta).
\end{aligned}
\tag{91}
$$

Let us make some additional comments about real values of θ such that $a<|\theta|<b$.

For $\Phi_3'(\theta)$ and $\Psi_3'(\theta)$ we obtain values with nonzero real parts. From the first equation in (90) it follows that the rays of longitudinal vibrations corresponding to these values of θ are located in the plane S_1. The second equation defines a family of planes, on which $\Psi_3(\theta)$ remains constant. It is easy to verify the boundary conditions by considering the potentials in domains of the plane S_1, filled with the rays of longitudinal vibrations. Let us consider these domains in detail.

For every real value of θ from the inequality $a<|\theta|<b$, the equation of the corresponding ray located in the plane S_1 is

$$t - \theta x - 3\sqrt{b^2 - \theta^2}\, f = 0, \qquad y = 2f.$$

Substituting into the second equation in (90) θ for θ_3' and putting $y = 2f$, we obtain the same equation . Hence the indicated ray of longitudinal vibrations coincide with the section of the plane, where $\Psi_3(\theta)$ is constant. The same can be obtained putting $y = 2f$ in the second equation in (85), i.e., the plane, on which $\Psi_1(\theta)$ is constant, crosses the plane S_1 along the same ray l_θ. When the rays of the potential φ_3 reflect from the plane S_0, the range of θ is the upper half-plane with the interval $(-a, +a)$ of the real axis. The real part of $\Phi_3'(\theta)$ is equal to zero along this interval.

Using the same argument, we can obtain solutions of problems with different boundary conditions, for example, with the absence of the displacements, etc.

15. Using the above method in the case when the source is located inside the medium, it is easy to solve also the first problem in the very compact form: the two-dimensional problem on vibrations of the half-space under the action of an impact concentrated on the surface.

Let the source of vibrations be located at the point

$$O(x = 0,\ y = 0,\ t = 0)$$

of the space S, let the complex potentials $\Phi(\theta_1)$ and $\Psi(\theta_1')$ of longitudinal and transverse vibrations correspond to this source. Consider two cones T_1 and T_2 with apex O and angles $\arctan\dfrac{1}{a}$ and $\arctan\dfrac{1}{b}$ at the apex. Write down the equations for θ_1 and θ_1',

$$t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\, y = 0, \tag{92}$$

$$t - \theta_1' x - \sqrt{b^2 - {\theta_1'}^2}\, y = 0. \tag{93}$$

Complex values of θ_1 from the upper half-plane correspond to rays passing through the point O and moving inside the cone T_1 in the domain $y > 0$, $t > 0$ of the space S. Real values of θ_1 such that $|\theta_1| > a$ correspond to rays located in the plane $y = 0$. Finally, real values of θ_1 from the interval $(-a, +a)$ correspond to generators of the cone T_1. Completely analogous correspondence will take place between rays inside the cone T_2 and complex values of θ_1'.

Let OA and OA_1 be generators of T_1 in the plane $y = 0$, let OB and OB_1 be generators for T_2. Using (38), one can write the condition that the stress is equal to zero inside the angle BOB_1 at all points of the plane $y = 0$

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\Psi'(\theta)]\right\} = 0,\\
&\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi'(\theta)]\right\} = 0.
\end{aligned} \tag{94}$$

Note that points inside the angle BOB_1 correspond to real values of θ such that $|\theta| > b$. Consider now the angles AOB and A_1OB_1. Here we have the potential $\Phi_1(\theta)$ of longitudinal vibrations. To satisfy the boundary conditions, we have to apply transverse vibrations. This corresponds to the fact that longitudinal vibrations propagating over the surface generate transverse vibrations inside. In this case, the argument of the function $\Phi_1(\theta)$ takes real values from the intervals (a, b) and $(-b, -a)$. For these values of θ, equation (93) defines planes tangent to the cone T_2. Let us take the parts of these planes between the plane $y = 0$ and the generators of the cone T_2.

Denote by U_θ these parts. On each U_θ the potential of transverse vibrations must be constant, and we have to choose the functions $\omega(\theta)$ such that the boundary conditions are satisfied in the angles AOB and A_1OB_1. Since $|\theta| \leq b$, we can write these conditions in the form

$$\begin{aligned} &\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\omega'(\theta)]\right\} = 0, \\ &\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\omega'(\theta)]\right\} = 0. \end{aligned} \tag{95}$$

Since the potential is continuous, the value of $\omega(\theta)$ must coincide with the real part of $\Psi(\theta)$ on the generator of the cone T_2, along which U_θ touches the cone. Then conditions (95) coincide with conditions (94), i.e., conditions (94) must be satisfied also for $a \leq |\theta| \leq b$. Since velocity of vibrations cannot be greater than $\dfrac{1}{a}$, we must assume that the potentials of longitudinal and transverse vibrations must vanish for $-a \leq \theta \leq a$, i.e., conditions (94) will be also satisfied for these values of θ. Thus, these conditions must be satisfied on the entire real axis. Calculating the functions $\Phi(\theta)$, $\Psi(\theta)$, and the potentials

$$\varphi = \mathrm{Re}\,[\Phi(\theta)], \quad \psi = \mathrm{Re}\,[\Psi(\theta)],$$

we have to continue ψ into the exterior of the cone T_2 along the planes U_θ.

The establishment of conditions (94) for all real values of θ is the essential fact in solving the first problem.

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[5] Paper [1] of Part I of this book. – *Ed.*

3. On Application of a New Method to Study Elastic Vibrations in a Space with Axial Symmetry*

V. I. Smirnov and S. L. Sobolev

1. In our paper [1] we considered elastic vibrations of special type in the two-dimensional case. Using the theory of functions of a complex variable for such vibrations, we obtained the general law of reflection in the case of a rectilinear boundary. This led us to solving the problem on propagation of elastic vibrations in a half-plane or in parallel layers, if the source of these vibrations has a type described in [1]. This source could be located either on the surface or inside the medium.

In the present work, using results of the previous article [1], we are going to consider the problem on propagation of elastic vibrations in a space with axial symmetry. As in the plane case, the main object of our studies will be vibrations of special type, namely, vibrations which can be constructed by superposing plane vibrations discussed in the previous article. Using the reflection law for plane vibrations, we will also have a law of reflection for vibrations obtained by superposition of plane vibrations. This will lead us to solving the problem on elastic vibrations in the case of the three-dimensional half-space and parallel layers. All these problems are similar to the problems discussed in our previous article for the two-dimensional case. We will also discuss the Lamb problem [2] about vibrations of the half-space under the action of a force applied to a point on the surface of the half-space along the normal vector to this surface. The fundamental problem in the two-dimensional case was the problem on vibrations under the action of an impact focused at some moment of time. In the case of the space, we will have a force engaged at some moment of time. Let us note also that in the Lamb problem we will obtain formulas for displacements at an arbitrary point of the half-space.

2. Let us introduce the cylindrical coordinates (ϱ, z, ϑ). Assume that the displacement vector in each point is located in the plane passing through this point and the z-axis. Suppose also that the displacement components q and w on the ϱ-axis and the z-axis do not depend on ϑ. In this case, we have the

* Tr. Seism. Inst., **29** (1933), 49 p.

formulas[1]

$$q = \frac{\partial \varphi}{\partial \varrho} - \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial \varrho} + \frac{1}{\varrho}\psi, \tag{1}$$

where φ and ψ must satisfy the equations

$$a^2 \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial \varphi}{\partial \varrho} + \frac{\partial^2 \varphi}{\partial z^2}, \tag{2}$$

$$b^2 \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial \psi}{\partial \varrho} - \frac{1}{\varrho^2}\psi + \frac{\partial^2 \psi}{\partial z^2}. \tag{3}$$

The constants a and b are expressed by the formulas

$$a = \sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad b = \sqrt{\frac{\rho}{\mu}}, \tag{4}$$

where λ and μ are the Lame coefficients, and ρ is the density of the medium.

The function $\varphi(\varrho, z, t)$ is the scalar potential of the displacement field, and the corresponding terms in (1) are the potential part of this field. Equation (2) is the wave equation

$$a^2 \frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi, \tag{5}$$

where φ does not depend on ϑ.

The terms in expression (1), containing the function ψ, give the solenoid part of the displacement field. This part is the curl of a vector field, and this field is the vector potential of the displacement field. This vector potential must satisfy the wave equation

$$b^2 \frac{\partial^2 \omega}{\partial t^2} = \nabla^2 \omega, \tag{6}$$

in other words, the components of this field in every direction must satisfy this equation. In the case of the axial symmetry with respect to the z-axis, this vector potential is directed along the ϑ-axis, and the function ψ gives the length of this vector. Assume that the vector field is composed from vectors of length ψ directed along the ϑ-axis, where ψ does not depend on ϑ. In this case, by (3), the vector field satisfies the wave equation (6).

3. Let us construct now a class of solutions of equations (2) and (3) that we use later. First of all, we recall some results from the previous article about solutions of the wave equation in the case of two variables

$$c^2 \frac{\partial^2 \omega}{\partial t^2} = \frac{\partial^2 \omega}{\partial x_1^2} + \frac{\partial^2 \omega}{\partial z^2} \qquad (c = a \text{ or } b). \tag{7}$$

[1] For details see the paper by S. L. Sobolev "Some questions of the theory of propagation of vibrations" in the book: Frank, F., Mises, R.: Differential and Integral Equations of Mathematical Physics. Vol. 2. ONTI, Leningrad – Moscow (1937). – *Ed.*

In the mentioned article we obtained one class of solutions of this equation. Let us recall these results. Let us construct the equation determining a complex variable θ, as a function of the variables (x_1, z, t),

$$t - \theta x_1 \pm \sqrt{c^2 - \theta^2}\, z = \chi(\theta), \tag{8}$$

where $\chi(\theta)$ is an analytic function. If in a domain of the space S with coordinates (x_1, z, t) equation (8) has an imaginary root, then the real or imaginary part of any analytic function of θ gives a solution of equation (7) in this domain of the space S. If in a domain of the space S equation (8) has a real root, then any twice-differentiable function of θ is a solution of equation (7). Also, note that each solution of equation (7), being a homogeneous function of order zero with respect to the arguments $t-\alpha$, $x_1-\beta$, $z-\gamma$, can be obtained by the method described above. In this case equation (8) has the form

$$(t-\alpha) - \theta(x_1-\beta) \pm \sqrt{c^2-\theta^2}(z-\gamma) = 0. \tag{8.1}$$

We construct now a class of solutions of equation (2). Let (x, y, z) be the Cartesian coordinates expressed in cylindrical coordinates according to the formulas

$$x = \varrho\cos\vartheta, \quad y = \varrho\sin\vartheta, \quad z = z.$$

Let us take an x_1-axis in the (x, y)-plane. Let μ be the angle between the x-axis and the x_1-axis. For a point (ϱ, z, ϑ), we have $x_1 = \varrho\cos(\vartheta-\mu)$. We can construct a solution of (5) by taking the real or imaginary part of an arbitrary analytic function of a complex variable $\theta_{\vartheta-\mu}$ determined by the equation

$$t - \theta_{\vartheta-\mu}\,\varrho\cos(\vartheta-\mu) \pm \sqrt{a^2 - \theta^2_{\vartheta-\mu}}\, z = \chi(\theta_{\vartheta-\mu}).$$

Integrating with respect to μ from 0 to 2π, we obtain a solution of equation (5), which does not depend on ϑ, i.e., we obtain a solution of equation (2). Let $\vartheta - \mu = \lambda$. Let us integrate with respect to λ from 0 to 2π. Then, we obtain a solution of equation (2) in the form

$$\varphi(\varrho, z, t) = \int_0^{2\pi} \Phi(\theta_\lambda)d\lambda, \tag{9}$$

where θ_λ is determined by the equation

$$t - \theta_\lambda\varrho\cos\lambda \pm \sqrt{a^2-\theta_\lambda^2}\, z = \chi(\theta_\lambda), \tag{10}$$

and $\Phi(\theta_\lambda)$ is an analytic function of θ_λ. The interval of integration $(0, 2\pi)$ can be reduced to $(0, \pi)$. If we want to obtain a real solution, we have to take a real or imaginary part of expression (9), i.e., we can write

$$\varphi = \operatorname{Re} \int_0^\pi \Phi(\theta_\lambda) d\lambda \quad \text{or} \quad \varphi = \operatorname{Im} \int_0^\pi \Phi(\theta_\lambda) d\lambda. \tag{11}$$

Consider now equation (3). First, assume that we deal with the two-dimensional case of elastic vibrations. Suppose that in the coordinate system (x_1, y_1, z) displacements do not depend on y_1 and occur on planes parallel to the (x_1, z)-plane. In this case, the vector potential at every point is parallel to the y_1-axis, and its length must satisfy equation (7) for $c = b$. Let us take a solution of this equation of the type described above, i.e., suppose that the length of the vector potential is equal to the real part of an analytic function $\Psi(\theta_{\vartheta-\mu})$, where $\theta_{\vartheta-\mu}$ is determined by the equation

$$t - \theta_{\vartheta-\mu}\, \varrho \cos(\vartheta - \mu) \pm \sqrt{b^2 - \theta_{\vartheta-\mu}^2}\, z = \chi(\theta_{\vartheta-\mu}).$$

Along the axes ϑ and ϱ the components of this vector are

$$\operatorname{Re}[\Psi(\theta_{\vartheta-\mu})] \cos(\vartheta - \mu) \quad \text{and} \quad \operatorname{Re}[\Psi(\theta_{\vartheta-\mu})] \sin(\vartheta - \mu).$$

Integrating with respect to μ from 0 to 2π, we obtain a vector potential that does not depend on ϑ and satisfies equation (6). Along the axes ϑ and ϱ the components of this potential are

$$\operatorname{Re} \int_0^{2\pi} \Psi(\theta_\lambda) \cos \lambda d\lambda \quad \text{and} \quad \operatorname{Re} \int_0^{2\pi} \Psi(\theta_\lambda) \sin \lambda d\lambda, \tag{12}$$

where θ_λ satisfies the equation

$$t - \theta_\lambda\, \varrho \cos \lambda \pm \sqrt{b^2 - \theta_\lambda^2}\, z = \chi(\theta_\lambda). \tag{13}$$

The second integral in (12) is equal to zero. Hence the obtained vector potential is directed along the ϑ-axis. Taking into account that this potential does not depend on ϑ and satisfies equation (7), we can assert that its length satisfies equation (3). Thus, we obtain solutions of equation (3) in the form

$$\psi = \operatorname{Re} \int_0^\pi \Psi(\theta_\lambda) \cos \lambda d\lambda \quad \text{or} \quad \psi = \operatorname{Im} \int_0^\pi \Psi(\theta_\lambda) \cos \lambda d\lambda. \tag{14}$$

Obviously, formulas (11) and (14) do not present all solutions of equation (2) and (3). Let us return to equation (7). As noted above, each homogeneous solution of order zero of the arguments $t - \alpha$, $x_1 - \beta$, $z - \gamma$ of this equation can be expressed as a function of θ, where θ is defined by (8.1). We prove a similar theorem for equations (2) and (3). Suppose that a solution of the equation in question is a homogeneous function of order zero of the ϱ, z, t. In this case, for θ_λ we have the equation

$$t - \theta_\lambda\, \varrho \cos\lambda \pm \sqrt{a^2 - \theta_\lambda^2}\, z = 0. \tag{15}$$

In this section we study homogeneous solutions of equation (2). Introduce the variables

$$\xi = \frac{\varrho}{t}, \quad \eta = \frac{z}{t}, \tag{16}$$

and assume that the function φ determined by equation (2) depend only on the arguments ξ and η. According to this assumption, equation (2) takes the form

$$(a^2\xi^2 - 1)\frac{\partial^2\varphi}{\partial\xi^2} + 2a^2\xi\eta\frac{\partial^2\varphi}{\partial\xi\partial\eta} + (a^2\eta^2 - 1)\frac{\partial^2\varphi}{\partial\eta^2}$$
$$+2a^2\left(\xi\frac{\partial\varphi}{\partial\xi} + \eta\frac{\partial\varphi}{\partial\eta}\right) - \frac{1}{\xi}\frac{\partial\varphi}{\partial\xi} = 0. \tag{17}$$

The substitution of $-\xi$ for ξ or $-\varrho$ for ϱ is equivalent to the substitution of $(\vartheta + \pi)$ for ϑ. Taking into account the axial symmetry, we consider solutions of equation (17) which are even functions of ξ, i.e.,

$$\varphi(-\xi, \eta) = \varphi(\xi, \eta).$$

Let us study (11), where θ_λ is determined by equation (15) as a function of ξ and η. In this case formula (11) gives us solutions of equation (17). It is easy to see that these solutions are even functions of ξ. Indeed, from equation (15) it follows that the variable θ_λ depends on λ through the product $\xi\cos\lambda$. Instead of (11) we can write

$$\varphi(\xi, \eta) = \int\limits_0^\pi F(\xi\cos\lambda, \eta)d\lambda.$$

Hence,

$$\varphi(-\xi, \eta) = \int\limits_0^\pi F(-\xi\cos\lambda, \eta)d\lambda.$$

Putting $\lambda_1 = \pi - \lambda$, we have

$$\varphi(-\xi, \eta) = -\int\limits_\pi^0 F(\xi\cos\lambda_1, \eta)d\lambda_1 = \int\limits_0^\pi F(\xi\cos\lambda_1, \eta)d\lambda_1 = \varphi(\xi, \eta),$$

which is required.

Further, we will prove that every solution of (17), that is an even function of ξ, can be expressed by (11) and (15). We will present a method to determine $\Phi(\theta_\lambda)$ for a given $\varphi(\xi, \eta)$.

We study preliminary formulas (11), (15), and functions given by these formulas in detail. Let T be the plane of the complex variable θ with the cut $(-a, a)$ along the real axis. The function $\sqrt{a^2-\theta^2}$ is single-valued in the domain T. We assume that this function is positive for the positive imaginary θ, i.e., for $\theta = \alpha i$, $\alpha > 0$. In other words, we assume that $\sqrt{a^2-\theta^2}$ is negative imaginary for $\theta > a$. Let us write equation (15) in the form

$$1 - \theta_\lambda \, \xi \cos\lambda + \sqrt{a^2 - \theta_\lambda^2}\, \eta = 0, \tag{18}$$

hence,

$$\theta_\lambda = \frac{\xi\cos\lambda}{\xi^2\cos^2\lambda + \eta^2} - i\frac{\eta\sqrt{1 - a^2(\xi^2\cos^2\lambda + \eta^2)}}{\xi^2\cos^2\lambda + \eta^2}. \tag{19}$$

First, we consider the disk K on the plane of the variables (ξ, η) defined by the inequality

$$\xi^2 + \eta^2 < \frac{1}{a^2}. \tag{20}$$

For such values of (ξ, η), equation (17) is elliptic. Let us introduce also the variable θ determined by the equation

$$1 - \theta\xi + \sqrt{a^2-\theta^2}\,\eta = 0, \tag{21}$$

where

$$\theta = \frac{\xi}{\xi^2+\eta^2} - i\frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}. \tag{22}$$

Let us note that we choose positive roots in (19) and (22). First, let us take the upper part of the disk K, where $\eta > 0$. In view of (21) and (22), this part corresponds to the lower part of the domain T, where $\operatorname{Im}\theta < 0$. Denote by T_1 this part of the domain T. Let $\Phi(\theta)$ be a function holomorphic in T_1. When the parameter λ runs through the interval $0 \leq \lambda \leq \pi$, the variable θ_λ traces a contour l located in T_1 and symmetric with respect to the imaginary axis. Let us separate the real and imaginary parts

$$\theta = \alpha + i\beta, \quad \Phi(\theta) = \omega_1(\alpha,\beta) + i\omega_2(\alpha,\beta),$$

and introduce the new function holomorphic in T_1,

$$\Phi_1(\theta) = \omega_1(-\alpha,\beta) - i\omega_2(-\alpha,\beta).$$

Let

$$\Phi_2(\theta) = \frac{1}{2}[\Phi(\theta) + \Phi_1(\theta)].$$

Taking into account the definition of $\Phi_1(\theta)$, we can assert that $\Phi_2(\theta)$ is real on the imaginary axis, and the formula

$$\varphi = \operatorname{Re}\int_0^\pi \Phi(\theta_\lambda)\,d\lambda$$

is equivalent to the formula

$$\varphi = \int_0^\pi \Phi_2(\theta_\lambda) d\lambda.$$

Therefore, we can always write (11) in the form

$$\varphi = \int_0^\pi \Phi(\theta_\lambda) d\lambda, \tag{23}$$

where $\Phi(\theta)$ is real on the imaginary axis. We will use such choice of the function $\Phi(\theta)$ subsequently.

In further calculations we will use instead of θ the new variable w,

$$w = \sqrt{a^2 - \theta^2}, \quad w_\lambda = \sqrt{a^2 - \theta_\lambda^2}, \tag{24}$$

where

$$w_\lambda = -\frac{\eta}{\xi^2 \cos^2 \lambda + \eta^2} - i\xi \cos \lambda \frac{\sqrt{1 - a^2(\xi^2 \cos^2 \lambda + \eta^2)}}{\xi^2 \cos^2 \lambda + \eta^2}. \tag{25}$$

On the plane w the domain T_1 corresponds to the half-plane located on the left of the imaginary axis with the cut $(-a, 0)$ along the real axis (see Fig. 1). Let us denote by S_1 this half-plane with the cut.

In this domain we have a holomorphic function $\Phi(w)$ real on the interval $-\infty < w < -a$ of the real axis and admitting conjugate values at points symmetric with respect to this axis. Instead of (23) we have

$$\varphi(\xi, \eta) = \int_0^\pi \Phi(w_\lambda) d\lambda. \tag{26}$$

Fixing (ξ, η), we have the point

$$w = -\frac{\eta}{\xi^2 + \eta^2} - i\frac{\xi\sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}. \tag{27}$$

When λ changes from 0 to π, the point w_λ traces the contour l_w symmetric with respect to the real axis. The beginning of this contour is the point w, and the end is the conjugate point $\overline{w}$. This contour is located in the domain S_1 (see Fig. 1).

Let us return to (26) and introduce instead of λ the new variable w_1, running the contour l_w. By (24),

$$w_1 = \sqrt{a^2 - \theta_\lambda^2} \quad \text{and} \quad \theta_\lambda = -\sqrt{a^2 - w_1^2},$$

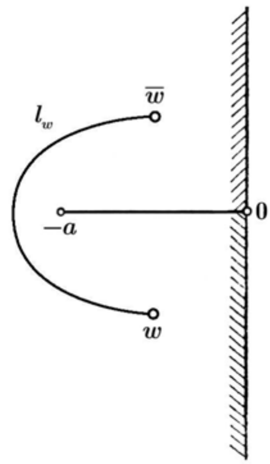

Fig. 1.

where the radical $\sqrt{a^2 - w_1^2}$ is negative imaginary for $w_1 > a$. Instead of equation (18) we have

$$1 + \sqrt{a^2 - w_1^2}\, \xi \cos \lambda + w_1 \eta = 0.$$

Consequently,

$$\cos \lambda = -\frac{1 + w_1 \eta}{\xi \sqrt{a^2 - w_1^2}}, \quad \lambda = \arccos\left(-\frac{1 + w_1 \eta}{\xi \sqrt{a^2 - w_1^2}} \right).$$

By simple calculations, we obtain

$$d\lambda = \frac{w_1 + a^2 \eta}{(a^2 - w_1^2)\sqrt{\xi^2 (a^2 - w_1^2) - (1 + w_1 \eta)^2}} dw_1.$$

The roots of the equation

$$\xi^2 (a^2 - w_1^2) - (1 + w_1 \eta)^2 = 0$$

are

$$w_1 = \frac{-\eta - i\xi \sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2} = w, \quad w_1 = \frac{-\eta + i\xi \sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2} = \overline{w}.$$

Hence,

$$d\lambda = \frac{w_1 + a^2 \eta}{\sqrt{\xi^2 + \eta^2}(a^2 - w_1^2)\sqrt{(w_1 - w)(\overline{w} - w_1)}} dw_1.$$

The radical $\sqrt{\xi^2 + \eta^2}$ has to be taken positive. To make the function

$$\sqrt{(w_1 - w)(\overline{w} - w_1)} \tag{28}$$

single-valued, we make two straight cuts parallel to the imaginary axis, going from the points w and $\overline{w}$ to infinity, without crossing the real axis. If ξ is positive, then the point w is located in the lower part of the domain S_1, and dw_1 is positive imaginary at the point where the contour l_w intersects the interval $(-\infty, -a)$ of the real axis. On the other hand, we have $w_1 < -a$ at this point. Hence, $a^2 - w_1^2 < 0$ and $w_1 + a^2\eta < 0$, because $0 < \eta < \dfrac{1}{a}$. The function $d\lambda$ must be positive. Consequently, in the formula for $d\lambda$ we should select function (28) positive imaginary for $w_1 < -a$, if $\xi > 0$. In the same way, we can prove that we should take this function negative imaginary for $w_1 < -a$, if $\xi < 0$. We consider the case $\xi = 0$ later. Instead of (26) we have

$$\varphi(\xi, \eta) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \int\limits_{l_w} \frac{(w_1 + a^2\eta)\Phi(w_1)}{(a^2 - w_1^2)\sqrt{(w_1 - w)(\overline{w} - w_1)}} dw_1. \tag{29}$$

Obviously, we can deform the contour of integration using the well-known Cauchy theorem.

For $\xi = 0$ formulas (25) and (26) give us

$$\varphi(0, \eta) = \pi\Phi\left(-\frac{1}{\eta}\right). \tag{30}$$

Assuming that the function $\Phi(w)$ is continuous up to the cut $-a \leq w \leq 0$, let us determine values of the function $\varphi(\xi, \eta)$ on the semicircle

$$\xi^2 + \eta^2 = \frac{1}{a^2}, \qquad \eta > 0.$$

In this case $w = \overline{w} = -a^2\eta$, and (29) gives us

$$\varphi(\xi, \eta)\Bigg|_{\xi^2+\eta^2=\frac{1}{a^2}} = \mp\frac{a}{i} \int\limits_{l_w} \frac{\Phi(w_1)}{a^2 - w_1^2} dw_1, \tag{31}$$

where the upper sign corresponds to the case $\xi > 0$, and the lower sign to the case $\xi < 0$. In the first case, the contour l_w goes from the point $w = -a^2\eta$ on the lower lip of the cut into the opposite point on the upper lip. In the second case, the direction of circulation about the contour l_w reverses (see Fig. 2).

It still remains to study values of the function $\varphi(\xi, \eta)$ defined by (26) or (29) on the diameter $\eta = 0$ of the disk K. By (25), in this case w_λ becomes infinite for

$$\lambda = \frac{\pi}{2},$$

and we cannot apply (26). Let us take (29) and tend the point (ξ, η) to the point $(\xi_0, 0)$. The end points of the contour l_w will tend to the limits

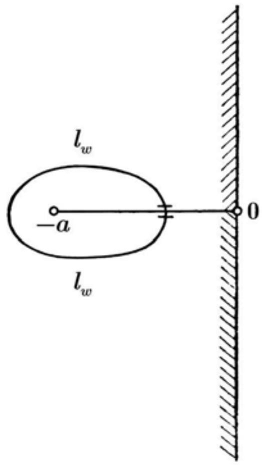

Fig. 2.

$$w_0 = -i\frac{\sqrt{1-a^2\xi_0^2}}{\xi_0}, \quad \overline{w}_0 = i\frac{\sqrt{1-a^2\xi_0^2}}{\xi_0}.$$

Using the Cauchy theorem, as we noticed above, we can assume that the contour l_w is always located within a finite distance. Finally, we have

$$\varphi(\xi_0, 0) = \frac{1}{\xi_0 i}\int\limits_{l_{w_0}} \frac{w_1\Phi(w_1)}{(a^2-w_1^2)\sqrt{w_1^2+\beta^2}}dw_1, \tag{32}$$

where

$$\beta = \frac{\sqrt{1-a^2\xi_0^2}}{\xi_0}, \tag{33}$$

and the radical $\sqrt{w_1^2+\beta^2}$ is positive for real values of w_1. The contour l_{w_0} with end-points $(-i\beta)$ and $(+i\beta)$ must be located inside of S_1. Until now, we studied the solution $\varphi(\xi, \eta)$ of equation (17), defined by (26), in the upper part of the disk K, where $\eta > 0$. Consider now the lower semidisk. Formula (22) tells us that this semidisk corresponds to the upper part of the domain T of the plane θ or, in view of (25), to the part of the plane w, located on the right of the imaginary axis, with the cut $(0, a)$ along the real axis. Let us denote this domain by S_2. Using (26) or (29), we suppose that the function $\Phi(w)$ is holomorphic in the domain S_2, and the contour l_w must be located in this domain.

The formulas obtained above are valid in this case with some sign changes. Let us point out these changes. We should take the radical $\sqrt{(w_1-w)(\overline{w}-w_1)}$ negative imaginary for $w_1 > 0$, if $\xi > 0$, and positive imaginary, if $\xi < 0$. Formulas (30) and (31) preserve their form, and instead of (32) we have

$$\varphi(\xi_0, 0) = -\frac{1}{\xi_0 i} \int\limits_{l_{w_0}} \frac{w_1 \Phi(w_1)}{(a^2 - w_1^2)\sqrt{w_1^2 + \beta^2}} dw_1, \tag{34}$$

where β is given by (33).

Let us note also the significance of the diameter $\eta = 0$ in representation (26) (or (29)) for solutions of equation (17), even with respect to ξ. Suppose, for example, that the function $\Phi(w)$ is holomorphic in the entire plane of the variable w with the cut $(-a, +a)$. Apply (29) in the semidisk $\eta > 0$ and in the semidisk $\eta < 0$. Then, we obtain two solutions of equation (17)

$$\varphi_1(\xi, \eta), \quad \eta > 0; \qquad \varphi_2(\xi, \eta), \quad \eta < 0.$$

Equation (17) is elliptic inside the disk K. Hence these solutions are analytic functions of ξ and η; however, generally speaking, $\varphi_2(\xi, \eta)$ is not an analytic continuation of $\varphi_1(\xi, \eta)$, i.e., by (29), analytic continuation of the function $\Phi(w)$ across the imaginary axis does not give analytic continuation $\varphi(\xi, \eta)$ across the diameter $\eta = 0$. We will come back to this question later.

4. In the previous section we studied the solutions of equation (17), defined by (26) and (29). We now prove that every solution of equation (17) inside of the disk K, that is an even function with respect to ξ, can be given by (26). We present also a way to determine the function $\Phi(w)$ for a given $\varphi(\xi, \eta)$.

First, consider the semidisk $\eta > 0$. Let $\varphi(\xi, \eta)$ be a solution of (17), even with respect to ξ. This solution is an analytic function of (ξ, η), and we can express it in the form

$$\varphi(\xi, \eta) = \varphi_0(\eta) + \varphi_2(\eta)\xi^2 + \varphi_4(\eta)\xi^4 + \cdots \tag{35}$$

in a neighborhood of the ray $\xi = 0$.

It is easy to prove that the coefficients $\varphi_2(\eta), \varphi_4(\eta), \ldots$ are uniquely defined by the coefficient $\varphi_0(\eta)$. Indeed, substituting (35) in (17) and equating the coefficient at ξ^n to zero, we have

$$n(n-1)a^2\varphi_n(\eta) - (n+2)(n+1)\varphi_{n+2}(\eta) + 2a^2 n\eta\varphi_n'(\eta)$$

$$+(a^2\eta^2 - 1)\varphi_n''(\eta) + 2a^2 n\varphi_n(\eta) + 2a^2\eta\varphi_n'(\eta) - (n+2)\varphi_{n+2}(\eta) = 0.$$

These equations completely determine the functions $\varphi_2(\eta)$, $\varphi_4(\eta)$, ... Hence the solution $\varphi(\xi, \eta)$ satisfying the condition of the form

$$\varphi(\xi, \eta)|_{\xi=0} = \varphi_0(\eta) \tag{36}$$

is unique. In condition (36) the function $\varphi_0(\eta)$ must be analytic for $0 < \eta < \dfrac{1}{a}$. It is easy to find a function $\Phi(w)$ such that the solution defined by (26) satisfies condition (36). Indeed, in view of (30), this condition gives us

$$\varphi_0(\eta) = \pi\Phi\left(-\frac{1}{\eta}\right)$$

or

$$\Phi(w) = \frac{1}{\pi}\varphi_0\left(-\frac{1}{w}\right) \qquad \text{for } -a > w > -\infty. \tag{37}$$

Then, we have analytic real values of the function $\Phi(w)$ along the interval $-a > w > -\infty$ of the real axis, and this function is holomorphic in a neighborhood of this interval. Hence (29) gives us the function $\varphi(\xi, \eta)$ in a neighborhood of the ray $\xi = 0$. However, as we already observed, condition (36) determines the solution uniquely. Hence every solution $\varphi(\xi, \eta)$ even with respect to ξ can be presented by (26) or (29). Formula (37) gives us a method to determine $\Phi(w)$ for a given $\varphi(\xi, \eta)$. Suppose that the given solution $\varphi(\xi, \eta)$ is analytic in the entire semidisk $\eta > 0$. Prove that $\Phi(w)$ is holomorphic in the entire domain T_1. Consider (26) and introduce instead of λ the new variable

$$\tau = \xi \cos \lambda. \tag{38}$$

For $\xi > 0$ we have

$$\varphi(\xi, \eta) = \int\limits_{-\xi}^{+\xi} \frac{\Phi(w_\tau)}{\sqrt{\xi^2 - \tau^2}} d\tau, \tag{39}$$

where

$$w_\tau = -\frac{\eta}{\tau^2 + \eta^2} - i\frac{\tau\sqrt{1 - a^2(\tau^2 + \eta^2)}}{\tau^2 + \eta^2}.$$

Equation (39) is the Abel integral equation on the function $\Phi(w)$, and we can solve it by the usual method. Let us multiply both parts of equation (39) by $\dfrac{\xi}{\sqrt{\mu^2 - \xi^2}}$ and integrate with respect to ξ from 0 to μ;

$$\int\limits_0^\mu \frac{\varphi(\xi, \eta)\xi}{\sqrt{\mu^2 - \xi^2}} d\xi = \int\limits_0^\mu \left[\int\limits_{-\xi}^{+\xi} \frac{\Phi(w_\tau) d\tau}{\sqrt{\xi^2 - \tau^2}}\right] \frac{\xi d\xi}{\sqrt{\mu^2 - \xi^2}}.$$

Changing order of integration on the right side and applying the well-known Dirichlet formula, we have

$$\int\limits_0^\mu \frac{\varphi(\xi, \eta)\xi}{\sqrt{\mu^2 - \xi^2}} d\xi = \int\limits_0^\mu \left[\int\limits_\tau^\mu \frac{\xi d\xi}{\sqrt{(\xi^2 - \tau^2)(\mu^2 - \xi^2)}}\right] \Phi(w_\tau) d\tau$$

$$+ \int\limits_{-\mu}^0 \left[\int\limits_{-\tau}^\mu \frac{\xi d\xi}{\sqrt{(\xi^2 - \tau^2)(\mu^2 - \xi^2)}}\right] \Phi(w_\tau) d\tau.$$

Consider the internal integral and introduce instead of ξ the new variable of integration σ,

$$\sigma^2 = \frac{\xi^2 - \tau^2}{\mu^2 - \tau^2} \quad \text{or} \quad \xi^2 = \sigma^2(\mu^2 - \tau^2) + \tau^2.$$

We have

$$\int\limits_{\tau}^{\mu} \frac{\xi d\xi}{\sqrt{(\xi^2 - \tau^2)(\mu^2 - \xi^2)}} = \int\limits_{0}^{1} \frac{d\sigma}{\sqrt{1 - \sigma^2}} = \frac{\pi}{2}.$$

An analogous result is valid for the second internal integral. The previous formula gives us

$$\int\limits_{0}^{\mu} \frac{\varphi(\xi, \eta)\xi d\xi}{\sqrt{\mu^2 - \xi^2}} = \frac{\pi}{2} \int\limits_{-\mu}^{+\mu} \Phi(w_\tau) d\tau,$$

or, differentiating with respect to μ,

$$\frac{\pi}{2}[\Phi(w_1) + \Phi(w_2)] = \frac{d}{d\mu} \int\limits_{0}^{\mu} \frac{\varphi(\xi, \eta)\xi d\xi}{\sqrt{\mu^2 - \xi^2}}, \tag{40}$$

where

$$w_1 = -\frac{\eta}{\mu^2 + \eta^2} - i\frac{\mu\sqrt{1 - a^2(\mu^2 + \eta^2)}}{\mu^2 + \eta^2},$$

$$w_2 = -\frac{\eta}{\mu^2 + \eta^2} + i\frac{\mu\sqrt{1 - a^2(\mu^2 + \eta^2)}}{\mu^2 + \eta^2}.$$

We change the limits of integration in (39) for the case $\xi < 0$; however, repeating computations, it is easy to verify that (40) is also valid for $\xi < 0$.

Let us note that the function $\Phi(w)$, real on the interval $-\infty < w < -a$ of the real axis, admits complex conjugate values at conjugate points w_1 and w_2. Substituting τ for ξ and ξ for μ, by (40), we have

$$\mathrm{Re}\,[\Phi(w)] = \frac{1}{\pi}\frac{d}{d\xi} \int\limits_{0}^{\xi} \frac{\varphi(\tau, \eta)\tau d\tau}{\sqrt{\xi^2 - \tau^2}}, \tag{41}$$

where

$$w = -\frac{\eta}{\xi^2 + \eta^2} - i\frac{\xi\sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}.$$

Introducing instead of τ the new variable of integration σ,

$$\tau = \xi\sigma,$$

we can reduce (41) to the form

$$\mathrm{Re}\,[\Phi(w)] = \frac{1}{\pi}\frac{d}{d\xi} \int\limits_{0}^{1} \frac{\varphi(\xi\sigma, \eta)\xi\sigma d\sigma}{\sqrt{1 - \sigma^2}}, \tag{42}$$

or, integrating by parts,

$$\operatorname{Re}[\varPhi(w)] = \frac{1}{\pi}\frac{d[\varphi(\xi\sigma,\eta)\xi]}{d\xi}\bigg|_{\sigma=0} + \frac{1}{\pi}\frac{d}{d\xi}\int\limits_0^1 \varphi'(\xi\sigma,\eta)\xi^2\sqrt{1-\sigma^2}d\sigma, \tag{43}$$

where φ' is the derivative of the function φ with respect to the first argument.

By assumption, the function $\varphi(\xi,\eta)$ is analytic in the upper part of the disk K, whence it follows that the right side of (43) is also analytic in this domain. If ξ is close to zero, the right side of (43) gives us the real part of the holomorphic function $\varPhi(w)$ defined by (37), i.e., it gives us the harmonic function of the variables

$$u = -\frac{\eta}{\xi^2+\eta^2}, \quad v = -\frac{\xi\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}.$$

Obviously, this fact holds for all values of ξ and η from the upper part of the disk K, and (43) gives us the real part of a function, holomorphic in the domain S_1, coinciding with $\varPhi(w)$, and defined by (37) in a neighborhood of the ray $\xi = 0$. We can repeat the same arguments for the lower part of the disk K.

Let us note one case, important in applications, when the function $\varphi(\xi,\eta)$ is analytic in the disk K with singular point $\xi = \eta = 0$. In this case, we can apply our arguments to the upper and lower parts of the disk K and obtain two functions $\varPhi_1(w)$ and $\varPhi_2(w)$ holomorphic in the domains S_1 and S_2, respectively. However, $\varPhi_2(w)$ is not an analytic continuation of $\varPhi_1(w)$. This is related to the fact that (43) cannot be applied for $\eta = 0$. If $\varphi(\xi,\eta)$ does not have singular points in K, then (43) presents the function $\varPhi(w)$ holomorphic in the domain (S_1+S_2). In this case, $\varphi(\xi,\eta)$ can be given by (26) in the entire disk K by means of the same holomorphic function $\varPhi(w)$.

5. Let us consider now two special cases, when $\varphi(\xi,\eta)$ has the singular point $\xi = \eta = 0$, and an analytic continuation of the function $\varphi(\xi,\eta)$ across the diameter $\eta = 0$ leads to a simple law of continuation of the function $\varPhi(w)$.

First, suppose that the function $\varphi(\xi,\eta)$ is analytic in the upper half of K and on the diameter $\eta = 0$, excluding $\xi = \eta = 0$, satisfies the condition

$$\frac{\partial\varphi(\xi,\eta)}{\partial\eta}\bigg|_{\eta=0} = 0. \tag{44}$$

Introduce the notation

$$\varphi(\xi,\eta)|_{\eta=0} = f(\xi). \tag{45}$$

As noted above, this function $\varphi(\xi,\eta)$ leads us to the function $\varPhi(w)$ holomorphic in the domain S_1. We assume that this function is continuous up to the imaginary axis. Using (32) and taking into account (45), we have

$$\frac{1}{\xi i}\int_{l_w}\frac{\Phi(w_1)w_1dw_1}{(a^2-w_1^2)\sqrt{w_1^2+\beta^2}}=f(\xi), \tag{46}$$

where

$$\beta=\frac{\sqrt{1-a^2\xi^2}}{\xi},$$

and l_w is the contour located in the domain S_1 and connecting the points $(-i\beta)$ and $(+i\beta)$.

In the domain S_2 consider the holomorphic function $\Phi_1(w)$ given by the formula

$$\Phi_1(w)=\Phi(-w). \tag{47}$$

Consider the solution $\varphi_1(\xi,\eta)$ of equation (17), analytic in the lower part of the disk K,

$$\varphi_1(\xi,\eta)=\int_0^\pi \Phi_1(w_\lambda)d\lambda.$$

Formula (34) gives us

$$\varphi_1(\xi,0)=-\frac{1}{\xi i}\int_{l'_w}\frac{\Phi_1(w_2)w_2dw_2}{(a^2-w_2^2)\sqrt{w_2^2+\beta^2}},$$

where l'_w is a contour located in the domain S_2 and connecting the points $(-i\beta)$ and $(i\beta)$. Introducing the new variable of integration $w_1=-w_2$, in view of (47), we have

$$\varphi_1(\xi,0)=\frac{1}{\xi i}\int_{l_w}\frac{\Phi(w_1)w_1dw_1}{(a^2-w_1^2)\sqrt{w_1^2+\beta^2}},$$

i.e.,

$$\varphi_1(\xi,\eta)|_{\eta=0}=f(\xi). \tag{48}$$

We now show that $\varphi_1(\xi,\eta)$ also satisfies the condition

$$\left.\frac{\partial\varphi_1(\xi,\eta)}{\partial\eta}\right|_{\eta=0}=0. \tag{49}$$

Formulas (48) and (49) show that, on the diameter $\eta=0$, $\varphi_1(\xi,\eta)$ satisfies the same Cauchy conditions as well as the function $\varphi(\xi,\eta)$. Therefore, $\varphi_1(\xi,\eta)$ is a continuation of $\varphi(\xi,\eta)$, i.e., in the present case formula (47) gives the law of continuation of the function $\Phi(w)$.

To prove (49), we need to obtain expressions for the derivatives $\dfrac{\partial\varphi}{\partial\eta}$ and $\dfrac{\partial\varphi_1}{\partial\eta}$. For this, let us take (29) in the form

$$\varphi(\xi, \eta) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \int\limits_{l_w} \frac{(w_1 + a^2\eta)\Phi(w_1)}{(-w_1 + \widetilde{a})(a^2 - w_1^2)} d\sqrt{(w_1 - w)(\overline{w} - w_1)},$$

where

$$\widetilde{a} = -\frac{\eta}{\xi^2 + \eta^2}.$$

Integrating by parts, we have

$$\varphi(\xi, \eta) = \frac{1}{\sqrt{\xi^2 + \eta^2}} \int\limits_{l_w} \frac{(w_1 + a^2\eta)\Phi'(w_1)}{(-w_1 + \widetilde{a})(a^2 - w_1^2)} \sqrt{(w_1 - w)(\overline{w} - w_1)} dw_1$$

$$+ \frac{1}{\sqrt{\xi^2 + \eta^2}} \int\limits_{l_w} \Phi(w_1)\sqrt{(w_1 - w)(\overline{w} - w_1)} \frac{d}{dw_1}\left\{\frac{w_1 + a^2\eta}{(-w_1 + \widetilde{a})(a^2 - w_1^2)}\right\} dw_1.$$

Differentiating with respect to η and putting $\eta = 0$, we have

$$\left.\frac{\partial\varphi(\xi, \eta)}{\partial\eta}\right|_{\eta=0} = \frac{i(1 - a^2\xi^2)}{\xi^3} \int\limits_{l_w} \frac{\sqrt{w_1^2 + \beta^2}\,\Phi'(w_1) dw_1}{w_1(a^2 - w_1^2)}$$

$$+ \frac{1}{i\xi^3} \int\limits_{l_w} \frac{w_1\Phi'(w_1)}{(a^2 - w_1^2)\sqrt{w_1^2 + \beta^2}} dw_1$$

$$+ \frac{i(1 - a^2\xi^2)}{\xi^3} \int\limits_{l_w} \frac{(3w_1^2 - a^2)\sqrt{w_1^2 + \beta^2}\,\Phi(w_1)}{w_1^2(a^2 - w_1^2)^2} dw_1$$

$$+ \frac{1}{i\xi^3} \int\limits_{l_w} \frac{2w_1^2\Phi(w_1)}{(a^2 - w_1^2)^2\sqrt{w_1^2 + \beta^2}} dw_1,$$

where β is given by (33); the contour l_w located in the domain S_1 connects the points $(-i\beta)$ and $(i\beta)$. We should take the radical $\sqrt{w_1^2 + \beta^2}$ positive for the real w_1, if $\xi > 0$, and negative, if $\xi < 0$. We have a similar expression for

$$\left.\frac{\partial\varphi_1(\xi, \eta)}{\partial\eta}\right|_{\eta=0}.$$

However, in this case we should take instead of $\Phi(w_1)$ the function $\Phi_1(w_1)$ determined by formula (47); the contour l_w must be located in the domain S_2, and the radical $\sqrt{w_1^2 + \beta^2}$ must have the opposite sign. Substituting in the expression

$$\left.\frac{\partial\varphi_1(\xi, \eta)}{\partial\eta}\right|_{\eta=0}$$

the variable of integration $w_2 = -w_1$ and taking into account (44), we obtain (49). If we have the condition

$$\varphi(\xi,\eta)|_{\eta=0}=0 \tag{50}$$

instead of condition (44), then we should replace (47) by the formula

$$\Phi_1(w)=-\Phi(-w).$$

We can prove this assertion in the same way as above.

6. In the previous sections we studied solutions of equation (17) inside the disk K, where equation (17) is elliptic. We now move on to the problem of continuation of these solutions into the exterior of this disk. Let, for example, $\varphi(\xi,\eta)$ be a solution of equation (17), analytic in the upper part of the disk K, including some arcs of the semicircle $\xi^2+\eta^2=\frac{1}{a^2}$ $(\eta>0)$. Let $\Phi(w)$ be the corresponding function holomorphic in the domain S_1. By (43), this function is also holomorphic on the lips of the cut $(-a,0)$ corresponding to the arcs mentioned above. For values of $\varphi(\xi,\eta)$ on this circle, we have (31).

By

$$w=-\frac{\eta}{\xi^2+\eta^2}-i\frac{\xi\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},$$

the real and imaginary parts of the function $\Phi(w)$ are functions of ξ and η. In our previous article [1] we proved that these functions are solutions of the equation

$$(a^2\xi^2-1)\frac{\partial^2 f}{\partial\xi^2}+2a^2\xi\eta\frac{\partial^2 f}{\partial\xi\partial\eta}+(a^2\eta^2-1)\frac{\partial^2 f}{\partial\eta^2}+2a^2\left(\xi\frac{\partial f}{\partial\xi}+\eta\frac{\partial f}{\partial\eta}\right)=0. \tag{51}$$

This equation as well as equation (17) is elliptic inside K and hyperbolic outside K. In our mentioned article we proved that characteristics of equations (51) and (17) are tangents to the circle $\xi^2+\eta^2=\dfrac{1}{a^2}$, and a solution of (51) can be obtained if we chose $f(\xi,\eta)$ to be constant along the characteristics. The real and imaginary parts of $\Phi(w)$ are known inside K. We apply the following principle of continuation of these solutions into the exterior of K: we take $f(\xi,\eta)$ to be constant along every tangent to the semicircle between the point of tangency and the axis $\eta=0$. Obviously, this constant is equal to the value of the function $f(\xi,\eta)$ at the point of tangency.

Applying (26) or (29) to the function $\Phi(w_\lambda)$ outside the disk K, we have a continuation of the solution $\varphi(\xi,\eta)$ of equation (17). As follows from our previous article, this continuation method has direct mechanical sense for the vector potential.

Let us consider the above continuation in detail. For example, consider a point $M(\xi,\eta)$ such that

$$\xi>0,\quad \frac{1}{a}>\eta>0,\quad \xi^2+\eta^2>\frac{1}{a^2}. \tag{52}$$

The coordinates of the tangency point for the tangent to the semicircle

$$\xi^2 + \eta^2 = \frac{1}{a^2}, \quad \eta > 0,$$

passing through the point M, are

$$\xi_0 = -\frac{\xi + \eta\sqrt{a^2(\xi^2+\eta^2)-1}}{a^2(\xi^2+\eta^2)}, \quad \eta_0 = \frac{\eta + \xi\sqrt{a^2(\xi^2+\eta^2)-1}}{a^2(\xi^2+\eta^2)}.$$

By the formula

$$w = -\frac{\eta + i\xi\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},$$

we have

$$w = -\frac{\eta_0}{\xi_0^2+\eta_0^2} = -a^2\eta_0 = -\frac{\eta + \xi\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}.$$

Thus, our continuation method gives us the following value of the function $\Phi(w)$ at the point $M(\xi, \eta)$:

$$\Phi\left(-\frac{\eta + \xi\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}\right). \tag{53}$$

To obtain the value of the function $\varphi(\xi, \eta)$ at the point M, we should substitute $\xi \cos\lambda$ for ξ in expression (53) and integrate with respect to λ from 0 to π. For a value of $\lambda = \lambda_0$, the radical

$$\sqrt{a^2(\xi^2\cos^2\lambda + \eta^2) - 1} \tag{54}$$

becomes pure imaginary, and the point with coordinates $(\xi\cos\lambda, \eta)$ is located inside K for $\lambda_0 < \lambda < \pi - \lambda_0$. For these values of λ, we should choose radical (54) to be negative imaginary. Then the argument in expression (53) is a complex number. For $\pi - \lambda_0 < \lambda < \pi$, this argument is

$$-\frac{\eta - \xi\cos\lambda\sqrt{a^2(\xi^2\cos^2\lambda+\eta^2)-1}}{\xi^2\cos^2\lambda+\eta^2},$$

where the radical should be chosen positive. This last expression is equal to $(-a^2\eta_0)$, where η_0 is the ordinate of the tangency point of the tangent to the upper semicircle, passing through the point $(\xi\cos\lambda, \eta)$.

Then, we have a contour l_w in the plane of the variable w. This contour starts from the point

$$w = -\frac{\eta + \xi\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}$$

located on the lower lip of the cut and goes along this cut until the point

$$w' = -\frac{\eta}{\xi^2\cos^2\lambda_0+\eta^2}, \qquad \lambda = \lambda^0.$$

Then, the contour describes the path located in S_1 until the point w' on the upper lip of the cut and, finally, goes along this lip until the point w.

Thus, for $\varphi(\xi,\eta)$ we have

$$\varphi(\xi,\eta)=\int_0^{\pi}\Phi\left(-\frac{\eta\pm\xi\cos\lambda\sqrt{a^2(\xi^2\cos^2\lambda+\eta^2)-1}}{\xi^2\cos^2\lambda+\eta^2}\right)d\lambda,$$

where the choice of the sign is given above. Introducing the integration variable w_1 instead of λ, as in Sect. 3, we have

$$d\lambda=\frac{w_1+a^2\eta}{(a^2-w_1^2)\sqrt{-(\xi^2+\eta^2)w_1^2-2\eta w_1+(\xi^2a^2-1)}}dw_1.$$

Putting

$$w=-\frac{\eta+\xi\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2},\quad w_0=-\frac{\eta-\xi\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2},$$

we have

$$d\lambda=\frac{w_1+a^2\eta}{\sqrt{\xi^2+\eta^2}(a^2-w_1^2)\sqrt{(w_1-w)(w_0-w_1)}}dw_1$$

and

$$\varphi(\xi,\eta)=\frac{1}{\sqrt{\xi^2+\eta^2}}\int_{l_w}\frac{(w_1+a^2\eta)\Phi(w_1)}{(a^2-w_1^2)\sqrt{(w_1-w)(w_0-w_1)}}dw_1,\qquad(55)$$

where l_w is the contour described above.

Also, let us make an assumption about the function $\varphi(\xi,\eta)$. We suppose that this function is analytic and equal to zero on an arc AA_1 of the semicircle

$$\xi^2+\eta^2=\frac{1}{a^2},\quad\eta>0,$$

symmetric with respect to the axis $\xi=0$. Formula (30) shows us that in this case $\Phi(w)$ is single-valued and holomorphic in a neighborhood of the point $w=-a$, and

$$\Phi(-a)=0.\qquad(56)$$

Indeed, in (30) $\varphi(0,\eta)$ is an analytic function of η in a neighborhood of the point $\eta=-a^{-1}$, since $\xi=0$, $\eta=-a^{-1}$ is the midpoint of the arc AA_1. By assumption, $\varphi(\xi,\eta)$ is analytic on the arc AA_1. Hence $\Phi(w)$ is analytic on the lips of the cut corresponding to this arc. As we have already noted, this function is single-valued in a neighborhood of the point $w=-a$. By the principle of analytic continuation, $\Phi(w)$ is single-valued and holomorphic on the interval $(-a,c)$ of the cut, where the point c corresponds to the points A and A_1 of the semicircle.

Let us consider formula (55). Assume that a tangent to the semicircle passing through a point (ξ,η) has a tangency point on the arc AA_1. In this case, the end points of the contour l_w are located on the interval $(-a, c)$ of the cut. By (56), the integrand does not have a pole at the point $w = -a$, and its branching points $w_1 = w$ and $w_1 = w_0$ are located as follows: one on the contour, and another one outside this contour. Hence, according to the Cauchy theorem, formula (56) gives us

$$\varphi(\xi, \eta) = 0,$$

i.e., according to our continuation rule, $\varphi(\xi, \eta)$ is equal to zero on tangents to the semicircle, whose tangency points are located on the arc AA_1, symmetric with respect to $\xi = 0$, and where $\varphi(\xi, \eta)$ is zero. We can obtain a similar result in the case of $\eta < 0$.

7. To clarify the described theory, let us give some examples.

Let us take a solution of equation (2), depending only on the argument $\dfrac{r}{t}$, where r is the distance from the origin, i.e., $r = \sqrt{\varrho^2 + z^2}$.

In this case, we can let

$$\varphi(\xi, \eta) = f(\omega),$$

where $\omega = \xi^2 + \eta^2$.

Then,

$$\frac{\partial \varphi}{\partial \xi} = 2\xi f'(\omega), \quad \frac{\partial \varphi}{\partial \eta} = 2\eta f'(\omega),$$

$$\frac{\partial^2 \varphi}{\partial \xi^2} = 2f'(\omega) + 4\xi^2 f''(\omega), \quad \frac{\partial^2 \varphi}{\partial \xi \partial \eta} = 4\xi\eta f''(\omega),$$

$$\frac{\partial^2 \varphi}{\partial \eta^2} = 2f'(\omega) + 4\eta^2 f''(\omega).$$

Substituting the expressions into equation (17), we have

$$f''(\omega)\,[4\xi^2(a^2\xi^2 - 1) + 8a^2\xi^2\eta^2 + 4\eta^2(a^2\eta^2 - 1)]$$

$$+f'(\omega)[2(a^2\xi^2 - 1) + 2(a^2\eta^2 - 1) + 4a^2(\xi^2 + \eta^2) - 2] = 0$$

or

$$2\omega f''(\omega) + 3f'(\omega) = 0.$$

Integrating, we have

$$\varphi(\xi, \eta) = \frac{C_1}{\sqrt{\xi^2 + \eta^2}} + C_2.$$

Let us take the solution

$$\varphi(\xi, \eta) = \frac{1}{\sqrt{\xi^2 + \eta^2}} - a, \tag{57}$$

which is equal to zero on the circle

$$\xi^2 + \eta^2 = \frac{1}{a^2}.$$

This solution has the singular point $\xi = \eta = 0$. We define the corresponding function $\Phi(w)$. In the upper semidisk $\eta > 0$, we have by (30),

$$\frac{1}{\eta} - a = \pi\Phi\left(-\frac{1}{\eta}\right).$$

Hence,

$$\Phi(w) = -\frac{1}{\pi}(w + a), \qquad \eta > 0. \tag{57.1}$$

For the semidisk $\eta < 0$, we have

$$-\frac{1}{\eta} - a = \pi\Phi_1\left(-\frac{1}{\eta}\right)$$

and

$$\Phi_1(w) = \frac{1}{\pi}(w - a). \tag{57.2}$$

It is easy to see that solution (57) satisfies the condition

$$\left.\frac{\partial\varphi(\xi,\eta)}{\partial\eta}\right|_{\eta=0} = 0,$$

and functions (57.1) and (57.2) satisfy condition (47). According to our continuation rule, solution (57) is equal to zero outside the disk K.

Let us consider a solution of equation (17), depending only on η. Such a solution satisfies the equation

$$(a^2\eta^2 - 1)\frac{d^2\varphi}{d\eta^2} + 2a^2\eta\frac{d\varphi}{d\eta} = 0.$$

Let us take the solution

$$\varphi = C\ln\frac{1 - a\eta}{1 + a\eta}. \tag{58}$$

Formula (30) gives us

$$C\ln\frac{1 - a\eta}{1 + a\eta} = \pi\Phi\left(-\frac{1}{\eta}\right).$$

Consequently,

$$\Phi(w) = \frac{C}{\pi}\ln\frac{1 + \frac{a}{w}}{1 - \frac{a}{w}} \tag{59}$$

in the entire disk K.

Let us note that solution (58) does not have singular points inside K. According to this fact, the function $\Phi(w)$ is an analytic function in S_1 and S_2. Solution (58) satisfies also the condition

$$\varphi(\xi, \eta)|_{\eta=0} = 0,$$

and function $\Phi(w)$ at points symmetric with respect to $w = 0$ satisfies the condition

$$\Phi(w) = -\Phi(-w)$$

mentioned in Sect. 5.

Let us consider one more solution of equation (17),

$$\varphi(\xi, \eta) = \frac{\eta}{\sqrt{1 - a^2\xi^2}}, \tag{60}$$

which does not have singular points inside K. Formula (30) gives us

$$\eta = \pi\Phi\left(-\frac{1}{\eta}\right).$$

Whence we have the expression for $\Phi(w)$ in the entire disk K,

$$\Phi(w) = -\frac{1}{\pi w}. \tag{61}$$

8. We now consider solutions of equation (3), homogeneous of order zero of the arguments ϱ, z and t, i.e., functions of the arguments

$$\xi = \frac{\varrho}{t}, \quad \eta = \frac{z}{t}.$$

Instead of equation (17) for ψ, we have the equation

$$\begin{aligned}(b^2\xi^2 - 1)\frac{\partial^2\psi}{\partial\xi^2} + 2b^2\xi\eta\frac{\partial^2\psi}{\partial\xi\partial\eta} + (b^2\eta^2 - 1)\frac{\partial^2\psi}{\partial\eta^2}\\ +2b^2\left(\xi\frac{\partial\psi}{\partial\xi} + \eta\frac{\partial\psi}{\partial\eta}\right) - \frac{1}{\xi}\frac{\partial\psi}{\partial\xi} + \frac{1}{\xi^2}\psi = 0.\end{aligned} \tag{62}$$

By axial symmetry, we have to consider solutions corresponding to the displacement component w, which is an even function of ξ and ϱ. The second formula in (1) tells us that such solutions are odd functions of ξ, i.e.,

$$\psi(-\xi, \eta) = -\psi(\xi, \eta). \tag{63}$$

Let us consider the formula [2]

$$\psi = \mathrm{Re}\int_0^\pi \Psi(\theta_\lambda)\cos\lambda d\lambda, \tag{64}$$

where θ_λ is a function of ξ and η, and is defined by the equation

[2] See formulas from Sect. 3. – *Ed.*

$$t - \theta_\lambda \varrho \cos\lambda \pm \sqrt{b^2 - \theta_\lambda^2}\, z = 0$$

or (65)

$$1 - \theta_\lambda \xi \cos\lambda \pm \sqrt{b^2 - \theta_\lambda^2}\, \eta = 0.$$

Formula (64) gives us solutions of equation (62). It easy to see that these solutions are odd functions of ξ. Indeed, from equation (65) it follows that the variable θ_λ depends on λ through the product $\xi \cos\lambda$. Hence instead of (64) we can write

$$\psi(\xi, \eta) = \int_0^\pi F(\xi\cos\lambda, \eta)\cos\lambda d\lambda.$$

Consequently,

$$\psi(-\xi, \eta) = \int_0^\pi F(-\xi\cos\lambda, \eta)\cos\lambda d\lambda$$

or, putting $\lambda_1 = \pi - \lambda$,

$$\psi(-\xi, \eta) = \int_\pi^0 F(\xi\cos\lambda_1, \eta)\cos\lambda_1 d\lambda_1 = -\int_0^\pi F(\xi\cos\lambda, \eta)\cos\lambda d\lambda = -\psi(\xi, \eta),$$

which is required.

Introducing new variables

$$w = \sqrt{b^2 - \theta^2}, \quad w_1 = \sqrt{b^2 - \theta_\lambda^2}$$

instead of (64), we have

$$\psi(\xi, \eta) = \mathrm{Re} \int_0^\pi \Psi(w_1)\cos\lambda d\lambda, \tag{66}$$

and

$$\begin{aligned} w &= -\frac{\eta}{\xi^2 + \eta^2} - i\frac{\xi\sqrt{1 - b^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}, \\ w_1 &= -\frac{\eta}{\xi^2\cos^2\lambda + \eta^2} - i\frac{\xi\cos\lambda\sqrt{1 - b^2(\xi^2\cos^2\lambda + \eta^2)}}{\xi^2\cos^2\lambda + \eta^2}. \end{aligned} \tag{67}$$

When the variable λ moves along the interval $(0, \pi)$, then w_1 describes a contour symmetric with respect to the real axis. Symmetric points correspond to values of $\cos\lambda$ with the same absolute values, but with opposite signs.

Separating the real and imaginary parts, we can write

$$w = \alpha + i\beta, \quad \Psi(w) = \omega_1(\alpha, \beta) + i\omega_2(\alpha, \beta).$$

Let us introduce the holomorphic functions

$$\Psi_1(w) = \omega_1(\alpha, -\beta) - i\omega_2(\alpha, -\beta),$$

$$\Psi_2(w) = \frac{1}{2}[\Psi(w) + \Psi_1(w)].$$

Instead of (66) we can write

$$\psi(\xi, \eta) = \frac{1}{i} \int\limits_0^\pi \Psi_2(w_1) \cos \lambda d\lambda,$$

where the function $\Psi_2(w)$ is real on the real axis. Then, we can always assume

$$\psi(\xi, \eta) = \frac{1}{i} \int\limits_0^\pi \Psi(w_1) \cos \lambda d\lambda, \tag{68}$$

where $\Psi(w)$ has complex conjugate values at points symmetric with respect to the real axis.

Further analysis will be completely analogous to the same one for the potential $\varphi(\xi, \eta)$. As noted above,

$$\cos \lambda = -\frac{1 + w_1\eta}{\xi\sqrt{b^2 - w_1^2}},$$

and an analog of formula (29) has the form

$$\psi(\xi, \eta) = \frac{i}{\xi\sqrt{\xi^2 + \eta^2}} \int\limits_{l_w} \frac{(w_1 + b^2\eta)(1 + \eta w_1)\Psi(w_1)}{(b^2 - w_1^2)\sqrt{b^2 - w_1^2}\sqrt{(w_1 - w)(\overline{w} - w_1)}} dw_1, \tag{69}$$

where the contour l_w and the sign of the radical $\sqrt{(w_1 - w)(\overline{w} - w_1)}$ were defined in Sect. 3. The radical $\sqrt{b^2 - w_1^2}$ should be taken positive, if w_1 is located on the upper part of the imaginary axis. If we take $\xi = 0$ in (68), then

$$\psi(0, \eta) = \frac{1}{i} \int\limits_0^\pi \Psi\left(-\frac{1}{\eta}\right) \cos \lambda d\lambda.$$

Differentiating formula (68) with respect to ξ and putting $\xi = 0$, by (67), we get

$$\frac{\partial \psi(\xi, \eta)}{\partial \xi}\bigg|_{\xi=0} = -\int\limits_0^\pi \Psi'\left(-\frac{1}{\eta}\right) \frac{\sqrt{1 - b^2\eta^2}}{\eta^2} \cos^2 \lambda d\lambda$$

or

$$\left.\frac{\partial\psi(\xi,\eta)}{\partial\xi}\right|_{\xi=0} = -\frac{\pi}{2}\frac{\sqrt{1-b^2\eta^2}}{\eta^2}\Psi'\left(-\frac{1}{\eta}\right). \tag{70}$$

This formula is similar to formula (30). It gives us the function $\Psi(w)$ with an additive constant. However, the substitution $\Psi(w) = \text{const}$ into (68) gives us $\psi(\xi,\eta) = 0$. Therefore, the indicated uncertainty of the function $\Psi(w)$ is unessential.

An analog of formula (31) for values of $\psi(\xi,\eta)$ on the circle

$$\xi^2+\eta^2 = \frac{1}{b^2}$$

has the form

$$\left.\psi(\xi,\eta)\right|_{\xi^2+\eta^2=\frac{1}{b^2}} = \mp\frac{b}{\xi}\int\limits_{l_w}\frac{(1+\eta w_1)\Psi(w_1)}{(b^2-w_1^2)\sqrt{b^2-w_1^2}}dw_1, \qquad \eta>0, \tag{71}$$

where the upper sign corresponds to the case $\xi > 0$, the lower to the case $\xi < 0$, and the contour l_w is the same as in (31). An analog of (32) has the form

$$\psi(\xi,0) = \frac{1}{\xi^2}\int\limits_{l_w}\frac{w_1\Psi(w_1)}{(b^2-w_1^2)^{3/2}\sqrt{w_1^2+\beta^2}}dw_1, \tag{72}$$

where

$$\beta = \frac{\sqrt{1-b^2\xi^2}}{\xi},$$

and the radical $\sqrt{w_1^2+\beta^2}$ must be taken positive, if w_1 is real. If $\eta < 0$, then we have similar formulas with other signs, as in the case of the potential φ.

9. In the previous section, we studied solutions of equation (62), defined by (68) and (69). We now prove that any solution of equation (62), even with respect to ξ, can be represented by (68) inside the disk K,

$$\xi^2+\eta^2 < \frac{1}{b^2}.$$

We give a method to determine the function $\Psi(w)$ for a given $\psi(\xi,\eta)$. Our analysis is similar to the one in Sect. 4.

Consider the semidisk for $\eta > 0$. Let $\psi(\xi,\eta)$ be a solution of (62), odd with respect to ξ. This solution can be represented in the form

$$\psi(\xi,\eta) = \psi_1(\eta)\xi + \psi_3(\eta)\xi^3 + \psi_5(\eta)\xi^5 + \cdots \tag{73}$$

in a neighborhood of the ray $\xi = 0$.

Substituting this expression in (62) and equating the coefficient at ξ^n to zero, we obtain

$$n(n-1)b^2\psi_n(\eta) - (n+2)(n+1)\psi_{n+2}(\eta) + 2b^2n\eta\psi_n'(\eta)$$

$$+(b^2\eta^2-1)\psi_n''(\eta)+2b^2n\psi_n(\eta)+2b^2\eta\psi_n'(\eta)$$
$$-(n+2)\psi_{n+2}(\eta)+\psi_{n+2}(\eta)=0.$$

These equations completely determine the functions $\psi_3(\eta)$, $\psi_5(\eta)$, ..., and the solution satisfying the condition

$$\left.\frac{\partial\psi(\xi,\eta)}{\partial\xi}\right|_{\xi=0}=\psi_1(\eta) \qquad (74)$$

is unique. The function $\psi_1(\eta)$ must be analytic for $0<\eta<\dfrac{1}{a}$. It is easy to find $\Psi(w)$ such that the solution determined by formula (68) satisfies condition (74). Indeed, in view of (70), we have

$$\psi_1(\eta)=-\frac{\pi}{2}\frac{\sqrt{1-b^2\eta^2}}{\eta^2}\Psi'\left(-\frac{1}{\eta}\right) \qquad (75)$$

or

$$\Psi'(w)=-\frac{2}{\pi}\frac{\psi_1\left(-\dfrac{1}{w}\right)}{w\sqrt{w^2-b^2}}=i\frac{2}{\pi}\frac{\psi_1\left(-\dfrac{1}{w}\right)}{w\sqrt{b^2-w^2}},$$

where the radical $\sqrt{w^2-b^2}$ is positive, and $\sqrt{b^2-w^2}$ is positive imaginary for $w<-b$.

Formula (75) gives us $\Psi'(w)$ along the interval $-\infty<w<-a$ of the real axis, and this function is a function holomorphic in a neighborhood of this interval.

We can prove now that $\Psi(w)$ is holomorphic in the domain T_1, if $\psi(\xi,\eta)$ is analytic in the semidisk for $\eta>0$.

Let us introduce the new variable $\tau=\xi\cos\lambda$ instead of λ in (68). In the case $\xi>0$, we have

$$\psi(\xi,\eta)=\frac{1}{i}\int\limits_{-\xi}^{+\xi}\frac{\Psi(w_\tau)\tau\,d\tau}{\xi\sqrt{\xi^2-\tau^2}}, \qquad (76)$$

where

$$w_\tau=-\frac{\eta}{\tau^2+\eta^2}-i\frac{\tau\sqrt{1-b^2(\tau^2+\eta^2)}}{\tau^2+\eta^2}.$$

Multiplying both parts of (76) by

$$\frac{\xi^2}{\sqrt{\mu^2-\xi^2}}$$

and integrating with respect to ξ from 0 to μ, applying transformations similar to the ones done in Sect. 4, we have

$$\int_0^\mu \frac{\psi(\xi,\eta)\xi^2 d\xi}{\sqrt{\mu^2-\xi^2}} = \frac{\pi}{2i}\int_{-\mu}^{+\mu} \Psi(w_\tau)\tau d\tau.$$

Differentiating with respect to μ and substituting τ for ξ and ξ for μ, we obtain

$$\operatorname{Im}[\Psi(w)] = \frac{i}{\pi\xi}\frac{d}{d\xi}\int_0^\xi \frac{\psi(\tau,\eta)\tau^2 d\tau}{\sqrt{\xi^2-\tau^2}}, \tag{77}$$

where

$$w = -\frac{\eta}{\xi^2+\eta^2} - i\frac{\xi\sqrt{1-b^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}. \tag{78}$$

The following formula is similar to formula (42) of Sect. 4,

$$\operatorname{Im}[\Psi(w)] = \frac{i}{\pi\xi}\frac{d}{d\xi}\int_0^1 \frac{\psi(\xi\sigma,\eta)\xi^2\sigma^2 d\sigma}{\sqrt{1-\sigma^2}}. \tag{79}$$

Hence, as in Sect. 4, $\Psi(w)$ is a holomorphic function in the domain S_1, if $\psi(\xi,\eta)$ is analytic in the semidisk for $\eta > 0$.

If the function $\psi(\xi,\eta)$ is analytic on some arcs of the semicircle

$$\xi^2+\eta^2 = \frac{1}{b^2}, \quad \eta > 0,$$

then the function $\Psi(w)$ is holomorphic on the corresponding intervals of the cut $(-b,0)$.

Suppose, for example, that $\psi(\xi,\eta)$ is analytic on the arc AA_1 containing the point $\xi = 0, \quad \eta = \dfrac{1}{b}$.

In this case, the function $\psi_1(\eta)$ is analytic in a neighborhood of the point $\eta = \dfrac{1}{b}$. In a neighborhood of the point $w = -b$ formula (75) gives us the expression

$$\Psi'(w) = \frac{f_1(w)}{\sqrt{b^2-w^2}},$$

where $f_1(w)$ is holomorphic in this neighborhood. Integrating and putting $\Psi(-b) = 0$, we have

$$\Psi(w) = \sqrt{b^2-w^2}f(w), \tag{80}$$

where $f(w)$ is holomorphic at the point $w = -b$.

Suppose that $\psi(\xi,\eta)$ is equal to zero on the arc AA_1 symmetric with respect to the ray $\xi = 0$. In this case, we can make an analytic continuation of the function

$$f(w) = \frac{\Psi(w)}{\sqrt{b^2-w^2}}$$

along the corresponding lips of the cut $(-b, 0)$.

However, as we observed above, $f(w)$ is single-valued and holomorphic at the point $w = -b$ corresponding to the midpoint of the arc AA_1. Consequently, this function has the same values on the intervals of two lips of the cut.

It is easy to prove that $f(-b) = 0$ in the present case. Indeed, let us apply formula (71) at points of the arc AA_1,

$$\psi(\xi, \eta)\Bigg|_{\xi^2+\eta^2=\frac{1}{b^2}} = -\frac{b}{\xi}\int\limits_{l_w} \frac{(1+\eta w_1)f(w_1)}{b^2 - w_1^2}\,dw_1,$$

where l_w is a closed contour, inside which the function $f(w)$ is single-valued and holomorphic. Computing the residue of the integrand at the point $w_1 = -b$, we have

$$\psi(\xi, \eta)\Bigg|_{\xi^2+\eta^2=\frac{1}{b^2}} = -\frac{b}{\xi}\frac{(1-\eta b)f(-b)}{b}\pi i.$$

However, by assumption, the left side is equal to zero. Hence, $f(-b) = 0$.

It remains to construct a continuation of the solution $\psi(\xi, \eta)$ into the exterior of the disk

$$\xi^2 + \eta^2 < \frac{1}{b^2}.$$

We can use the same continuation method, as in the case of the potential $\varphi(\xi, \eta)$, since the real and imaginary parts of the function $\Psi(w)$ also satisfy equation (51), where we need only substitute b for a. We set $\Psi(w)$ to be a constant along every tangent to the semicircle between the tangency point and the axis $\eta = 0$, and this constant is equal to the value of $\Psi(w)$ at the tangency point. Instead of formula (55) for the points (ξ, η),

$$\xi > 0, \quad \frac{1}{b} > \eta > 0, \quad \xi^2 + \eta^2 > \frac{1}{b^2},$$

we have

$$\psi(\xi, \eta) = -\frac{1}{\xi\sqrt{\xi^2+\eta^2}\,i}\int\limits_{l_w} \frac{(w_1 + a^2\eta)(1+\eta w_1)\Psi(w_1)\,dw_1}{(a^2 - w_1^2)\sqrt{a^2 - w_1^2}\sqrt{(w_1 - w)(\overline{w} - w_1)}},$$

where l_w is the contour indicated in Sect. 6,

$$w = -\frac{\eta + \xi\sqrt{b^2(\xi^2+\eta^2) - 1}}{\xi^2+\eta^2}, \quad w_1 = -\frac{\eta - \xi\sqrt{b^2(\xi^2+\eta^2) - 1}}{\xi^2+\eta^2}.$$

Repeating the arguments of Sect. 6, we can show that $\psi(\xi, \eta)$ is equal to zero on the tangents such that the tangency points are located on the arc AA_1 symmetric with respect to $\xi = 0$, and $\psi(\xi, \eta)$ is equal to zero. All above results are also valid for $\eta < 0$.

10. Let us give now some examples. Searching for solutions of equation (62) in the form

$$\psi(\xi, \eta) = \xi^n f(\xi^2 + \eta^2),$$

by simple calculations, we can obtain the solution

$$\psi(\xi, \eta) = \frac{1 - b^2(\xi^2 + \eta^2)}{(\xi^2 + \eta^2)^{3/2}} \xi, \tag{81}$$

which is an odd function of ξ. For this solution,

$$\psi_1(\eta) = \frac{\partial \psi(\xi, \eta)}{\partial \xi}\bigg|_{\xi=0} = \frac{1 - b^2\eta^2}{\eta^3} \qquad \text{for} \quad \eta > 0,$$

$$\psi_1(\eta) = \frac{\partial \psi(\xi, \eta)}{\partial \xi}\bigg|_{\xi=0} = -\frac{1 - b^2\eta^2}{\eta^3} \qquad \text{for} \quad \eta < 0.$$

Applying formula (75), for $\eta > 0$ we have

$$\frac{1 - b^2\eta^2}{\eta^3} = -\frac{\pi}{2} \frac{\sqrt{1 - b^2\eta^2}}{\eta^2} \Psi'\left(-\frac{1}{\eta}\right).$$

Let

$$\eta = -\frac{1}{w}, \qquad w < 0.$$

Since the radical $\sqrt{1 - b^2\eta^2}$ must be positive, we obtain

$$w(b^2 - w^2) = \frac{\pi}{2}\sqrt{w^2 - b^2}\, w\Psi'(w),$$

where the radical $\sqrt{w^2 - b^2}$ is positive for $w < -b$. Also, we can write

$$w(b^2 - w^2) = -i\frac{\pi}{2}\sqrt{b^2 - w^2}\, w\Psi'(w),$$

where $\sqrt{b^2 - w^2}$ is positive imaginary for $w < -b$, and

$$\Psi'(w) = \frac{2}{\pi}\sqrt{b^2 - w^2} i.$$

For $\eta < 0$, we have

$$w(b^2 - w^2) = -\frac{\pi}{2}\sqrt{w^2 - b^2}\, w\Psi'(w),$$

where $\sqrt{w^2 - b^2}$ is positive for $w > b$. If the radical $\sqrt{b^2 - w^2}$ is positive imaginary for $w < -b$, then we have to take it negative imaginary for $w > b$, and the previous formula takes the form

$$w(b^2 - w^2) = -i\frac{\pi}{2}\sqrt{b^2 - w^2}\, w\Psi'(w),$$

and, consequently, for $\eta < 0$,

$$\Psi'(w) = \frac{2}{\pi}\sqrt{b^2 - w^2}i,$$

i.e., we have for $\eta > 0$ and for $\eta < 0$ the same analytic function $\Psi(w)$.

Also, note that our solution (81) is equal to zero on the circle

$$\xi^2 + \eta^2 = \frac{1}{b^2}.$$

Therefore, a continuation of this solution into the exterior of the disk is equal to zero for $|\eta| \leq \frac{1}{b}$. We put $\psi(\xi, \eta) = 0$ for $|\eta| > \frac{1}{b}$ too. We will always make this assumption, when $\psi(\xi, \eta)$ is equal to zero on the arc AA_1 of the circle $\xi^2 + \eta^2 = \frac{1}{b^2}$, symmetric with respect to $\xi = 0$.

11. We consider now some mechanical problems. First, let us clarify the mechanical sense of the homogeneous solutions of equations (2) and (3). As proved in our article [1], in the case of the two-dimensional problem, the homogeneous potentials give, for example, elastic vibrations under the action of an impact focused at some point and at some moment of time. In the given case, homogeneous potentials give us elastic vibrations under the action of an impact. Without loss of generality, we can assume that this point is the coordinate origin, and the moment of time is $t = 0$. Similarly to the two-dimensional case, we consider a force applied to some point as the limiting case of a force applied to points of a certain surface, convergent to the given point.

Let us consider a surface of revolution

$$f(\varrho, z) = 0,$$

where ϱ and z belong to bounded intervals. Let us construct homothetic surfaces S_ε,

$$f\left(\frac{\varrho}{\varepsilon}, \frac{z}{\varepsilon}\right) = 0 \tag{82}$$

as $\varepsilon \to 0$. We say that points of two surfaces S_{ε_1} and S_{ε_2} correspond, if the coordinates of these points are connected by the homothetic transformation

$$x_2 = \frac{\varepsilon_2}{\varepsilon_1}x_1, \quad y_2 = \frac{\varepsilon_2}{\varepsilon_1}y_1, \quad z_2 = \frac{\varepsilon_2}{\varepsilon_1}z_1.$$

Let us assume that a stress is applied at points of the surface S_ε at the moment $t = 0$. This stress does not depend on the angle ϑ whose components are products of $\frac{1}{\varepsilon^2}$ by a quantity independent of ε and constant at the corresponding points of the surfaces S_ε, as the parameter ε tends to zero. Also, suppose that the initial state of the elastic space is at rest at the moment $t = 0$. In this case, for every value of ε we have elastic vibrations with axial symmetry. Let $\varphi_\varepsilon(\varrho, z, t)$ and $\psi_\varepsilon(\varrho, z, t)$ be corresponding potentials.

These potentials must satisfy equations (2) and (3). Components of stresses are given on the surface S_ε. These boundary conditions have the form

$$\begin{gathered} D_1(\varphi_\varepsilon, \psi_\varepsilon) = \frac{1}{\varepsilon^2} X(x,y,z), \quad D_2(\varphi_\varepsilon, \psi_\varepsilon) = \frac{1}{\varepsilon^2} Y(x,y,z), \\ D_3(\varphi_\varepsilon, \psi_\varepsilon) = \frac{1}{\varepsilon^2} Z(x,y,z), \end{gathered} \tag{83}$$

where D_1, D_2, and D_3 are homogeneous linear functions of the second-order derivatives of the functions φ_ε and ψ_ε with respect to x, y, z with constant coefficients, and X, Y, and Z are functions, defined on the surface S_ε independent of the angle ϑ. We should also take into account the initial conditions which define the rest at $t = 0$. Then, the functions φ_ε and ψ_ε satisfy conditions (2), (3), boundary conditions (83) and the initial data, mentioned above. Let us construct functions

$$\varphi(\varrho, z, t) = \varphi_\varepsilon(k\varrho, kz, kt), \qquad \psi(\varrho, z, t) = \psi_\varepsilon(k\varrho, kz, kt), \tag{84}$$

where k is a constant. We have

$$\frac{\partial \varphi}{\partial \varrho} = \frac{\partial \varphi_\varepsilon(k\varrho, kz, kt)}{\partial (k\varrho)} k, \qquad \frac{\partial^2 \varphi}{\partial \varrho^2} = \frac{\partial^2 \varphi_\varepsilon(k\varrho, kz, kt)}{\partial (k\varrho)^2} k^2.$$

Obviously, functions (84) satisfy equations (2) and (3). Since $D_l(\varphi, \psi)$ are homogeneous linear functions of the second-order derivatives of φ and ψ with respect to x, y, and z with constant coefficients, we can write

$$D_l(\varphi, \psi)|_{(x,y,z)} = k^2 D_l(\varphi_\varepsilon, \psi_\varepsilon)|_{(kx,ky,kz)},$$

where the indexes (x, y, z) and (kx, ky, kz) denote the points where we should take the corresponding expressions. However, the functions φ_ε and ψ_ε satisfy conditions (83). Consequently,

$$\begin{gathered} D_1(\varphi, \psi) = \left(\frac{\varepsilon}{k}\right)^{-2} X(kx, ky, kz), \quad D_2(\varphi, \psi) = \left(\frac{\varepsilon}{k}\right)^{-2} Y(kx, ky, kz), \\ D_3(\varphi, \psi) = \left(\frac{\varepsilon}{k}\right)^{-2} Z(kx, ky, kz), \end{gathered}$$

i.e., φ and ψ satisfy boundary conditions on the surface $S_{\frac{\varepsilon}{k}}$ whose points have the coordinates

$$x' = kx, \quad y' = ky, \quad z' = kz,$$

where (x, y, z) is the point on S_ε, corresponding to the point (x', y', z') on the surface $S_{\frac{\varepsilon}{k}}$. By the given condition, the initial state for the potentials $\varphi_\varepsilon(\varrho, z, t)$ and $\psi_\varepsilon(\varrho, z, t)$ is the rest state.

Obviously, the potentials φ and ψ satisfy the same conditions as the potentials $\varphi_{\frac{\varepsilon}{k}}$ and $\psi_{\frac{\varepsilon}{k}}$,

$$\varphi_\varepsilon(k\varrho, kz, kt) = \varphi_{\frac{\varepsilon}{k}}(\varrho, z, t), \quad \psi_\varepsilon(k\varrho, kz, kt) = \psi_{\frac{\varepsilon}{k}}(\varrho, z, t).$$

As $\varepsilon \to 0$, we obtain potentials $\varphi_0(\varrho, z, t)$ and $\psi_0(\varrho, z, t)$ corresponding to the case of force applied at the origin at the time moment $t = 0$. The previous formulas give us

$$\varphi_0(k\varrho, kz, kt) = \varphi_0(\varrho, z, t), \quad \psi_0(k\varrho, kz, kt) = \psi_0(\varrho, z, t),$$

i.e., $\varphi_0(\varrho, z, t)$ and $\psi_0(\varrho, z, t)$ are homogeneous functions of order zero of the arguments (ϱ, z, t).

In conclusion, we note that we take the components of stresses of order $\dfrac{1}{\varepsilon^2}$, because the area of the surface S_ε is equal to the product of ε^2 by a constant independent of ε.

12. We now consider the problem on vibrations of the half-space $z > 0$ under the action of a force applied at the point $\varrho = 0$, $z = f$ at the moment of time $t = 0$. Assume that this force has axial symmetry and produces vibrations of longitudinal type. This source of vibrations is given by a potential φ which, as we observed above, is a homogeneous function of order zero of the arguments $(\varrho, z - f, t)$, i.e., this potential is a function of

$$\xi = \frac{\varrho}{t}, \quad \eta = \frac{z - f}{t}.$$

Taking into account that the initial state is at rest, we can assert that this function $\varphi(\xi, \eta)$ must be defined only inside the disk

$$\xi^2 + \eta^2 < \frac{1}{a^2}$$

and vanish on the boundary of this disk, corresponding to the front of propagation of a longitudinal wave. As proved above, one can represent this given potential in the form[3]

$$\varphi(\xi, \eta) = \int_0^\pi \Phi(\theta_\lambda) d\lambda, \tag{85}$$

where

$$\theta_\lambda = \frac{\xi \cos\lambda - i\eta\sqrt{1 - a^2(\xi^2 \cos^2\lambda + \eta^2)}}{\xi^2 \cos^2\lambda + \eta^2}. \tag{86}$$

Potential (85) determines completely the displacement for $0 < t < af$.

Outside this time interval we have to add two more reflected potentials: one for reflected longitudinal waves, and another for reflected transverse waves. These reflected potentials can be defined by the condition that there are no stresses on the surface $z = 0$. We can represent them as a combination of potentials of two-dimensional problems [1]. For these potentials, we have the expressions

[3] For similar formulas see Sect. 3. – *Ed.*

$$\varphi_1 = \int_0^\pi \Phi_1(\theta_\lambda^{(1)}) d\lambda, \qquad \psi_1 = \int_0^\pi \Psi_1(\theta_\lambda^{(2)}) \cos\lambda d\lambda,$$

where $\theta_\lambda^{(1)}$ and $\theta_\lambda^{(2)}$ are determined by the equations

$$t - \theta_\lambda^{(1)} \varrho \cos\lambda - \sqrt{a^2 - {\theta_\lambda^{(1)}}^2}(z+f) = 0,$$

$$t - \theta_\lambda^{(2)} \varrho \cos\lambda - \sqrt{b^2 - {\theta_\lambda^{(2)}}^2}\, z - \sqrt{a^2 - {\theta_\lambda^{(2)}}^2}\, f = 0.$$

For the functions $\Phi_1(\theta)$ and $\Psi_1(\theta)$, we have

$$\begin{aligned} \Phi_1'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\Phi'(\theta), \\ \Psi_1'(\theta) &= -\frac{4\theta(2\theta^2 - b^2)\sqrt{a^2-\theta^2}}{(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}\Phi'(\theta). \end{aligned} \tag{87}$$

Similarly, we can consider the case of transverse type source. In this case, the function ψ_1 defining the reflected transverse potential is a function of the arguments

$$\xi_1 = \frac{\varrho}{t}, \quad \eta_1 = \frac{z+f}{t}.$$

This function is not zero on two arcs of the semicircle

$$\xi_1^2 + \eta_1^2 = \frac{1}{b^2}, \quad \eta_1 > 0.$$

A continuation ψ_1 into the exterior of the semidisk $\xi_1^2 + \eta_1^2 < \dfrac{1}{b^2}$ gives us transverse vibrations produced by longitudinal vibrations propagating over the surface $z = 0$ with a speed greater than for the transverse vibrations. In our previous article, we already observed a similar phenomenon in the two-dimensional case. Our method also gives a solution of the problem on vibrations of a layer under the action of the source described above.

Let us make some essential additions to the previous arguments about the reflection of elastic waves in the three-dimensional space with axial symmetry. Suppose that the potential $\varphi(\xi, \eta)$ corresponding to longitudinal incident waves vanishes on the circle

$$\xi^2 + \eta^2 = \frac{1}{a^2},$$

or on a half-circle, or on some arc AA_1 symmetric with respect to the point $\xi = 0$ of the circle. As noted in Sect. 6, in this case, expressing $\varphi(\xi, \eta)$ in the integral form

$$\varphi(\xi, \eta) = \int_0^\pi \Phi(w_\lambda) d\lambda,$$

the function $\Phi(w)$, real on the interval $(-\infty, -a)$ of the real axis, is also regular at the point $w = -a$ and, therefore, is real and regular on some interval $(-\infty, -c)$, where $-a < -c \leq 0$. If $\varphi(\xi, \eta)$ is equal to zero on the semicircle, then, as we observed earlier, $c = 0$. Moreover, we must have $\Phi(-a) = 0$, i.e., in a neighborhood of the point $w = -a$ we have an expansion of the form

$$\Phi(w) = \sum_{n=1}^{\infty} a_n (w + a)^n,$$

where the coefficients a_n are real. Vice versa, if we have such expansion for Φ, then $\varphi(\xi, \eta)$ vanishes on some arc AA_1. Using the variable

$$w = \sqrt{a^2 - \theta^2},$$

where $w = -a$ for $\theta = 0$, we have the expansion

$$\Phi(\theta) = \sum_{n=1}^{\infty} b_n \theta^{2n},$$

where the coefficients b_n are real. The two expansions are equivalent. In a similar case, when the potential $\psi(\xi, \eta)$ vanishes on some arc AA_1 such that values of θ satisfy the condition $-b \leq \theta \leq b$, we have an expansion in the form (see Sect. 9)

$$\Psi(w) = (b^2 - w^2)^{3/2} \sum_{n=1}^{\infty} a_n (w + b)^n,$$

where the coefficients a_n are real. Using the variable

$$w = \sqrt{b^2 - \theta^2},$$

where $w = -b$ for $\theta = 0$, we obtain the equivalent expansion

$$\Psi(\theta) = \sum_{n=1}^{\infty} b_n \theta^{2n+1}$$

with real coefficients. And vice versa, having such expansion at the origin, the potential $\psi(\xi, \eta)$ vanishes on some arc AA_1.

Until now we studied the case when the equation on θ associates the real interval $-a \leq \theta \leq a$ or $-b \leq \theta \leq b$ to a semicircle. We do not always have such case under reflection. However, it is easy to prove that the previous result always occurs. In this case, AA_1 is a part of a front propagation of disturbances. Since fractions in (87) and in similar formulas for the reflection of transverse vibrations are functions regular for $\theta = 0$, even or odd with respect to θ, we can assert that after the reflection the functions $\Phi_1(\theta)$ and $\Psi_1(\theta)$ behave as described above, i.e., the reflected potentials $\varphi_1(x, y, t)$ and $\psi_1(x, y, t)$ vanish on parts of the front corresponding to the same values of θ, as before the reflection of the arc AA_1, i.e., for $-a \le \theta \le a$.

These results can be formulated differently. Namely, if the imaginary part of $\Phi(\theta)$ vanishes on some interval $-c < \theta < c$ $(c \le a)$ of the real axis, and $\Phi(0) = 0$, then

$$\operatorname{Re} \int_0^\pi \Phi(\theta_\lambda) d\lambda$$

gives us the longitudinal potential, where θ_λ is a root of the equation

$$t - \theta_\lambda \varrho \cos\lambda \pm \sqrt{a^2 - \theta_\lambda^2}\, z - \chi(\theta_\lambda) = 0.$$

This potential is equal to zero on the surface obtained by rotating the arc AA_1 around the z-axis. This arc corresponds to the interval $-c < \theta < c$, in view of the equation

$$t - \theta x \pm \sqrt{a^2 - \theta^2}\, z - \chi(\theta) = 0.$$

The function $\chi(\theta)$ is real on the indicated interval. A similar result occurs for the reflection of transverse waves. We come back to the study of these problems in detail later.

Finally, let us note that if the force acting at the point $\varrho = z = 0$ is a given function of time, then we can present it as a sum of forces acting at the given point at different moments of time. In this case, the potentials are the Stieltjes integrals of expressions $\varphi dQ_1(t - \tau_1)$ and $\psi dQ_2(t - \tau_1)$, where φ and ψ are elementary potentials produced by a force applied at a moment τ_1.

13. We solve now the Lamb problem [2] about the vibrations of the half-space $z > 0$ under the action of a force at the point $\varrho = z = 0$, parallel to the z-axis. We use the variable θ and consider the potentials φ and ψ in the form

$$\varphi = \int_0^\pi \Phi(\theta_1) d\lambda, \quad \psi = \int_0^\pi \Psi(\theta_2) \cos\lambda d\lambda, \tag{88}$$

where θ_1 and θ_2 are determined by the equations

$$\begin{aligned} \delta_1 &= t - \theta_1 \varrho \cos\lambda - \sqrt{a^2 - \theta_1^2}\, z = 0, \\ \delta_2 &= t - \theta_2 \varrho \cos\lambda - \sqrt{b^2 - \theta_2^2}\, z = 0. \end{aligned} \tag{89}$$

In these equations the radical signs are chosen such that values of θ from the upper half-plane correspond to rays in the half-space $z > 0$. The imaginary part of the function $\Phi(\theta)$ is an odd function at the points symmetric with respect to the imaginary axis. It occurs for the real part of the function $\Psi(\theta)$. These properties of the functions $\Phi(\theta)$ and $\Psi(\theta)$ follow from the fact that the imaginary axis of θ corresponds to the real axis of w, and the expression for ψ does not contain the factor $\dfrac{1}{i}$.

Let u, v, w be components of the displacement along the axes ϱ, ϑ, z. It is known that the components of the strain tensor are expressed by the formulas

$$\varepsilon_\varrho = \frac{\partial u}{\partial \varrho}, \quad \varepsilon_\vartheta = \frac{1}{\varrho}\frac{\partial v}{\partial \vartheta} + \frac{u}{\varrho}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{\varrho\vartheta} = \frac{1}{\varrho}\frac{\partial u}{\partial \vartheta} + \frac{\partial v}{\partial \varrho} - \frac{v}{\varrho}, \quad \gamma_{\vartheta z} = \frac{\partial v}{\partial z} + \frac{1}{\varrho}\frac{\partial w}{\partial \vartheta}, \quad \gamma_{z\varrho} = \frac{\partial w}{\partial \varrho} + \frac{\partial u}{\partial z}.$$

In the case of axial symmetry $v = 0$, and the components $u = q$ and w do not depend on θ. Hence,

$$\varepsilon_\varrho = \frac{\partial q}{\partial \varrho}, \quad \varepsilon_\vartheta = \frac{q}{\varrho}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{\varrho\vartheta} = \gamma_{\vartheta z} = 0, \quad \gamma_{z\varrho} = \frac{\partial w}{\partial \varrho} + \frac{\partial u}{\partial z}.$$

For the stress components, we have

$$T_{zz} = \lambda\left[\frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho q) + \frac{\partial w}{\partial z}\right] + 2\mu\frac{\partial w}{\partial z},$$

$$T_{\varrho z} = \mu\left(\frac{\partial w}{\partial \varrho} + \frac{\partial q}{\partial z}\right),$$

$$T_{\vartheta z} = 0,$$

or, in view of (1),

$$\frac{1}{\mu}T_{zz} = \frac{\lambda}{\mu}\left[\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho\frac{\partial \varphi}{\partial \varrho} - \varrho\frac{\partial \psi}{\partial \varrho}\right) + \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \varrho \partial z} + \frac{1}{\varrho}\frac{\partial \psi}{\partial z}\right]$$
$$+2\left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \varrho \partial z} + \frac{1}{\varrho}\frac{\partial \psi}{\partial z}\right),$$

$$\frac{1}{\mu}T_{\varrho z} = \frac{\partial^2 \varphi}{\partial z \partial \varrho} + \frac{\partial^2 \psi}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial \psi}{\partial \varrho} - \frac{1}{\varrho^2}\psi + \frac{\partial^2 \varphi}{\partial \varrho \partial z} - \frac{\partial^2 \psi}{\partial z^2}.$$

By simple transformations, we obtain

$$\frac{1}{\mu}T_{zz} = \frac{\lambda}{\mu}\left(\frac{1}{\varrho}\frac{\partial \varphi}{\partial \varrho} + \frac{\partial^2 \varphi}{\partial \varrho^2} + \frac{\partial^2 \varphi}{\partial z^2}\right) + 2\left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \varrho \partial z} + \frac{1}{\varrho}\frac{\partial \psi}{\partial z}\right),$$

$$\frac{1}{\mu}T_{\varrho z} = 2\frac{\partial^2 \varphi}{\partial \varrho \partial z} + \frac{\partial^2 \psi}{\partial \varrho^2} - \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{\varrho}\frac{\partial \psi}{\partial \varrho} - \frac{1}{\varrho^2}\psi.$$

Using equations (2), (3), and the formula

$$\frac{\lambda}{\mu} = \frac{b^2}{a^2} - 2,$$

we have

$$\begin{aligned} \frac{1}{\mu} T_{zz} &= (b^2 - 2a^2)\frac{\partial^2 \varphi}{\partial t^2} + 2\frac{\partial^2 \varphi}{\partial z^2} + 2\frac{\partial^2 \psi}{\partial \varrho \partial z} + \frac{2}{\varrho}\frac{\partial \psi}{\partial z}, \\ \frac{1}{\mu} T_{\varrho z} &= 2\frac{\partial^2 \varphi}{\partial \varrho \partial z} + b^2 \frac{\partial^2 \psi}{\partial t^2} - 2\frac{\partial^2 \psi}{\partial z^2}. \end{aligned} \tag{90}$$

Let us consider formulas (88) and construct expressions for the derivatives of φ and ψ with respect to ϱ and z.

In our previous article we studied the equation

$$\delta = t - \theta x - \sqrt{c^2 - \theta^2}\, y = 0$$

and obtained the formulas for the derivatives of a smooth function $f(\theta)$ of θ with respect to x, y and t,

$$\frac{\partial f(\theta)}{\partial y} = f'(\theta)\frac{\sqrt{c^2-\theta^2}}{\delta'}, \qquad \frac{\partial^2 f(\theta)}{\partial x^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{\theta^2}{\delta'}\right],$$

$$\frac{\partial^2 f(\theta)}{\partial x \partial y} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{\theta\sqrt{c^2-\theta^2}}{\delta'}\right], \qquad \frac{\partial^2 f(\theta)}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{c^2-\theta^2}{\delta'}\right],$$

$$\frac{\partial^2 f(\theta)}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{1}{\delta'}\right],$$

where

$$\delta' = \frac{\partial \delta}{\partial \theta} = -x + \frac{\theta}{\sqrt{c^2-\theta^2}}y.$$

Using formulas of such type in the case of equations (89), for the function $\Phi(\theta_1)$ we have

$$\begin{aligned} \frac{\partial^2 \Phi(\theta_1)}{\partial \varrho \partial z} &= \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{\theta_1\sqrt{a^2-\theta_1^2}}{\delta_1'}\right]\cos\lambda, \\ \frac{\partial^2 \Phi(\theta_1)}{\partial z^2} &= \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{a^2-\theta_1^2}{\delta_1'}\right], \\ \frac{\partial^2 \Phi(\theta_1)}{\partial t^2} &= \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{1}{\delta_1'}\right], \end{aligned} \tag{91}$$

and for the function $\Psi(\theta_2)$ we obtain

$$\begin{aligned}
\frac{\partial \Psi(\theta_2)}{\partial z} &= \Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'},\\
\frac{\partial^2 \Psi(\theta_2)}{\partial \varrho \partial z} &= \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{\theta_2\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\cos\lambda,\\
\frac{\partial^2 \Psi(\theta_2)}{\partial z^2} &= \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{b^2-\theta_2^2}{\delta_2'}\right],\\
\frac{\partial^2 \Psi(\theta_2)}{\partial t^2} &= \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{1}{\delta_2'}\right].
\end{aligned} \tag{92}$$

Let us denote by $T_{\varrho z}^{(1)}$, $T_{zz}^{(1)}$ the parts of stresses arising from the potential φ, and by $T_{\varrho z}^{(2)}$, $T_{zz}^{(2)}$ from the potential ψ, respectively. Taking into account (88), (90), and (91), we have

$$\frac{1}{\mu}T_{\varrho z}^{(1)} = \int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{2\theta_1\sqrt{a^2-\theta_1^2}}{\delta_1'}\right]\cos\lambda d\lambda,$$

$$\frac{1}{\mu}T_{zz}^{(1)} = \int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{b^2-2a^2}{\delta_1'}\right]d\lambda + \int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{2a^2-2\theta_1^2}{\delta_1'}\right]d\lambda$$

or

$$\begin{aligned}
\frac{1}{\mu}T_{\varrho z}^{(1)} &= \int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{2\theta_1\sqrt{a^2-\theta_1^2}}{\delta_1'}\right]\cos\lambda d\lambda,\\
\frac{1}{\mu}T_{zz}^{(1)} &= \int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial \theta_1}\left[\Phi'(\theta_1)\frac{b^2-2\theta_1^2}{\delta_1'}\right]d\lambda.
\end{aligned} \tag{93}$$

Similarly, by (88), (90), and (92),

$$\frac{1}{\mu}T_{\varrho z}^{(2)} = \int_0^\pi \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{b^2}{\delta_2'}\right]\cos\lambda d\lambda - \int_0^\pi \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{2b^2-2\theta_2^2}{\delta_2'}\right]\cos\lambda d\lambda$$

or

$$\frac{1}{\mu}T_{\varrho z}^{(2)} = \int_0^\pi \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2^2-b^2}{\delta_2'}\right]\cos\lambda d\lambda \tag{94}$$

and

$$\frac{1}{\mu}T_{zz}^{(2)} = \int_0^\pi \frac{1}{\delta_2'}\frac{\partial}{\partial \theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\cos^2\lambda d\lambda$$

$$+\frac{2}{\varrho}\int\limits_0^{\pi}\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\cos\lambda d\lambda. \tag{95}$$

Integrating the second integral by parts and taking into account that

$$\delta_2' = -\varrho\cos\lambda + \frac{\theta_2}{\sqrt{b^2-\theta_2^2}}z$$

depends on λ through θ_2, we have

$$\frac{2}{\varrho}\int\limits_0^{\pi}\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\cos\lambda d\lambda = -\frac{2}{\varrho}\int\limits_0^{\pi}\sin\lambda\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\frac{\partial\theta_2}{\partial\lambda}d\lambda$$

$$+\frac{2}{\varrho}\int\limits_0^{\pi}\sin\lambda\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{(\delta_2')^2}\varrho\sin\lambda d\lambda.$$

However, it is obvious that

$$\frac{\partial\theta_2}{\partial\lambda} = -\frac{\theta_2\varrho\sin\lambda}{\delta_2'}$$

and, consequently,

$$\frac{2}{\varrho}\int\limits_0^{\pi}\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\cos\lambda d\lambda = 2\int\limits_0^{\pi}\left\{\frac{\theta_2}{\delta_2'}\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\right.$$

$$\left.+\frac{1}{\delta_2'}\Psi'(\theta_2)\frac{\sqrt{b^2-\theta_2^2}}{\delta_2'}\right\}\sin^2\lambda d\lambda = \int\limits_0^{\pi}\frac{1}{\delta_2'}\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\sin^2\lambda d\lambda.$$

Substituting the equality into expression (95), we have

$$\frac{1}{\mu}T_{zz}^{(2)} = \int\limits_0^{\pi}\frac{1}{\delta_2'}\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]d\lambda, \tag{96}$$

and formulas (93), (94), and (96) give us the following expressions for the stresses inside the half-space:

$$\begin{aligned}\frac{1}{\mu}T_{\varrho z} &= \int\limits_0^{\pi}\left\{\frac{1}{\delta_1'}\frac{\partial}{\partial\theta_1}\left[\Phi'(\theta_1)\frac{2\theta_1\sqrt{a^2-\theta_1^2}}{\delta_1'}\right] + \frac{1}{\delta_2'}\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2^2-b^2}{\delta_2'}\right]\right\}\cos\lambda d\lambda,\\ \frac{1}{\mu}T_{zz} &= \int\limits_0^{\pi}\left\{\frac{1}{\delta_1'}\frac{\partial}{\partial\theta_1}\left[\Phi'(\theta_1)\frac{b^2-2\theta_1^2}{\delta_1'}\right] + \frac{1}{\delta_2'}\frac{\partial}{\partial\theta_2}\left[\Psi'(\theta_2)\frac{2\theta_2\sqrt{b^2-\theta_2^2}}{\delta_2'}\right]\right\}d\lambda,\end{aligned} \tag{97}$$

where

$$\delta_1' = \frac{\theta_1 z - \sqrt{a^2-\theta_1^2}\,\varrho\cos\lambda}{\sqrt{a^2-\theta_1^2}}, \quad \delta_2' = \frac{\theta_2 z - \sqrt{b^2-\theta_2^2}\,\varrho\cos\lambda}{\sqrt{b^2-\theta_2^2}}. \tag{98}$$

Let us study the limits of expressions (97), as z tends to zero, i.e., the stresses on the surface $z = 0$. Consider, for example, the first term in the expression for T_{zz},

$$\int_0^\pi \frac{1}{\delta_1'}\frac{\partial}{\partial\theta_1}\left[\varPhi'(\theta_1)\frac{b^2-2\theta_1^2}{\delta_1'}\right]d\lambda. \tag{99}$$

Let us introduce instead of λ a new variable of integration θ_1. The first equation in (89) gives us

$$\cos\lambda = \frac{t-\sqrt{a^2-\theta_1^2}\,z}{\varrho\theta_1},$$

$$\lambda = \arccos\frac{t-\sqrt{a^2-\theta_1^2}\,z}{\varrho\theta_1} = \arccos\frac{1-\sqrt{a^2-\theta_1^2}\,\eta}{\xi\theta_1}.$$

Hence we have

$$\begin{aligned} d\lambda &= \pm\frac{a^2z - t\sqrt{a^2-\theta_1^2}}{\theta_1\sqrt{a^2-\theta_1^2}}\,\frac{1}{\sqrt{\varrho^2\theta_1^2-(t-\sqrt{a^2-\theta_1^2}\,z)^2}}d\theta_1 \\ &= \pm\frac{a^2\eta - \sqrt{a^2-\theta_1^2}}{\theta_1\sqrt{a^2-\theta_1^2}}\,\frac{1}{\sqrt{\xi^2\theta_1^2-(1-\sqrt{a^2-\theta_1^2}\,\eta)^2}}d\theta_1. \end{aligned} \tag{100}$$

Substituting $w_1 = \sqrt{a^2-\theta_1^2}$ for θ_1 in the expression

$$\xi^2\theta_1^2 - (1-\sqrt{a^2-\theta_1^2}\,\eta)^2,$$

we obtain

$$-(\xi^2+\eta^2)w_1^2 + 2\eta w_1 + (a^2\xi^2-1).$$

This expression has the roots

$$w = \frac{\eta \pm i\xi\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2} = \frac{zt \pm i\varrho\sqrt{t^2-a^2(\varrho^2+z^2)}}{\varrho^2+z^2}.$$

Similarly, the expression

$$\xi^2\theta_1^2 - (1-\sqrt{a^2-\theta_1^2}\,\eta)^2$$

has the roots

$$\theta = \frac{\xi + i\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}, \quad \theta^{(*)} = \frac{-\xi + i\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}.$$

This fact follows immediately from the formula $w = \sqrt{a^2-\theta^2}$, which associates the upper half-plane of θ to a domain S_2 [4] of the plane w.

The expression in (100) has the branching points θ and $\theta^{(*)}$ on the upper half-plane. We can make it single-valued by using cuts parallel to the real axis. The cuts go from the points θ and $\theta^{(*)}$ to infinity and do not cross the imaginary axis. For the variable θ_1, we have

$$\theta_1 = \frac{\xi\cos\lambda + i\eta\sqrt{1-a^2(\xi^2\cos^2\lambda+\eta^2)}}{\xi^2\cos^2\lambda+\eta^2}. \tag{101}$$

Putting $\lambda = \dfrac{\pi}{2}$, we obtain

$$\theta_1^{(0)} = \frac{i\sqrt{1-a^2\eta^2}}{\eta}, \quad \sqrt{a^2-\theta_1^{(0)\,2}} = \frac{1}{\eta}.$$

For this value of λ, $d\theta_1$ is negative for $\xi > 0$ and positive for $\xi < 0$. Formula (100) gives us

$$d\lambda = \frac{a^2\eta - \dfrac{1}{\eta}}{i\dfrac{\sqrt{1-a^2\eta^2}}{\eta^2}} \frac{d\theta_1}{\sqrt{\xi^2\theta_1^{(0)\,2} - \left(1-\sqrt{a^2-\theta_1^{(0)\,2}}\,\eta\right)^2}}.$$

The difference $a^2\eta - \dfrac{1}{\eta}$ is negative, and we have to take the radical

$$\sqrt{\xi^2\theta_1^{(0)\,2} - \left(1-\sqrt{a^2-\theta_1^{(0)\,2}}\,\eta\right)^2}$$

negative imaginary in the case of $\xi > 0$ and positive imaginary in the case of $\xi < 0$.

Obviously, this rule will be valid for the radical

$$\sqrt{\xi^2\theta_1^2 - (1-\sqrt{a^2-\theta_1^2}\,\eta)^2}$$

in the case of positive imaginary values of θ_1. Integral (99) has the form

$$\int_{l_\theta} \frac{1}{\delta_1'}\frac{\partial}{\partial\theta_1}\left[\varPhi'(\theta_1)\frac{b^2-2\theta_1^2}{\delta_1'}\right]\frac{a^2\eta-\sqrt{a^2-\theta_1^2}}{\theta_1\sqrt{a^2-\theta_1^2}}$$

[4] S_2 is the part of the complex plane w, located on the right of the imaginary axis, with the cut $(0, a)$. – *Ed.*

$$\times \frac{1}{\sqrt{\xi^2\theta_1^2 - (1 - \sqrt{a^2 - \theta_1^2}\,\eta)^2}} d\theta_1, \tag{102}$$

where l_θ is a contour going from the point θ to the point $\theta^{(*)}$. On the plane w this contour corresponds to the contour l_w often mentioned above. Also, note that δ_1' is expressed by the formula

$$\delta_1' = \frac{\theta_1 z - \sqrt{a^2 - \theta_1^2}\,\varrho\cos\lambda}{\sqrt{a^2 - \theta_1^2}}$$

$$= \frac{\theta_1 z - \varrho\sqrt{a^2 - \theta_1^2}\,\dfrac{t - \sqrt{a^2 - \theta_1^2}\,z}{\varrho\theta_1}}{\sqrt{a^2 - \theta_1^2}} = \frac{a^2 z - t\sqrt{a^2 - \theta_1^2}}{\theta_1\sqrt{a^2 - \theta_1^2}},$$

and this function does not have roots on the upper half-plane. Therefore, the integrand in (102) is a holomorphic function over the line $\theta\theta^{(*)}$. Thus, by the Cauchy theorem, we can deform the contour of integration and, for example, always take for this contour the upper semicircle, such that the interval $\theta\theta^{(*)}$ is the diameter. Letting z go to zero, we have

$$\eta \to 0, \quad \delta_1' \to -\frac{t}{\theta_1}, \quad \theta \to \frac{1}{\xi}, \quad \theta^{(*)} \to -\frac{1}{\xi}, \quad \cos\lambda \to \frac{t}{\varrho\theta_1},$$

and expression (102) becomes

$$-\frac{1}{t^2}\int\limits_{1/\xi}^{-1/\xi} \frac{\partial}{\partial\theta_1}\Big(\varPhi'(\theta_1)\theta_1(b^2 - 2\theta_1^2)\Big)\frac{d\theta_1}{\sqrt{\xi^2\theta_1^2 - 1}},$$

where the radical $\sqrt{\xi^2\theta_1^2 - 1}$ is negative imaginary, if $\xi > 0$, and θ_1 is located on the upper part of the imaginary axis, and positive imaginary for $\xi < 0$.

In a similar way, for the second term for T_{zz} in (97) we have

$$-\frac{1}{t^2}\int\limits_{1/\xi}^{-1/\xi} \frac{\partial}{\partial\theta_2}\Big(\varPsi'(\theta_2)2\theta_2^2\sqrt{b^2 - \theta_2^2}\Big)\frac{d\theta_2}{\sqrt{\xi^2\theta_2^2 - 1}},$$

and, denoting the variable of integration by the same letter θ_1, we obtain

$$-\frac{1}{\mu}T_{zz} = \frac{1}{t^2}\int\limits_{1/\xi}^{-1/\xi} \frac{\omega_1(\theta_1)d\theta_1}{\sqrt{\xi^2\theta_1^2 - 1}},$$

where

$$\omega_1(\theta_1) = \frac{\partial}{\partial\theta_1}\Big(\varPhi'(\theta_1)\theta_1(b^2 - 2\theta_1^2) + \varPsi'(\theta_1)2\theta_1^2\sqrt{b^2 - \theta_1^2}\Big). \tag{103}$$

For the stress $T_{\varrho z}$ we have

$$-\frac{1}{\mu}T_{\varrho z} = \frac{1}{t\varrho}\int\limits_{1/\xi}^{-1/\xi}\frac{\omega_2(\theta_1)d\theta_1}{\theta_1\sqrt{\xi^2\theta_1^2-1}},$$

where

$$\omega_2(\theta_1) = \frac{\partial}{\partial\theta_1}\left(\Phi'(\theta_1)2\theta_1^2\sqrt{a^2-\theta_1^2}+\Psi'(\theta_1)\theta_1(2\theta_1^2-b^2)\right). \tag{104}$$

The boundary conditions on the surface $z=0$ give us

$$\int\limits_{1/\xi}^{-1/\xi}\frac{\omega_1(\theta_1)d\theta_1}{\sqrt{\xi^2\theta_1^2-1}} = 0, \qquad \int\limits_{1/\xi}^{-1/\xi}\frac{\omega_2(\theta_1)d\theta_1}{\theta_1\sqrt{\xi^2\theta_1^2-1}} = 0. \tag{105}$$

In the case of the Lamb problem, as we observed in our previous article, the front of the longitudinal wave is the circle $x^2+y^2=\dfrac{1}{a^2}t^2$, and the front of the transverse wave is the arc of the circle $x^2+y^2=\dfrac{1}{b^2}t^2$, symmetric with respect to $y=0$, and two segments of the tangents in the ends of this arc. This fact tells us that in the present case the potential $\varphi(\xi,\eta)$ is equal to zero on the entire semicircle

$$\xi^2+\eta^2=\frac{1}{a^2}, \qquad \eta>0,$$

and the potential $\psi(\xi,\eta)$ is equal to zero on the arc of the semicircle

$$\xi^2+\eta^2=\frac{1}{b^2}, \qquad \eta>0,$$

symmetric with respect to the axis $\xi=0$. However, we observed above (see Sect. 6) that in this case the function $\Phi(w)$ has the zero $w=-a$ and can be expressed in its neighborhood in the form

$$\Phi(w) = \alpha_1(w+a)+\alpha_2(w+a)^2+\cdots.$$

Similarly, the function $\Psi(w)$ can be expressed in a neighborhood of $w=-b$ in the form (see Sect. 9)

$$\Psi(w) = \sqrt{b^2-w^2}\left(\beta_1(w+b)+\beta_2(w+b)^2+\cdots\right).$$

Introducing for $\Phi(w)$ instead of w the new variable θ

$$w=\sqrt{a^2-\theta^2}, \quad w+a=\gamma_2\theta^2+\gamma_4\theta^4+\cdots,$$

in a neighborhood of the point $\theta = 0$, we have

$$\Phi(\theta) = \alpha_2'\theta^2 + \alpha_4'\theta^4 + \cdots .$$

Similarly, introducing for $\Psi(w)$,

$$w = \sqrt{b^2 - \theta^2},$$

we have

$$\Psi(\theta) = \beta_3'\theta^3 + \beta_5'\theta^5 + \cdots .$$

Formulas (103) and (104) give us

$$\begin{aligned} \omega_1(\theta) &= \alpha_1''\theta + \alpha_3''\theta^3 + \cdots , \\ \omega_2(\theta) &= \beta_2''\theta^2 + \beta_4''\theta^4 + \cdots . \end{aligned} \tag{106}$$

We assume that these functions do not have singularities in a bounded domain.

By assumption, the imaginary part of the function $\Phi(\theta)$ and the real part of the function $\Psi(\theta)$ must vanish on the imaginary axis. Therefore, in view of (103) and (104), the real part of $\omega_1(\theta_1)$ and the imaginary part of $\omega_2(\theta_1)$ must vanish on the imaginary axis. Thus, the coefficients α_n'' and β_n'' must be real. Obviously, boundary conditions (105) hold. Substituting expression (106) into formulas (103) and (104), we obtain for $\Phi'(\theta)$ and $\Psi'(\theta)$ the equations

$$\begin{aligned} (b^2 - 2\theta^2)\Phi'(\theta) + 2\theta\sqrt{b^2 - \theta^2}\Psi'(\theta) &= \omega_3(\theta), \\ 2\theta\sqrt{a^2 - \theta^2}\Phi'(\theta) - (b^2 - 2\theta^2)\Psi'(\theta) &= \omega_4(\theta), \end{aligned}$$

where

$$\omega_3(\theta) = \frac{1}{2}\alpha_1''\theta + \frac{1}{4}\alpha_3''\theta^3 + \cdots , \quad \omega_4(\theta) = \frac{1}{3}\beta_2''\theta^2 + \frac{1}{5}\beta_4''\theta^4 + \cdots . \tag{107}$$

These equations give us

$$\begin{aligned} \Phi'(\theta) &= \frac{(b^2 - 2\theta^2)\omega_3(\theta) + 2\theta\sqrt{b^2 - \theta^2}\omega_4(\theta)}{F(\theta)}, \\ \Psi'(\theta) &= \frac{2\theta\sqrt{a^2 - \theta^2}\omega_3(\theta) - (b^2 - 2\theta^2)\omega_4(\theta)}{F(\theta)}, \end{aligned} \tag{108}$$

where

$$F(\theta) = (b^2 - 2\theta^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}. \tag{109}$$

We see that the functions $\Phi'(\theta)$ and $\Psi'(\theta)$ have poles at points where $F(\theta)$ is equal to zero. As noted in the previous work, these points $\theta = \pm c$ correspond to the Rayleigh waves. Moreover, functions (108) have one more singular point $\theta = \infty$ corresponding to the value $\xi = \eta = 0$, i.e., the point where the force is applied. Let us assume that this point is a pole, and the order of this pole is small as possible. Formulas (107) and (108) show us that these conditions include the case

$$\omega_4(\theta) \equiv 0, \quad \alpha_3'' = \alpha_5'' = \cdots = 0.$$

The number $\frac{1}{2}\alpha_1''$ is real. Denoting it by β, we have

$$\Phi'(\theta) = \beta \frac{\theta(b^2 - 2\theta^2)}{F(\theta)}, \quad \Psi'(\theta) = \beta \frac{2\theta^2\sqrt{a^2 - \theta^2}}{F(\theta)}. \tag{110}$$

14. Let us construct formulas for the displacement components q and w. For the potentials we have

$$\varphi = \int_0^\pi \Phi(\theta_1) d\lambda, \quad \psi = \int_0^\pi \Psi(\theta_2) \cos\lambda d\lambda, \tag{111}$$

where θ_1 and θ_2 satisfy the equations

$$\begin{aligned} \delta_1 &= t - \theta_1 \varrho \cos\lambda - \sqrt{a^2 - \theta_1^2}\, z = 0, \\ \delta_2 &= t - \theta_2 \varrho \cos\lambda - \sqrt{b^2 - \theta_2^2}\, z = 0. \end{aligned} \tag{112}$$

From (1) we have

$$q = \int_0^\pi \Phi'(\theta_1) \frac{\partial\theta_1}{\partial\varrho} d\lambda - \int_0^\pi \Psi'(\theta_2) \frac{\partial\theta_2}{\partial z} \cos\lambda d\lambda,$$

$$w = \int_0^\pi \Phi'(\theta_1) \frac{\partial\theta_1}{\partial z} d\lambda + \int_0^\pi \Psi'(\theta_2) \frac{\partial\theta_2}{\partial\varrho} \cos\lambda d\lambda + \frac{1}{\varrho} \int_0^\pi \Psi(\theta_2) \cos\lambda d\lambda.$$

Integrating by parts in the last integral, we obtain

$$w = \int_0^\pi \Phi'(\theta_1) \frac{\partial\theta_1}{\partial z} d\lambda + \int_0^\pi \Psi'(\theta_2) \frac{\partial\theta_2}{\partial\varrho} \cos\lambda d\lambda - \frac{1}{\varrho} \int_0^\pi \Psi'(\theta_2) \frac{\partial\theta_2}{\partial\lambda} \sin\lambda d\lambda.$$

By (112), for the derivatives we have

$$\frac{\partial\theta_1}{\partial t} = -\frac{1}{\delta_1'}, \qquad \frac{\partial\theta_1}{\partial\varrho} = \frac{\theta_1\cos\lambda}{\delta_1'} = -\theta_1\cos\lambda \frac{\partial\theta_1}{\partial t},$$

$$\frac{\partial\theta_1}{\partial z} = \frac{\sqrt{a^2 - \theta_1^2}}{\delta_1'} = -\sqrt{a^2 - \theta_1^2}\, \frac{\partial\theta_1}{\partial t},$$

$$\frac{\partial\theta_2}{\partial t} = -\frac{1}{\delta_2'}, \qquad \frac{\partial\theta_2}{\partial\varrho} = -\theta_2\cos\lambda \frac{\partial\theta_2}{\partial t}, \qquad \frac{\partial\theta_2}{\partial z} = -\sqrt{b^2 - \theta_2^2}\, \frac{\partial\theta_2}{\partial t},$$

$$\frac{\partial\theta_2}{\partial\lambda} = -\frac{\theta_2\varrho\sin\lambda}{\delta_2'} = \theta_2\varrho\sin\lambda\frac{\partial\theta_2}{\partial t}.$$

Using these expressions, we obtain

$$q = \int_0^\pi \left[-\theta_1\Phi'(\theta_1)\frac{\partial\theta_1}{\partial t} + \sqrt{b^2-\theta_2^2}\Psi'(\theta_2)\frac{\partial\theta_2}{\partial t}\right]\cos\lambda d\lambda,$$

$$w = \int_0^\pi \left[-\sqrt{a^2-\theta_1^2}\Phi'(\theta_1)\frac{\partial\theta_1}{\partial t} - \theta_2\Psi'(\theta_2)\frac{\partial\theta_2}{\partial t}\right]d\lambda$$

or, taking into account (110),

$$\begin{aligned} q &= \beta\int_0^\pi \left[\frac{\theta_1^2(2\theta_1^2-b^2)}{F(\theta_1)}\frac{\partial\theta_1}{\partial t} + \frac{2\theta_2^2\sqrt{a^2-\theta_2^2}\sqrt{b^2-\theta_2^2}}{F(\theta_2)}\frac{\partial\theta_2}{\partial t}\right]\cos\lambda d\lambda, \\ w &= \beta\int_0^\pi \left[\frac{\theta_1(2\theta_1^2-b^2)\sqrt{a^2-\theta_1^2}}{F(\theta_1)}\frac{\partial\theta_1}{\partial t} - \frac{2\theta_2^3\sqrt{a^2-\theta_2^2}}{F(\theta_2)}\frac{\partial\theta_2}{\partial t}\right]d\lambda. \end{aligned} \tag{113}$$

To determine the constant β we need to use the value of force applied at the point $\varrho = z = 0$. Let P be the value of this force. Expand asymptotically q and w as $t \to +\infty$. Computing the limit, we must obtain the displacement components for the static problem, where the vertical force acts at the point $\varrho = z = 0$ on the surface of the half-space $z > 0$. For this static problem, the solution is known [3]

$$\begin{aligned} q &= u\cos\vartheta + v\sin\vartheta = \frac{P}{4\pi\mu}\left\{\frac{\varrho z}{(\varrho^2+z^2)^{3/2}} + \frac{a^2}{a^2-b^2}\left(\frac{1}{\varrho} - \frac{z}{\varrho\sqrt{\varrho^2+z^2}}\right)\right\}, \\ w &= \frac{P}{4\pi\mu}\left(\frac{z^2}{(\varrho^2+z^2)^{3/2}} - \frac{b^2}{a^2-b^2}\frac{1}{\sqrt{\varrho^2+z^2}}\right). \end{aligned} \tag{114}$$

Let us take formulas (113) and expand the integrands in powers of $\frac{1}{t}$. For the variable θ_1, we have the equation

$$t - \theta_1\varrho\cos\lambda - \sqrt{a^2-\theta_1^2}\,z = 0. \tag{115}$$

Hence the expansion for θ_1 in a neighborhood of the point $t = \infty$ has the form

$$\theta_1 = c_1 t + c_0 + c_{-1}\frac{1}{t} + \cdots, \qquad \frac{\partial\theta_1}{\partial t} = c_1 - c_{-1}\frac{1}{t^2} - \cdots. \tag{116}$$

Equation (115) can be written in the form

$$t - \theta_1 \varrho \cos\lambda + i\theta_1 \left(1 - \frac{a^2}{\theta_1^2}\right)^{1/2} z = 0$$

or

$$t - \theta_1 \varrho \cos\lambda + i\theta_1 z - i\frac{a^2 z}{2\theta_1} - i\frac{a^4 z}{8\theta_1^3} - \cdots = 0.$$

Using (116), we have

$$t - (\varrho\cos\lambda - iz)\left(c_1 t + c_0 + c_{-1}\frac{1}{t} + \cdots\right)$$

$$-i\frac{a^2 z}{2c_1 t}\left[1 + \left(\frac{c_0}{c_1 t} + \frac{c_{-1}}{c_1 t^2} + \cdots\right)\right]^{-1} - \cdots = 0.$$

Equating the coefficients of t, t^0, and t^{-1} to zero, we obtain

$$c_1 = \frac{1}{\varrho\cos\lambda - iz}, \quad c_0 = 0, \quad c_{-1} = -i\frac{a^2 z}{2}. \tag{117}$$

Expand now the function

$$\frac{\theta_1^2(2\theta_1^2 - b^2)}{(b^2 - 2\theta_1^2)^2 + 4\theta_1^2\sqrt{a^2 - \theta_1^2}\sqrt{b^2 - \theta_1^2}}$$

in a neighborhood of the point $\theta = \infty$ or $t = \infty$,

$$\frac{2\theta_1^4 - b^2\theta_1^2}{4\theta_1^4 - 4b^2\theta_1^2 + b^4 - 4\theta_1^4\left(1 - \frac{a^2}{\theta_1^2}\right)^{1/2}\left(1 - \frac{b^2}{\theta_1^2}\right)^{1/2}}$$

$$= \frac{2\theta_1^4 - b^2\theta_1^2}{(2a^2 - 2b^2)\theta_1^2 + \left[\frac{(b^2 - a^2)^2}{2} + b^4\right] + \cdots}$$

$$= \frac{1}{a^2 - b^2}\theta_1^2 - \frac{a^4 + b^4}{4(a^2 - b^2)^2} + \cdots$$

$$= \frac{c_1^2}{a^2 - b^2}t^2 + \left[\frac{2c_1 c_{-1}}{a^2 - b^2} - \frac{a^4 + b^4}{4(a^2 - b^2)^2}\right] + \cdots .$$

Hence,

$$\frac{\theta_1^2(2\theta_1^2 - b^2)}{F(\theta_1)}\frac{\partial\theta_1}{\partial t} = \frac{c_1^3}{a^2 - b^2}t^2 + \left[-\frac{c_1^2 c_{-1}}{a^2 - b^2} + \frac{2c_1^2 c_{-1}}{a^2 - b^2} - \frac{(a^4 + b^4)c_1}{4(a^2 - b^2)^2}\right] + \cdots .$$

Similarly,

$$\theta_2 = d_1 t + d_{-1}\frac{1}{t} + \cdots , \quad \frac{\partial\theta_2}{\partial t} = d_1 - d_{-1}\frac{1}{t^2} - \cdots ,$$

where

$$d_1 = \frac{1}{\varrho\cos\lambda - iz}; \quad d_{-1} = -i\frac{b^2 z}{2} \tag{118}$$

and

$$\frac{2\theta_2^2\sqrt{a^2-\theta_2^2}\sqrt{b^2-\theta_2^2}}{F(\theta_2)} = \frac{1}{2} - \frac{(b^2-2\theta_2^2)^2}{2F(\theta_2)}$$

$$= \frac{1}{2} - \frac{4\theta_2^4 - 4b^2\theta_2^2 + b^4}{2(2a^2-2b^2)\theta_2^2 + [(b^2-a^2)^2 + 2b^4] + \cdots}$$

$$= -\frac{1}{a^2-b^2}\theta_2^2 + \frac{b^4 - 2a^2b^2 + 3a^4}{4(a^2-b^2)^2} + \cdots$$

$$= -\frac{d_1^2}{a^2-b^2}t^2 + \left[-\frac{d_1 d_{-1}}{a^2-b^2} + \frac{b^4-2a^2b^2+3a^4}{4(a^2-b^2)^2} + \cdots\right].$$

Consequently,

$$\frac{2\theta_2^2\sqrt{a^2-\theta_2^2}\sqrt{b^2-\theta_2^2}}{F(\theta_2)}\,\frac{\partial\theta_2}{\partial t}$$

$$= -\frac{d_1^3}{a^2-b^2}t^2 + \left[\frac{d_1^2 d_{-1}}{a^2-b^2} - 2\frac{d_1^2 d_{-1}}{a^2-b^2} + \frac{(b^4-2a^2b^2+3a^4)d_1}{4(a^2-b^2)^2}\right] + \cdots .$$

Substituting expressions (117) and (118) for c_1, c_{-1}, d_1, and d_{-1}, we have

$$\frac{\theta_1^2(2\theta_1^2-b^2)}{F(\theta_1)}\,\frac{\partial\theta_1}{\partial t} + \frac{2\theta_2^2\sqrt{a^2-\theta_2^2}\sqrt{b^2-\theta_2^2}}{F(\theta_2)}\,\frac{\partial\theta_2}{\partial t}$$

$$= \frac{c_1^2 c_{-1} - d_1^2 d_{-1}}{a^2-b^2} + \frac{2a^4-2a^2b^2}{4(a^2-b^2)^2}c_1 + \cdots$$

$$= -\frac{iz}{2(\varrho\cos\lambda - iz)^2} + \frac{a^2}{2(a^2-b^2)(\varrho\cos\lambda - iz)} + \cdots$$

$$= \frac{a^2\varrho\cos\lambda - iz(2a^2-b^2)}{2(a^2-b^2)(\varrho\cos\lambda - iz)^2} + \cdots ,$$

where the sum of omitted terms has order $\dfrac{1}{t}$. Hence, in view of the first formula in (113), we have

$$q|_{t=\infty} = \beta\int_0^\pi \frac{a^2\varrho\cos\lambda - iz(2a^2-b^2)}{2(a^2-b^2)(\varrho\cos\lambda - iz)^2}\cos\lambda\, d\lambda.$$

By similar computations, we can also obtain

$$w|_{t=\infty} = \beta\int_0^\pi \frac{ib^2\varrho\cos\lambda - z(a^2-2b^2)}{2(a^2-b^2)(\varrho\cos\lambda - iz)^2}\, d\lambda.$$

Integrating with respect to λ, we have

$$q|_{t=\infty} = \frac{\pi\beta}{2}\left[\frac{\varrho z}{(\varrho^2+z^2)^{3/2}} + \frac{a^2}{a^2-b^2}\left(\frac{1}{\varrho} - \frac{z}{\varrho\sqrt{\varrho^2+z^2}}\right)\right],$$

$$w|_{t=\infty} = \frac{\pi\beta}{2}\left[\frac{z^2}{(\varrho^2+z^2)^{3/2}} - \frac{b^2}{a^2-b^2}\frac{1}{\sqrt{\varrho^2+z^2}}\right].$$

Taking into account (114), we can determine the constant

$$\beta = \frac{P}{2\pi^2\mu}. \tag{119}$$

Substituting this expression for β in (113), we have the formulas presenting a solution of the Lamb problem.

For points of the surface $z = 0$, these formulas are given in H. Lamb's work cited. For the general case $z \geq 0$, they were obtained by S. L. Sobolev [4] by a method different from the method of the present work.

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[5] Paper [2] of Part I of this book. – *Ed.*
[6] Paper [1] of Part I of this book. – *Ed.*

4. On Vibrations of a Half-Plane and a Layer with Arbitrary Initial Conditions*

S. L. Sobolev

The plane problem on vibrations of a half-plane and an elastic layer has been repeatedly studied by many authors.

In the majority of papers on this question particular solutions of the equations of elasticity are found in the form of stationary sinusoidal modes. However, this approach presents considerable difficulties if we want to obtain the general solution of the problem under arbitrary initial and boundary conditions.

Our paper is based on a different principle, namely, on the method of characteristics.

We apply the principle of reflected waves to construct a particular solution, and then use the well-known method of G. Green and B. Riemann.

The advantages of this approach are to avoid the use of Fourier integrals, which can be difficult in the case of continuous spectra.

V. Volterra was the first who applied this method to equations in the plane theory of elasticity. Our paper has points of contact with a famous memoir of this notable mathematician.

Another essential feature of our approach is the application of the theory of functions of one complex variable to problems of this type, which had already been used in the paper by V. I. Smirnov and S. L. Sobolev [1]. This new method allows us to construct a class of solutions of the equations of elasticity for which the reflection principle is easily established.

We show that by means of these solutions any arbitrary solution of the problem can be obtained. For this purpose we have to adapt the Volterra formula, which represents a direct generalization of the well-known Green formula.

1. To avoid complications we restrict ourselves to vibrations of the half-plane, because vibrations of the layer are in essence the same; the only difference is that for the layer several reflections have to be considered.

Our problem is to integrate the elasticity equations

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$$\begin{aligned} \varrho\frac{\partial^2 u}{\partial t^2} &= (\lambda+2\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+X,\\ \varrho\frac{\partial^2 v}{\partial t^2} &= (\lambda+2\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+Y \end{aligned} \tag{1}$$

with the initial conditions

$$\begin{aligned} u|_{t=T(x,y)} &= u_0(x,y), & \left.\frac{\partial u}{\partial t}\right|_{t=T(x,y)} &= u_0'(x,y),\\ v|_{t=T(x,y)} &= v_0(x,y), & \left.\frac{\partial v}{\partial t}\right|_{t=T(x,y)} &= v_0'(x,y) \end{aligned} \tag{2}$$

and the boundary conditions

$$\begin{aligned} X_y|_{y=0} &= \mu\left.\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right|_{y=0}=0,\\ Y_y|_{y=0} &= \left.\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y}\right]\right|_{y=0}=0. \end{aligned} \tag{3}$$

The mechanical meaning of the boundary conditions is that stresses vanish on the boundary of the medium. For the sake of simplicity, we limit ourselves in this paper only to the case when $T=0$, $X=0$, $Y=0$, although the proposed method allows us to obtain solutions for arbitrary T, X, and Y.

Let us now recall the Volterra fundamental formula, which generalizes the Green formula. We prove it in a somewhat simpler form than V. Volterra did.

Let (u,v) and (u_1,v_1) be any two solutions of (1).

For the first solution, we denote the stress components by X_x, X_y, Y_y and the components of the external forces by X, Y. We denote the corresponding quantities for the second solution (u_1,v_1) by $X_{x,1}$, $X_{y,1}$, $Y_{y,1}$ and X_1, Y_1. It is known that[1]

$$\begin{aligned} X_x &= \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x}, & X_{x,1} &= \lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\mu\frac{\partial u_1}{\partial x},\\ X_y &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), & X_{y,1} &= \mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right),\\ Y_y &= \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y}, & Y_{y,1} &= \lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\mu\frac{\partial v_1}{\partial y}. \end{aligned} \tag{4}$$

We consider a domain Ω bounded by a surface S in the space with the coordinates (x,y,t). Let ν be the direction of the inward normal to this surface.

[1] For details see S. L. Sobolev "Some questions of the theory of propagation of vibrations" in the book: Frank, F., Mises, R: Differential and Integral Equations of Mathematical Physics. Vol. 2. ONTI, Leningrad – Moscow (1937). – *Ed.*

Consider the following integral over the surface S,

$$I = \iint\limits_S \Big\{ [u_1 X_x + v_1 X_y - u X_{x,1} - v X_{y,1}] \cos \nu x$$

$$+ [u_1 X_y + v_1 Y_y - u X_{y,1} - v Y_{y,1}] \cos \nu y$$

$$- \varrho \left[u_1 \frac{\partial u}{\partial t} + v_1 \frac{\partial v}{\partial t} - u \frac{\partial u_1}{\partial t} - v \frac{\partial v_1}{\partial t} \right] \cos \nu t \Big\} \, dS.$$

Replacing the stresses by their expressions (4) and applying the Gauss formula, we obtain

$$I = \iiint\limits_\Omega \Big\{ u \left[(\lambda + 2\mu) \frac{\partial}{\partial x} \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} \right) - \varrho \frac{\partial^2 u_1}{\partial t^2} \right]$$

$$+ v \left[(\lambda + 2\mu) \frac{\partial}{\partial y} \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) - \mu \frac{\partial}{\partial x} \left(\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} \right) - \varrho \frac{\partial^2 v_1}{\partial t^2} \right]$$

$$- u_1 \left[(\lambda + 2\mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \varrho \frac{\partial^2 u}{\partial t^2} \right]$$

$$- v_1 \left[(\lambda + 2\mu) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \varrho \frac{\partial^2 v}{\partial t^2} \right] \Big\} \, dx \, dy \, dt.$$

Using equation (1), we arrive at the final result[2]

$$\iint\limits_S \Big\{ [u_1 X_x + v_1 X_y - u X_{x,1} - v X_{y,1}] \cos \nu x$$

$$+ [u_1 X_y + v_1 Y_y - u X_{y,1} - v Y_{y,1}] \cos \nu y$$

$$- \varrho \left[u_1 \frac{\partial u}{\partial t} + v_1 \frac{\partial v}{\partial t} - u \frac{\partial u_1}{\partial t} - v \frac{\partial v_1}{\partial t} \right] \cos \nu t \Big\} \, dS$$

$$= \iiint\limits_\Omega \{ u_1 X + v_1 Y - u X_1 - v Y_1 \} \, dx \, dy \, dt. \tag{5}$$

In all that follows formula (5) plays a main role. In our case the formula is simplified, because the triple integral in it vanishes.

2. Formula (5) has been derived under the assumption that equations (1) are valid throughout Ω. However, we have to apply this formula also when certain partial derivatives of the solution (u_1, v_1) are discontinuous or even become infinite on isolated surfaces inside Ω.

Let us clarify now conditions under which the formula remains valid even for this case.

[2] Formula (5) is known as the Volterra formula. – *Ed.*

For definiteness we suppose that there is just one surface of discontinuity Σ inside Ω. Suppose also that at every point this surface has a continuously changing tangent plane, and that the components of displacement (u_1, v_1), as well as their derivatives in the direction tangential to Σ, all vary continuously on passage across Σ[3]. We also need one more condition. We require that the quantities

$$\begin{gathered} \varrho\frac{\partial u_1}{\partial t}\cos\nu t - X_{x,1}\cos\nu x - X_{y,1}\cos\nu y, \\ \varrho\frac{\partial v_1}{\partial t}\cos\nu t - X_{y,1}\cos\nu x - Y_{y,1}\cos\nu y \end{gathered} \tag{6}$$

are also continuous on Σ. In other words, quantities (6) tend to a limit at any point M of Σ, and these limits are the same when Σ is approached from either side.

Under these hypotheses we can divide Ω into two parts by Σ, and apply formula (5) to each half. Adding up the obtained relations, the integrals over Σ disappear and we again arrive at (5) for the entire domain Ω.

Continuity conditions (6) are sometimes referred to as the *dynamic conditions of compatibility* of the elasticity problem. They are also consequences of the equations of elasticity in the integral form

$$\begin{gathered} \iint\limits_S \left(\varrho\frac{\partial u_1}{\partial t}\cos\nu t - X_{x,1}\cos\nu x - X_{y,1}\cos\nu y\right) dS = -\iiint\limits_\Omega X_1\,dx\,dy\,dt, \\ \iint\limits_S \left(\varrho\frac{\partial v_1}{\partial t}\cos\nu t - X_{y,1}\cos\nu x - Y_{y,1}\cos\nu y\right) dS = -\iiint\limits_\Omega Y_1\,dx\,dy\,dt. \end{gathered} \tag{7}$$

This point of view is extremely general; however, we cannot go into details here.

Continuity conditions for quantities (6) also admit a geometric interpretation, according to which the velocity of propagation of the line of discontinuity in the (x, y)-plane can only take two well-defined values. This fact is well known, however, the usual proof assumes that the derivatives of the components of displacement are finite on each side of the surface Σ. For our purposes it is very important to get rid of this restriction, because for solutions we encounter later these derivatives in fact go off to infinity on approaching Σ. For this reason we now give a full proof of the fact mentioned above.

We rewrite expressions (6) in the form

$$\begin{aligned} \varrho\frac{\partial u_1}{\partial t}\cos\nu t - \left[\lambda\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + 2\mu\frac{\partial u_1}{\partial x}\right]\cos\nu x \\ -\mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\cos\nu y, \end{aligned} \tag{6.1}$$

[3] This condition is sometimes called the *kinematic condition of compatibility*. – *Ed.*

$$\varrho\frac{\partial v_1}{\partial t}\cos\nu t-\left[\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\mu\frac{\partial v_1}{\partial y}\right]\cos\nu y-\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)\cos\nu x.$$

The continuity conditions for the tangential derivatives mean that the quantities

$$\begin{aligned}
&\frac{\partial u_1}{\partial x}\cos\nu y-\frac{\partial u_1}{\partial y}\cos\nu x, &&\frac{\partial v_1}{\partial x}\cos\nu y-\frac{\partial v_1}{\partial y}\cos\nu x,\\
&\frac{\partial u_1}{\partial y}\cos\nu t-\frac{\partial u_1}{\partial t}\cos\nu y, &&\frac{\partial v_1}{\partial y}\cos\nu t-\frac{\partial v_1}{\partial t}\cos\nu y,\\
&\frac{\partial u_1}{\partial t}\cos\nu x-\frac{\partial u_1}{\partial x}\cos\nu t, &&\frac{\partial v_1}{\partial t}\cos\nu x-\frac{\partial v_1}{\partial x}\cos\nu t
\end{aligned}\tag{8}$$

vary continuously on passage across Σ. Thus, our conditions yield the following eight quantities, which vary continuously on passage across Σ:

$$\begin{aligned}
&\varrho\frac{\partial u_1}{\partial t}\cos\nu t-(\lambda+2\mu)\frac{\partial u_1}{\partial x}\cos\nu x-\mu\frac{\partial u_1}{\partial y}\cos\nu y-\mu\frac{\partial v_1}{\partial x}\cos\nu y\\
&\qquad\qquad-\lambda\frac{\partial v_1}{\partial y}\cos\nu x=M_1,\\
&\varrho\frac{\partial v_1}{\partial t}\cos\nu t-\lambda\frac{\partial u_1}{\partial x}\cos\nu y-\mu\frac{\partial u_1}{\partial y}\cos\nu x-\mu\frac{\partial v_1}{\partial x}\cos\nu x\\
&\qquad\qquad-(\lambda+2\mu)\frac{\partial v_1}{\partial y}\cos\nu y=M_2,\\
&\frac{\partial u_1}{\partial x}\cos\nu y-\frac{\partial u_1}{\partial y}\cos\nu x=M_3, \qquad \frac{\partial v_1}{\partial x}\cos\nu y-\frac{\partial v_1}{\partial y}\cos\nu x=M_6,\\
&-\frac{\partial u_1}{\partial t}\cos\nu y+\frac{\partial u_1}{\partial y}\cos\nu t=M_4, \qquad -\frac{\partial v_1}{\partial t}\cos\nu y+\frac{\partial v_1}{\partial y}\cos\nu t=M_7,\\
&\frac{\partial u_1}{\partial t}\cos\nu x-\frac{\partial u_1}{\partial x}\cos\nu t=M_5, \qquad \frac{\partial v_1}{\partial t}\cos\nu x-\frac{\partial v_1}{\partial x}\cos\nu t=M_8.
\end{aligned}\tag{9}$$

Consider the matrix of coefficients at the derivatives of the functions u and v:

$$\left\|\begin{matrix}
\varrho\cos\nu t & 0 & -(\lambda+2\mu)\cos\nu x & -\mu\cos\nu y & -\mu\cos\nu y & -\lambda\cos\nu x\\
0 & \varrho\cos\nu t & -\lambda\cos\nu y & -\mu\cos\nu x & -\mu\cos\nu x & -(\lambda+2\mu)\cos\nu y\\
0 & 0 & \cos\nu y & -\cos\nu x & 0 & 0\\
-\cos\nu y & 0 & 0 & \cos\nu t & 0 & 0\\
\cos\nu x & 0 & -\cos\nu t & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cos\nu y & -\cos\nu x\\
0 & -\cos\nu y & 0 & 0 & 0 & \cos\nu t\\
0 & \cos\nu x & 0 & 0 & -\cos\nu t & 0
\end{matrix}\right\|. \tag{10}$$

If this matrix is of rank 6, then we can choose a subsystem of six equations from (9) for which all derivatives of the displacements can be defined uniquely as continuous functions. Hence for $\varSigma$ to be a surface of discontinuity it is necessary that matrix (10) have rank less than 6.

It is easy to verify that matrix (10) may have six determinants of order 6 not vanishing identically; their values are $Q\cos^2 \nu t$, $Q\cos^2 \nu x$, $Q\cos^2 \nu y$, $Q\cos \nu x \cos \nu t$, $Q\cos \nu y \cos \nu t$, $Q\cos \nu x \cos \nu y$, respectively, where Q is an expression of the form

$$[\varrho\cos^2 \nu t - (\lambda+2\mu)(\cos^2 \nu x + \cos^2 \nu y)][\varrho\cos^2 \nu t - \mu(\cos^2 \nu x + \cos^2 \nu y)].$$

Taking into account that $\cos^2 \nu t$, $\cos^2 \nu x$ and $\cos^2 \nu y$ cannot vanish simultaneously, we obtain the condition

$$\begin{aligned}&[\varrho\cos^2 \nu t - (\lambda+2\mu)(\cos^2 \nu x + \cos^2 \nu y)]\\ &\times[\varrho\cos^2 \nu t - \mu(\cos^2 \nu x + \cos^2 \nu y)] = 0.\end{aligned} \tag{11}$$

This equation leads us to two types of surfaces of discontinuity,

$$\varrho\cos^2 \nu t - (\lambda+2\mu)(\cos^2 \nu x + \cos^2 \nu y) = 0, \tag{12}$$

$$\varrho\cos^2 \nu t - \mu(\cos^2 \nu x + \cos^2 \nu y) = 0. \tag{13}$$

The discontinuities of types (12) and (13) are called *longitudinal* and *transverse*, respectively.

The geometrical meaning of such classification will become quite clear after simple analysis.

Consider the coordinate system (x, y) chosen so that at a given point M of the surface $\varSigma$ the y-axis is tangent to the surface, i.e., we suppose that $\cos \nu y = 0$. Then, by (9), for a discontinuity of type (12) all derivatives of v_1 must be continuous at the point M.

Indeed, by virtue of the choice of the coordinate system, we can assert that the derivatives $\dfrac{\partial u_1}{\partial y}$ and $\dfrac{\partial v_1}{\partial y}$ are continuous. The second and the eighth equations of (9) take in our case the form

$$\varrho\frac{\partial v_1}{\partial t}\cos \nu t - \mu\frac{\partial v_1}{\partial x}\cos \nu x = M_2', \qquad \frac{\partial v_1}{\partial t}\cos \nu x - \frac{\partial v_1}{\partial x}\cos \nu t = M_8,$$

where M_2' and M_8 vary continuously on passage across $\varSigma$.

Because of (12), the determinant of this system is nonzero, so $\dfrac{\partial v_1}{\partial x}$ and $\dfrac{\partial v_1}{\partial t}$ can be found as continuous functions on passage across $\varSigma$. In general, we may claim that in the case given by equality (12) all derivatives of the displacement components in the tangential direction to $\varSigma$, in other words, all derivatives of a coordinate corresponding to the tangent to the line of discontinuity in the (x, y)-plane, are continuous on passage across this surface (this line).

Similarly, in the case when (13) holds, with the same choice of coordinates, all derivatives of u_1 are continuous, i.e., in this case the derivatives of the displacement coordinate corresponding to the normal to the line of discontinuity are continuous.

The preceding considerations had the purpose of deriving necessary consequences of the assumptions we have made. We may now claim that our results are also sufficient for (5) to be applicable.

More precisely, suppose that the surface Σ satisfies conditions (12), the tangential derivatives of (u_1, v_1) are continuous on passage across Σ, and, in addition, all derivatives of the displacement component corresponding to the tangent to the line of discontinuity in the (x, y)-plane are continuous on passage across this line. Then we may claim that the quantities in (6) are continuous. Hence formula (5) is applicable. If instead of (12) we have equality (13), then for the above conditions we need to replace the tangential component of the discontinuity line by the normal component of this line.

Indeed, consider condition (12) and all other conditions mentioned above. We choose the coordinate axes in the (x, y)-plane so that $\cos \nu y = 0$. Then the following derivatives are continuous: $\dfrac{\partial u_1}{\partial y}, \dfrac{\partial v_1}{\partial t}, \dfrac{\partial v_1}{\partial x}, \dfrac{\partial v_1}{\partial y}$.

The first of expressions (6.1) reduces in our case to the form

$$\varrho \frac{\partial u_1}{\partial t} \cos \nu t - (\lambda + 2\mu) \frac{\partial u_1}{\partial x} \cos \nu x + M,$$

where M is continuous on passage across Σ. By (12), we have

$$\varrho \frac{\partial u_1}{\partial t} \cos \nu t - (\lambda + 2\mu) \frac{\partial u_1}{\partial x} \cos \nu x = \varrho \frac{\cos \nu t}{\cos \nu x} \left(\frac{\partial u_1}{\partial t} \cos \nu x - \frac{\partial u_1}{\partial x} \cos \nu t \right).$$

Hence the first quantity in (6.1) is continuous. The continuity of the second quantity is obvious. In the case when condition (13) holds, the proof is obvious. We mention also two very useful special cases when formula (5) is valid. The first is when condition (12) holds, the tangential derivatives are continuous and, in addition, the vector (u_1, v_1) is potential, i.e., $\dfrac{\partial u_1}{\partial y} = \dfrac{\partial v_1}{\partial x}$. Choosing the x-axis and the y-axis as above, it is easy to see that $\dfrac{\partial v_1}{\partial x}$ and $\dfrac{\partial v_1}{\partial t}$ are continuous. Thus, all requirements indicated above are fulfilled. The second case is when (13) holds, the tangential derivatives are continuous and, in addition, the vector (u_1, v_1) is solenoidal, i.e., $\dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0$. The proof is similar to the first case.

We now point out a consequence of the preceding arguments. Consider the first case. Suppose that the vector (u_1, v_1) vanishes on Σ and is defined only on one side of the surface. It is easy to see that the vector (u_1, v_1) can be extended to the other side of Σ as $u_1 = 0$, $v_1 = 0$.

The previous arguments show that quantities (6.1) remain continuous, therefore they vanish on Σ. A similar result holds in the second case as well.

Finally, we point out that in both cases under the given hypotheses, the integrals over Σ in formula (5) vanish.

The surface Σ satisfying (12) or (13) is often called the characteristic of the equations of elasticity. The angle between the t-axis and the normal to the longitudinal characteristic is equal to $\arctan \frac{1}{a}$, while the angle between the t-axis and the normal to the transverse characteristic is $\arctan \frac{1}{b}$.

3. For further arguments we need to recall basic features of the new method of studying the plane problem on elastic vibrations as proposed in the paper [1]. This method consists in the application of the theory of functions of a complex variable to this problem.

We recall the main idea of the method in detail, because we have to apply it here with certain modifications.

Consider in the (x, y, t)-space the wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \nabla^2 \varphi \qquad (c = a, b). \tag{14}$$

In this space we take an equation that is linear with respect to x, y, t, whose coefficients are analytic functions of a complex variable Φ,

$$l(\Phi)t + m(\Phi)x + n(\Phi)y = q(\Phi). \tag{15}$$

If the coefficients l, m and n satisfy the equation

$$l^2(\Phi) = c^2[m^2(\Phi) + n^2(\Phi)], \tag{16}$$

then the complex-valued function Φ defined by (15) as a function of x, y, t satisfies equation (14).

A similar claim holds for the real and imaginary parts of Φ.

Thus, we can define a class of solutions of equation (14) in the form

$$\varphi = \operatorname{Re}(\Phi), \tag{17}$$

where Φ satisfies (15).

The proof is completely obvious. We have to find the second derivatives of Φ and verify that they satisfy equation (14). The above theorem is completely analogous to a corresponding theorem in the theory of characteristics for first-order equations.

It should be noted here that this theorem is valid also in the case when values of Φ defined by (15) are real for all x, y, t in some domain of the three-dimensional space and the coefficients l, m, n, and q are also real. This is a particular case of a more general situation.

It is useful to normalize the method for obtaining solutions of equation (14) in the above form.

We can divide equation (15) by $l(\Phi)$ and introduce the new variable $\theta = \dfrac{m(\Phi)}{l(\Phi)}$; then equation (15) takes the form

$$t + \theta x \pm \sqrt{\frac{1}{c^2} - \theta^2} y = \chi(\theta), \tag{18}$$

and φ is determined by the equation

$$\varphi = \operatorname{Re}(\varPhi(\theta)), \tag{19}$$

where $\varPhi$ is an analytic function of θ. Formula (19) expresses the fact that φ is a harmonic function of two variables: the real and the imaginary parts of θ.

Formula (19) provides a solution of equation (18) also when (18) has a real root in some domain of the (x, y, t)-space. In this case formula (19) gives us the real part of an analytic function on the real axis. In other words, in this case we obtain an arbitrary function of the real variable θ.

Formula (19) leads to a simple construction of a class of solutions of equations of the elasticity theory.

It is well known that in the absence of external forces the general solution of equations (1) can be written as the sum of two vectors

$$(u, v) = (u_1, v_1) + (u_2, v_2). \tag{20}$$

The potential vector (u_1, v_1) satisfies the condition

$$\frac{\partial u_1}{\partial y} = \frac{\partial v_1}{\partial x} \tag{21}$$

and the wave equation

$$\frac{\partial^2 (u_1, v_1)}{\partial t^2} = a^2 \nabla^2 (u_1, v_1), \qquad a^2 = \frac{\lambda + 2\mu}{\varrho}. \tag{22}$$

The solenoidal vector (u_2, v_2) satisfies the condition

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{23}$$

and the wave equation

$$\frac{\partial^2 (u_2, v_2)}{\partial t^2} = b^2 \nabla^2 (u_2, v_2), \qquad b^2 = \frac{\mu}{\varrho}. \tag{24}$$

Each component of both vectors can be obtained from corresponding equalities of type (19).

As we will see later, using this approach we will be able to obtain in the form of (19) a particular solution of the problem in question. This approach was used by V. Volterra in his famous memoir.

On the other hand, this method allows us easily to satisfy boundary conditions (3) with the help of some simple reflection operations. For this purpose it suffices to superimpose on one of the solutions of the equations of type (19) two more solutions of the same type (the reflected waves).

4. Now we consider in detail the theory of reflections.

First of all it is useful to recall some simple properties of solutions of the above type.

Consider the manifold $\theta = \text{const}$.

As we can see from equation (18), for real θ bounded between $-\dfrac{1}{c}$ and $+\dfrac{1}{c}$ the function $\chi(\theta)$ is also real, and the manifolds $\theta = \text{const}$ are planes in three-dimensional space making an angle $\arctan c$ with the t-axis.

On the other hand, the values of θ different from the above values define lines rather than planes; this is so because for real x, y and t we need to separate the real and imaginary parts in equation (18).

Hence we obtain a pair of real linear equations which determine a line.

The set of all lines corresponding to values of θ outside the interval $\left(-\dfrac{1}{c}, +\dfrac{1}{c}\right)$ fills out, generally speaking, a domain in the (x, y, t)-space. The boundary surface of this domain must be composed of lines corresponding to the deleted values of θ. For the study of elastic vibrations of some part of the plane, its boundary must correspond in the (x, y, t)-space to a cylindrical surface with generators parallel to the t-axis, and this surface also determines a part of the boundary of the mentioned above domain. Let us investigate more closely those parts of the boundary surface that are generated by real values of θ in the interval $\left(-\dfrac{1}{c}, +\dfrac{1}{c}\right)$. We find the equations for the limiting position of the line when the complex variable θ tends to the real value θ_0 from that interval. Taking into account that, by condition, $\chi(\theta)$ is real-valued on a part of or the whole interval $\left(-\dfrac{1}{c}, +\dfrac{1}{c}\right)$, we can write equations of the line corresponding to the complex value of θ in the form of two equations

$$t + \theta x \pm \sqrt{\frac{1}{c^2} - \theta^2}\, y - \chi(\theta) = 0, \quad t + \overline{\theta} x \pm \sqrt{\frac{1}{c^2} - \overline{\theta}^2}\, y - \chi(\overline{\theta}) = 0,$$

where $\overline{\theta}$ is the complex conjugate of θ and the square root is defined by continuation through the interval $\left(-\dfrac{1}{c}, +\dfrac{1}{c}\right)$.

In the limit when θ and $\overline{\theta}$ approach to θ_0, these equations lead to the following relations, which determine the position of the line for $\theta = \theta_0$:

$$t + \theta_0 x \pm \sqrt{\frac{1}{c^2} - \theta_0^2}\, y - \chi(\theta_0) = 0, \quad x \mp \frac{\theta_0}{\sqrt{\frac{1}{c^2} - \theta_0^2}} y - \chi'(\theta_0) = 0. \qquad (25)$$

Thus, we see that the surface generated by the above lines is the envelope of the planes corresponding to the real values of θ inside $\left(-\dfrac{1}{c}, +\dfrac{1}{c}\right)$. Furthermore, each plane forms an angle $\arctan c$ with the t-axis. Thus, we can claim that the normal to the surface Σ generated by these lines must satisfy the condition $\tan \nu t = \pm \frac{1}{c}$. If we let

$$c = a = \sqrt{\frac{\lambda + 2\mu}{\varrho}},$$

then this condition coincides with (12); on the other hand, if we let $c = b = \sqrt{\frac{\mu}{\varrho}}$, then it coincides with (13), i.e., Σ is a characteristic surface. If we write the equation of this surface in the form $v(x, y, t) = C$, then equations (12) and (13) can be reduced to the following form:

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 - \frac{1}{c^2}\left(\frac{\partial v}{\partial t}\right)^2 = 0 \qquad \left(c^2 = \frac{\lambda + 2\mu}{\varrho},\ \frac{\mu}{\varrho}\right). \tag{26}$$

Suppose that we are dealing with a vector of potential (respectively, solenoidal) type satisfying the elasticity equations, and that its components are expressed in form (19) in some domain D_1 of the (x, y, t)-space, where equation (18) has complex conjugate roots. The preceding arguments provide us with a simple method for continuation of this solution across Σ into a domain D_2 in which equation (18) has two real roots corresponding to the above imaginary roots. However, this continuation is not unique. As was shown above, Σ is a characteristic surface, hence quantities (6.1) remain continuous and formula (5) remains applicable after continuation in the case when this continuation is carried out preserving the continuity of (u_1, v_1) across Σ, and, thus, the continuity of the tangential derivatives of (u_1, v_1). We look for (u_1, v_1) in the domain D_2 in the form of (19) with the same analytic functions, as those defined on D_1. This ensures continuity of (u_1, v_1) on passage across Σ. In what follows one conclusion is essential concerning the planes

$$t + \theta x \pm \sqrt{\frac{1}{a^2} - \theta^2}\, y - \chi(\theta) = 0, \qquad -\frac{1}{a} \leq \theta \leq \frac{1}{a},$$

which fill out D_2.

Every plane corresponding to a value of θ in the interval $\left(-\frac{1}{a}, +\frac{1}{a}\right)$ is divided into two parts by the corresponding line of the surface Σ. Consequently outside D_2 we have two systems of half-planes.

From the point of view of the solutions of equations (18), these two systems correspond to the fact that the complex root in D_1 of this equation becomes multiple on the boundary and can be extended in two different ways beyond this domain. Thus, it is quite natural to define the function φ outside D_1 in two different ways.

As we have seen above, each point of D_2 corresponds to two different real values of θ. In (19) we can choose for (u_1, v_1) values of θ such that they belong to one or another of the above systems of half-planes.

In general, we may superimpose these two solutions and obtain a composite structure, which will turn out to be useful later. To obtain this composite structure we express u_1 and v_1 as the sum of two functions of θ chosen so that their sum on either half of the tangent plane is equal to u_1 and v_1 on the corresponding line of Σ.

We now study the connection between the values of the complex variable θ and the direction of the corresponding line. First, note that this direction does not depend on the choice of the function $\chi(\theta)$.

For definiteness we suppose that in formula (18) just one sign is chosen in front of the radical $\sqrt{\dfrac{1}{a^2} - \theta^2}$.

We cut the complex plane θ along the interval $\left(-\dfrac{1}{a}, +\dfrac{1}{a}\right)$ of the real line, and let the radical $\sqrt{\dfrac{1}{a^2} - \theta^2}$ have negative imaginary part for real $\theta > \dfrac{1}{a}$. Then this root is positive on the upper lip of our cut and negative on the lower lip. Suppose that the minus sign is taken in front of the radical in formula (18). Then along the lines corresponding to θ in the upper half-plane t and y are changing in the same direction. For a solution of the problem of vibration of the half-plane $y > 0$ with given initial conditions we take the parts of the lines for which $y > 0$ and we let t decrease. In accordance with this fact, the rays corresponding to values of θ in the upper half-plane go from the interior of the domain $y > 0$ and intersect the boundary $y = 0$. We call these rays incident rays. Similarly, along each ray corresponding to θ in the lower half-plane, t and y vary in opposite directions and these rays go from the boundary $y = 0$ into the domain $y > 0$. These are called reflected rays. If in formula (18) we take the plus sign in front of the radical, the rays corresponding to θ in the lower half-plane become the incident rays.

Besides the incident and reflected rays, the notions of incident and reflected waves are also useful.

To clarify these notions, we now study in detail the surface Σ generated by those lines that correspond to real values of θ in the interval $\left(-\dfrac{1}{a}, +\dfrac{1}{a}\right)$. The section L of Σ by the plane $t = t_0$ is the envelope of the lines

$$t_0 + \theta x - \sqrt{\frac{1}{a^2} - \theta^2}\, y - \chi(\theta) = 0,$$

and the slopes of these lines are determined by

$$\frac{dy}{dx} = \frac{\theta}{\sqrt{\frac{1}{a^2} - \theta^2}},$$

from which it follows that the tangent to L at the point, where $\theta = 0$, is parallel to the axis $y = 0$; also the part of L for which x is a monotone function of θ is convex. We examine just this part of L, assuming that the point, where $\theta = 0$, is in its interior.

The part of the plane $t = t_0$ lying in a neighborhood of this curve on its concave side belongs to the domain D of the (x, y, t)-space filled out by the lines corresponding to complex values of θ. Suppose that θ is in the upper half-plane, i.e., that the rays are incident.

We assume that as t varies over some interval $0 > t > T$ these rays move inside the half-plane $y > 0$.

We also assume that in the case of incident rays and for the part of L under consideration the quantity $\dfrac{d^2y}{dx^2}$ is positive. This is essential to our argument. Thus, the part of L in question has the shape illustrated in Fig. 1.

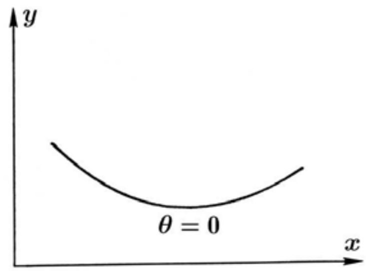

Fig. 1.

In the domain D the components of the vector (u_1, v_1) are determined by equations of type (19). We now define these functions outside the domain D as follows: on each tangent half-plane to Σ, whose traces on the plane $t = t_0$ are shown in Fig. 2, a), we define u_1, v_1 by equating them with their respective values at the point of tangency; in the domain beyond the horizontal tangent we obtain $u_1 = v_1 = 0$.

The rule by which we have defined u_1, v_1 outside D was chosen to ensure that in a neighborhood of the line $y = 0$ for $t > T$ the vector (u_1, v_1) remains constant.

In some domain D of the (x, y, t)-space consisting of incident rays let equation (18) for θ give a complex value, for instance, in the upper half-plane.

It is perfectly obvious that if $\operatorname{Re}(\Phi(0)) = 0$ (from now on we always suppose this to be the case), then our method allows us to continue u_1 and v_1 in a well-defined manner to the exterior of D so that their extensions are continuous functions.

We call such a wave an incident wave. Reflected waves are defined analogously. The part corresponding to complex values of θ consists of reflected rays. The corresponding curve has its convex side turned towards the half-space $y > 0$ and the values of the vector (u_1, v_1) are defined in such a way that this vector vanishes on each half-plane whose line of intersection with the plane $y = 0$ has points with the following property: the t-coordinate of such a point is greater than the t-coordinate of the point of intersection of the corresponding line and the plane $y = 0$.

This wave is shown in Fig. 2, b).

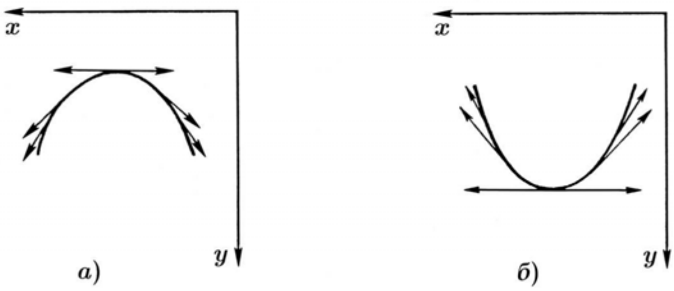

Fig. 2.

Note that in the case of a layer enclosed between two straight lines, a wave that is incident relative to one of its boundaries is reflected relative to the other.

The idea of our method is that to satisfy boundary conditions (3) we have to superimpose two reflected waves on the motion, given as an incident longitudinal or transverse wave. One wave is longitudinal and the other one is transverse.

As pointed out above, the reflected waves have been chosen so that for $t > T$ the fundamental mode remains stationary.

5. We now move on to the construction of the reflected waves. Consider the incident longitudinal wave defined by the equations

$$\delta_1 \equiv t + \theta_1 x - \sqrt{\frac{1}{a^2} - \theta_1^2}\, y - \chi(\theta_1) = 0, \tag{18.1}$$

$$u_1 = \operatorname{Re}(U_1(\theta_1)), \qquad v_1 = \operatorname{Re}(V_1(\theta_1)). \tag{27}$$

For this wave to be in fact longitudinal and to satisfy (21) it suffices that functions U_1 and V_1 satisfy the obvious relation:

$$-\frac{\partial U_1}{\partial \theta_1}\sqrt{\frac{1}{a^2} - \theta_1^2} = \frac{\partial V_1}{\partial \theta_1}\theta_1, \tag{28}$$

which we assume to hold from now on.

Let equations (27) give a solution satisfying boundary conditions (3) for $t > T$; more precisely, suppose that for such values of t there is no motion in a neighborhood of the plane $y = 0$. Suppose also that $\operatorname{Re}(V_1(0)) = \operatorname{Re}(U_1(0)) = 0$, which is necessary for the vector (u_1, v_1) to be continuous.

To meet boundary conditions (3) we superimpose on this solution two reflected waves, longitudinal and transverse, which do not alter the motion for $t > T$.

For this purpose it is natural to look for both waves in the same form

$$u_2 = \mathrm{Re}\,(U_2(\theta_2)), \qquad v_2 = \mathrm{Re}\,(V_2(\theta_2)), \tag{29}$$

$$u_3 = \mathrm{Re}\,(U_3(\theta_3)), \qquad v_3 = \mathrm{Re}\,(V_3(\theta_3)). \tag{30}$$

The simplest way to do it is to suppose that the values of θ_2 and θ_3 coincide with the values of θ_1 on the plane $y = 0$, i.e., that each incident ray generates two reflected rays determined by the same value of the complex variable θ.

For θ_2 and θ_3 to determine reflected rays we need to take in equation (18) the coefficient of y with the sign opposite to that of the incident wave.

Thus, we obtain the following two equations for θ_2 and θ_3:

$$\delta_2 \equiv t + \theta_2 x + \sqrt{\frac{1}{a^2} - \theta_2^2}\, y - \chi(\theta_2) = 0, \tag{31}$$

$$\delta_3 \equiv t + \theta_3 x + \sqrt{\frac{1}{b^2} - \theta_3^2}\, y - \chi(\theta_3) = 0. \tag{32}$$

It is obvious that θ_2 and θ_3 coincide with θ_1 for $y = 0$.

For wave (29) to be longitudinal it suffices that U_2 and V_2 satisfy the equation

$$\frac{\partial U_2}{\partial \theta_2}\sqrt{\frac{1}{a^2} - \theta_2^2} = \frac{\partial V_2}{\partial \theta_2}\theta_2, \tag{33}$$

and for wave (30) to be transverse it suffices that the equation

$$-\frac{\partial U_3}{\partial \theta_3}\theta_3 = \frac{\partial V_3}{\partial \theta_3}\sqrt{\frac{1}{b^2} - \theta_3^2} \tag{34}$$

holds.

We now consider boundary conditions (3). Substituting the obtained values of u and v into the expressions for stresses, we obtain

$$X_y|_{y=0} = \mu \mathrm{Re}\left\{ \frac{\partial U_1}{\partial \theta_1}\frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} + \frac{\partial V_1}{\partial \theta_1}\frac{-\theta_1}{\delta_1'} + \frac{\partial U_2}{\partial \theta_2}\frac{-\sqrt{\frac{1}{a^2} - \theta_2^2}}{\delta_2'} \right.$$

$$\left. + \frac{\partial V_2}{\partial \theta_2}\frac{-\theta_2}{\delta_2'} + \frac{\partial U_3}{\partial \theta_3}\frac{-\sqrt{\frac{1}{b^2} - \theta_3^2}}{\delta_3'} + \frac{\partial V_3}{\partial \theta_3}\frac{-\theta_3}{\delta_3'} \right\}\Bigg|_{y=0} = 0,$$

$$Y_y|_{y=0} = \mu \mathrm{Re}\left\{ \frac{a^2}{b^2}\left[\frac{\partial U_1}{\partial \theta_1}\frac{-\theta_1}{\delta_1'} + \frac{\partial V_1}{\partial \theta_1}\frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} + \frac{\partial U_2}{\partial \theta_2}\frac{-\theta_2}{\delta_2'} \right.\right.$$

$$
+\frac{\partial V_2}{\partial \theta_2}\frac{-\sqrt{\frac{1}{a^2}-\theta_2^2}}{\delta_2'}+\frac{\partial U_3}{\partial \theta_3}\frac{-\theta_3}{\delta_3'}+\frac{\partial V_3}{\partial \theta_3}\frac{-\sqrt{\frac{1}{b^2}-\theta_3^2}}{\delta_3'}\Bigg]
$$

$$
-2\left[\frac{\partial U_1}{\partial \theta_1}\frac{-\theta_1}{\delta_1'}+\frac{\partial U_2}{\partial \theta_2}\frac{-\theta_2}{\delta_2'}+\frac{\partial U_3}{\partial \theta_3}\frac{-\theta_3}{\delta_3'}\right]\Bigg\}\Bigg|_{y=0}=0,
$$

where δ_1', δ_2', and δ_3' denote the partial derivatives with respect to θ of the left sides of corresponding equations [4].

Taking into account (28), (33), and (34), we may replace the derivatives of the function V by the corresponding derivatives of the function U.

Observing that δ_1', δ_2', and δ_3' coincide for $y=0$, we obtain the following two equations sufficient for the validity of the boundary conditions:

$$
\begin{gathered}
2\frac{\partial U_1}{\partial \theta}\sqrt{\frac{1}{a^2}-\theta^2}-2\frac{\partial U_2}{\partial \theta}\sqrt{\frac{1}{a^2}-\theta^2}-\frac{\partial U_3}{\partial \theta}\frac{\left(\frac{1}{b^2}-2\theta^2\right)}{\sqrt{\frac{1}{b^2}-\theta^2}}=0,\\
\frac{\partial U_1}{\partial \theta}\frac{\left(2\theta^2-\frac{1}{b^2}\right)}{\theta}+\frac{\partial U_2}{\partial \theta}\frac{\left(2\theta^2-\frac{1}{b^2}\right)}{\theta}+\frac{\partial U_3}{\partial \theta}2\theta=0.
\end{gathered}
\tag{35}
$$

These equations allow us to find the unknown functions U_2 and U_3 from the following two equations:

$$
\begin{aligned}
\frac{\partial U_2}{\partial \theta}&=-\frac{\left(2\theta^2-\frac{1}{b^2}\right)^2-4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}{\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}\frac{\partial U_1}{\partial \theta},\\
\frac{\partial U_3}{\partial \theta}&=-\frac{4\left(2\theta^2-\frac{1}{b^2}\right)\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}{\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}\frac{\partial U_1}{\partial \theta}.
\end{aligned}
\tag{36.1}
$$

Then the functions V_2 and V_3 are found from

$$
\begin{aligned}
\frac{\partial V_2}{\partial \theta}&=\frac{\left(2\theta^2-\frac{1}{b^2}\right)^2-4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}{\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}\frac{\partial V_1}{\partial \theta},\\
\frac{\partial V_3}{\partial \theta}&=-\frac{4\theta^2\left(2\theta^2-\frac{1}{b^2}\right)}{\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}\frac{\partial V_1}{\partial \theta}.
\end{aligned}
\tag{36.2}
$$

It is easy to see that the reflected waves do not change the motion when $t>T$. Consequently the obtained motion satisfies the initial conditions.

The functions U_2, V_2, U_3, V_3 are continuous, because constants of integration can be chosen so that

---

[4] In view of (22), (24) we have $\frac{a^2}{b^2}=\frac{\lambda}{\mu}+2$. – *Ed.*

$$\mathrm{Re}\,(U_2(0)) = \mathrm{Re}\,(V_2(0)) = \mathrm{Re}\,(U_3(0)) = \mathrm{Re}\,(V_3(0)) = 0. \tag{37}$$

The case of a transverse incident wave can be studied in a completely analogous fashion.

For such wave we obtain the formulas

$$u_4 = \mathrm{Re}\,(U_4(\theta_4)), \qquad v_4 = \mathrm{Re}\,(V_4(\theta_4)), \tag{38}$$

$$\mathrm{Re}\,(U_4(0)) = 0, \qquad \mathrm{Re}\,(V_4(0)) = 0, \tag{39}$$

where θ_4 satisfies the equation

$$\delta_4 \equiv t + \theta_4 x - \sqrt{\frac{1}{b^2} - \theta_4^2}\, y - \chi(\theta_4) = 0, \tag{40}$$

and U_4, V_4 are connected by the relation

$$\frac{\partial U_4}{\partial \theta_4}\theta_4 = \frac{\partial V_4}{\partial \theta_4}\sqrt{\frac{1}{b^2} - \theta_4^2}. \tag{41}$$

Then the reflected waves are defined by the following equations:

$$u_5 = \mathrm{Re}\,(U_5(\theta_5)), \qquad v_5 = \mathrm{Re}\,(V_5(\theta_5)), \tag{42}$$

$$\delta_5 \equiv t + \theta_5 x + \sqrt{\frac{1}{a^2} - \theta_5^2}\, y - \chi(\theta_5) = 0, \tag{43}$$

$$u_6 = \mathrm{Re}\,(U_6(\theta_6)), \qquad v_6 = \mathrm{Re}\,(V_6(\theta_6)), \tag{44}$$

$$\delta_6 \equiv t + \theta_6 x + \sqrt{\frac{1}{b^2} - \theta_6^2}\, y - \chi(\theta_6) = 0, \tag{45}$$

$$\frac{\partial U_5}{\partial \theta_5}\sqrt{\frac{1}{a^2} - \theta_5^2} = \frac{\partial V_5}{\partial \theta_5}\theta_5, \tag{46}$$

$$\frac{\partial U_6}{\partial \theta_6}\theta_6 = -\frac{\partial V_6}{\partial \theta_6}\sqrt{\frac{1}{b^2} - \theta_6^2}, \tag{47}$$

$$\begin{aligned}
\frac{\partial U_5}{\partial \theta} &= \frac{4\theta^2\left(2\theta^2 - \frac{1}{b^2}\right)}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}\frac{\partial U_4}{\partial \theta},\\
\frac{\partial V_5}{\partial \theta} &= \frac{4\left(2\theta^2 - \frac{1}{b^2}\right)\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}\frac{\partial V_4}{\partial \theta},
\end{aligned} \tag{48.1}$$

$$\begin{aligned}
\frac{\partial U_6}{\partial \theta} &= \frac{\left(2\theta^2 - \frac{1}{b^2}\right)^2 - 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}\frac{\partial U_4}{\partial \theta},\\
\frac{\partial V_6}{\partial \theta} &= \frac{-\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}\frac{\partial V_4}{\partial \theta}.
\end{aligned} \tag{48.2}$$

To apply formula (5) to the reflected waves so constructed it is essential that the coefficients in equalities (36) and (48) do not have singular points in the open upper half-plane of the variable θ. Let us find all the singular points of these coefficients. First of all, there are the branch points $\pm\dfrac{1}{a}$, $\pm\dfrac{1}{b}$ on the real axis; these are of no interest. Let us study the poles of the coefficients in question. They can occur only at the point $\theta = \infty$ or at the roots of the equation

$$F(\theta) \equiv \left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2} = 0. \tag{49}$$

The series expansion of the denominator in the neighborhood of the point $\theta = \infty$ takes the simple form

$$\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2} = \left(\frac{2}{a^2} - \frac{2}{b^2}\right)\theta^2 + O(1), \tag{50}$$

where $O(1)$ is finite at $\theta = \infty$.

The expansion of all numerators begins with θ^4. Hence the coefficients have a second-order pole at $\theta = \infty$. However, no difficulties arise because of this pole, since the ray corresponding to $\theta = \infty$ is parallel to $y = 0$ and therefore is not reflected.

Now we consider the roots of equation (49). The inspection of this equation shows that it has only two real roots $\pm\dfrac{1}{c}$, where $c < b$.

Indeed, the existence of these roots follows from the change of the sign of the left side of equation (49) as θ varies in each of the intervals $\left(-\infty, -\dfrac{1}{b}\right)$ and $\left(\dfrac{1}{b}, +\infty\right)$.

The fact that these roots are the only real roots can be easily proved. To prove that there are no other real roots of equation (49), we have merely to investigate the sign of the derivative. For instance, for $\dfrac{1}{b} < \theta < \infty$,

$$F'(\theta) = \frac{\left(16\theta^3 - 8\frac{\theta}{b^2}\right)\sqrt{\theta^2 - \frac{1}{b^2}}\sqrt{\theta^2 - \frac{1}{a^2}} - 16\theta^5 + 12\theta^3\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{8\theta}{a^2b^2}}{\sqrt{\theta^2 - \frac{1}{a^2}}\sqrt{\theta^2 - \frac{1}{b^2}}}$$

$$< -\frac{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left(12\theta^3 + \frac{8\theta}{b^2}\right)}{\sqrt{\theta^2 - \frac{1}{a^2}}\sqrt{\theta^2 - \frac{1}{b^2}}} < 0,$$

and for $-\infty < \theta < -\dfrac{1}{b}$ the inequality signs are reversed.

Thus, $F'(\theta)$ has constant sign in each of these intervals.

It is easy to see that on the interval $\left(-\frac{1}{b}, +\frac{1}{b}\right)$ of the real axis the left side of equation (49) does not have roots. It is now easy to show that this expression has no roots in the open upper half-plane of the variable θ.

Indeed, if θ goes along the interval $\left(-\infty, -\frac{1}{b}\right)$ with an indentation in the upper half-plane to avoid the root $-\frac{1}{c}$, then the argument of our quantity undergoes a jump of π. On the interval $\left(-\frac{1}{b}, +\frac{1}{b}\right)$ the real part of our function is positive and there is no jump of its argument. When θ moves along the interval $\left(\frac{1}{b}, +\infty\right)$ there is again a jump of argument of π as for the interval $\left(-\infty, -\frac{1}{b}\right)$. Finally, in view of the estimate of our expression in the neighborhood of $\theta = \infty$ (formula (50)), the change of argument along a sufficiently large semicircle centered at the origin is -2π. Hence the required result now follows from the classical Cauchy theorem.

A similar argument shows that there are no roots of equation (49) in the lower half-plane.

As we see later, these poles give rise to the phenomenon known as "surface waves" or the Rayleigh waves.

6. After these preparatory remarks we can now move on to the solving of the problem stated about vibrations of the half-plane under arbitrary initial conditions.

First of all we make an important remark concerning the representation of particular solutions in elasticity theory.

It can be shown that if some solution φ of the wave equation is a homogeneous function of degree zero with respect to

$$(t_0 - t), \quad (x - x_0), \quad (y - y_0), \tag{51}$$

then it can be written in the form

$$\varphi = \operatorname{Re}(\Phi(\theta)), \tag{52}$$

where

$$(t_0 - t) - \theta(x - x_0) + \sqrt{\frac{1}{a^2} - \theta^2}(y - y_0) = 0. \tag{53}$$

The proof is obvious. Any homogeneous function of degree zero is a function of the arguments $\frac{x - x_0}{t_0 - t}$ and $\frac{y - y_0}{t_0 - t}$. Instead of these variables we can introduce two new variables:

$$\operatorname{Re}\theta \quad \text{and} \quad \operatorname{Im}\theta. \tag{54}$$

Replacing in the wave equation x, y and t by variables (54), we arrive at the Laplace equation. This proves the above assertion.

Following V. Volterra, we take a particular solution of the problem in the form

$$u_1 = \frac{\partial \varphi_1}{\partial x}, \qquad v_1 = \frac{\partial \varphi_1}{\partial y},$$

$$\varphi_1 = (t_0 - t) \ln \left(\frac{a(t_0 - t)}{r} + \sqrt{\frac{a^2 (t_0 - t)^2}{r^2} - 1} \right) - \sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}}, \quad (55)$$

where $a = \sqrt{\dfrac{\lambda + 2\mu}{\varrho}}$ and $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, and the point (x, y, t) is located inside the cone

$$a(t_0 - t) = r.$$

Here u_1 and v_1 are homogeneous functions of degree zero of variables (51), therefore we can express them in form (52) (or (27)).

A simple computation gives us

$$U_1 = -i\sqrt{\frac{1}{a^2} - \theta^2}, \qquad V_1 = -i\theta, \quad (56)$$

where θ satisfies equation (53).

It is easy to see that in this case the rays corresponding to complex values of θ generate the cone $t - t_0 < \dfrac{r}{a}$ with an apex at the point (x_0, y_0, t_0) and angle $\arctan a$ between its generator and the t-axis. At the boundary of this domain u_1 and v_1 vanish. The incident rays correspond to the half of the cone for which $y - y_0 < 0$, and the rays, which by our definition are reflected, correspond to the other half of the cone.

The formulas obtained in the preceding section enable us to find the reflected waves and to construct the solution for $t < t_0 - \dfrac{y_0}{a}$ as well.

For this purpose we have to take in (18.1), (31) and (32)

$$\chi(\theta) = t_0 + \theta x_0 - \sqrt{\frac{1}{a^2} - \theta^2}\, y_0. \quad (57)$$

The waves arising in this case are shown in Fig. 3.

The displacements u, v are nonzero inside the above-mentioned cone,

$$t_0 - t = \frac{1}{a}\sqrt{(x - x_0)^2 + (y - y_0)^2}. \quad (58)$$

Denote by Ω_1 the domain bounded by this cone, the plane $y = 0$ and the plane $t = 0$.

Let L_1 be the surface of the cone, Σ be the part of the boundary of Ω_1 in the plane $y = 0$, and S_1 be the part of the boundary of Ω_1 in the plane $t = 0$. Then it follows from the arguments of Sect. 2, that the displacements and their corresponding expressions (6.1) vanish on L_1.

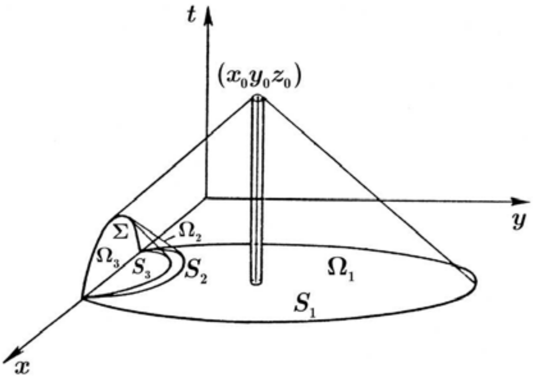

Fig. 3.

The displacements (u_2, v_2) are nonzero in the domain Ω_2 bounded by a part L_2 of the surface of the reflected cone

$$t_0 - t = \frac{1}{a}\sqrt{(x - x_0)^2 + (y + y_0)^2} \tag{59}$$

and a part S_2 of the plane $t = 0$.

These displacements together with expressions (6.1) vanish on L_2.

Finally, the displacements (u_3, v_3) are nonzero only in the domain Ω_3 bounded by the envelope of the planes

$$t_0 - t - \theta_3(x - x_0) - \sqrt{\frac{1}{b^2} - \theta_3^2}\, y - \sqrt{\frac{1}{a^2} - \theta_3^2}\, y_0 = 0, \tag{60}$$

a part of Σ in the plane $y = 0$, and a part S_3 of the plane $t = 0$.

On L_3 the displacements (u_3, v_3) and their corresponding expressions (6) vanish.

The sum of the three waves on Σ (and therefore on the entire plane $y = 0$) satisfies the equations

$$\begin{aligned} X_{y,1} + X_{y,2} + X_{y,3} &= 0, \\ Y_{y,1} + Y_{y,2} + Y_{y,3} &= 0. \end{aligned} \tag{61}$$

In precisely the same way we can examine the reflection of the second particular solution of V. Volterra:

$$\begin{gathered} u_4 = \frac{\partial \psi_4}{\partial y}, \qquad v_4 = -\frac{\partial \psi_4}{\partial x}, \\ \psi_4 = (t_0 - t)\ln\left(\frac{b(t_0 - t)}{r} + \sqrt{\frac{b^2(t_0 - t)^2}{r^2} - 1}\right) - \sqrt{(t_0 - t)^2 - \frac{r^2}{b^2}}, \end{gathered} \tag{62}$$

where $b = \sqrt{\dfrac{\mu}{\varrho}}$, $t_0 - \dfrac{y_0}{b} < t < t_0$.

By arguments that differ in no way from the preceding ones it can be shown that the displacements (u_4, v_4) can be written in the form

$$\begin{aligned} u_4 &= \operatorname{Re}(U_4(\theta_4)), \quad U_4(\theta_4) = -i\theta_4, \\ v_4 &= \operatorname{Re}(V_4(\theta_4)), \quad V_4(\theta_4) = i\sqrt{\frac{1}{b^2} - \theta_4^2}, \end{aligned} \tag{63}$$

where

$$(t_0 - t) - \theta_4(x - x_0) + \sqrt{\frac{1}{b^2} - \theta_4^2}(y - y_0) = 0. \tag{64}$$

Thus, the analysis carried out in the previous section allows us to construct reflected waves and to find a solution for $t < t_0 - \dfrac{y_0}{b}$.

A geometric picture of the waves is provided in Fig. 4.

To obtain the appropriate formulas in this case we have to choose for $\chi(\theta)$ in the corresponding equations of Sect. 5 ((40), (43) and (45)) the function

$$\chi(\theta) = t_0 + \theta x_0 - \sqrt{\frac{1}{b^2} - \theta^2}\, y_0. \tag{65}$$

The displacements (u_4, v_4) vanish outside the cone L_4:

$$t_0 - t = \frac{1}{b}\sqrt{(x - x_0)^2 + (y - y_0)^2}, \tag{66}$$

and are nonzero inside the domain Ω_4 bounded by this cone, a part Σ' of the surface $y = 0$ and a part S_4 of the plane $t = 0$; (u_4, v_4) vanish on L_4 together with their corresponding expressions (6.1).

Several observations have to be made concerning the reflected waves. First consider the longitudinal reflected wave. It is generated by rays corresponding to values of θ in the upper half-plane. The boundary of the domain generated by these rays consists of rays corresponding to values of θ in the interval $\left(-\dfrac{1}{b}, +\dfrac{1}{b}\right)$, i.e., the rays obtained as reflections of rays belonging to the contour of the incident wave.

However, the rays corresponding to the subintervals $\left(-\dfrac{1}{b}, -\dfrac{1}{a}\right)$ and $\left(\dfrac{1}{a}, \dfrac{1}{b}\right)$ lie in the plane $y = 0$, as can be seen from equations (43). It is also easy to see that they touch the hyperbola in the intersection of the cone (66) with the plane $y = 0$.

The components of displacements (u_5, v_5) and their corresponding expressions (6.1) vanish on the surface L_5 defined as the envelope of planes (43) for $-\dfrac{1}{a} < \theta < +\dfrac{1}{a}$.

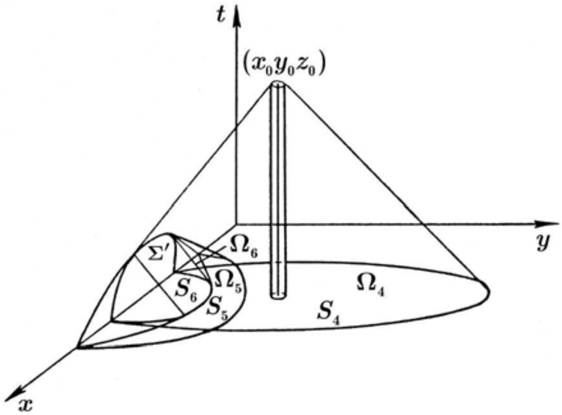

Fig. 4.

The reflected transverse wave is an example of a wave in which there are half-planes as well as rays.

Indeed, the boundary of the domain generated by the rays corresponding to complex values of θ is generated by the rays corresponding to values of θ in the interval $-\dfrac{1}{b} < \theta < +\dfrac{1}{b}$.

But on this interval the coefficients in reflection formulas (48) are not real. Therefore, the functions $u_6(\theta_6)$ and $v_6(\theta_6)$ cannot have zero real parts on this interval, since the real parts of the functions $u_4(\theta_4)$ and $v_4(\theta_4)$ vanish here, while their imaginary parts are definitely nonzero.

Hence, by the arguments in Sect. 5, (u_6, v_6) are nonzero on certain half-planes.

Thus, the domain Ω_6, where (u_6, v_6) are nonzero is bounded by the surface L_6 consisting of a part of the reflected cone

$$t_0 - t = \frac{1}{b}\sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{67}$$

and two planes tangent to cone (67) and corresponding to the values $\theta = \pm\dfrac{1}{a}$, and also of the surface Σ' and a part S_6 of the plane $t = 0$.

7. We now return to the main problem. Following V. Volterra, we apply formula (5) to the required solution as well as to the solution (u_1, v_1) constructed above in the domain Ω_1', which is obtained from Ω_1 by cutting out a small cylinder M of radius ε along the singular (for our solution) line $x = x_0$, $y = y_0$.

We obtain

$$\iint\limits_{S_1} \mathfrak{G}_1 dS + \iint\limits_{L_1} \mathfrak{G}_1 dS + \iint\limits_{M} \mathfrak{G}_1 dS + \iint\limits_{\Sigma} \mathfrak{G}_1 dS = 0, \tag{68}$$

where $\mathfrak{G}_1$ denotes the expression in the double integral in formula (5).

As noted earlier, the surface integral over L_1 is zero. The integral over S_1 is known, because its integrand contains only u, v, $\dfrac{\partial u}{\partial t}$, and $\dfrac{\partial v}{\partial t}$ when $t = 0$. Consider the integral over M. It is obvious that $\cos \nu t$ vanishes on M. Keeping in mind that ν is the inward normal for Ω_1', we obtain

$$\cos \nu x = \frac{x - x_0}{r}, \ \cos \nu y = \frac{y - y_0}{r}, \ r = \sqrt{(x-x_0)^2 + (y-y_0)^2} = \varepsilon. \tag{69}$$

Furthermore,

$$u_1 = -\frac{\sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}}}{r} \cos \nu x, \qquad v_1 = -\frac{\sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}}}{r} \cos \nu y,$$

$$\begin{aligned} &X_{x,1} \cos \nu x + X_{y,1} \cos \nu y \\ &\quad = \left\{ (\lambda + 2\mu) \frac{(t_0 - t)^2}{\sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}}} - \lambda \sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}} \right\} \frac{\cos \nu x}{r^2}, \\ &X_{y,1} \cos \nu x + Y_{y,1} \cos \nu y \\ &\quad = \left\{ (\lambda + 2\mu) \frac{(t_0 - t)^2}{\sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}}} - \lambda \sqrt{(t_0 - t)^2 - \frac{r^2}{a^2}} \right\} \frac{\cos \nu y}{r^2}. \end{aligned} \tag{70}$$

Consider first the integral

$$\iint\limits_M [u(X_{x,1} \cos \nu x + X_{y,1} \cos \nu y) + v(X_{y,1} \cos \nu x + Y_{y,1} \cos \nu y)]\, dt\, dl,$$

where dl denotes the element of arc length of the circle in the base of the cylinder. This integral can be transformed in an obvious way to the following:

$$\begin{aligned} \int\limits_0^{t_0 - \frac{\varepsilon}{a}} \Bigg\{ &\left[2\mu \sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}} + \frac{\varepsilon^2 (\lambda + 2\mu)}{a^2 \sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}}} \right] \\ &\times \frac{1}{\varepsilon^2} \int\limits_C (u \cos \nu x + v \cos \nu y)\, dl \Bigg\}\, dt, \end{aligned}$$

where C is the circle.

Applying to the integral over C first the Green formula and then the mean value theorem, we obtain

$$\pi \int\limits_0^{t_0 - \frac{\varepsilon}{a}} \left\{ \left[2\mu \sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}} + \frac{(\lambda + 2\mu) \frac{\varepsilon^2}{a^2}}{\sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}}} \right] \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_{\xi, \eta} \right\} dt,$$

where the point ξ, η depends on t and lies inside C.

We find the limit of this integral as $\varepsilon \to 0$. It is equal to

$$2\pi\mu \int_0^{t_0} (t_0 - t)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{x_0,\, y_0} dt$$

$$+ \lim_{\varepsilon\to 0}(\lambda + 2\mu)\frac{\pi}{a^2} \int_0^{t_0 - \frac{\varepsilon}{a}} \frac{\varepsilon^2}{\sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}}}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{\xi,\, \eta} dt.$$

Let K be the upper bound of the quantity $\left|\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right|$. It is obvious that the absolute value of the second term in the above expression does not exceed

$$(\lambda + 2\mu)\frac{\pi}{a^2} \int_0^{t_0 - \frac{\varepsilon}{a}} \frac{K\varepsilon^2}{\sqrt{(t_0 - t)^2 - \frac{\varepsilon^2}{a^2}}} dt.$$

Substituting the independent variable

$$t = t_0 - \frac{\varepsilon}{a} - z,$$

this integral reduces to the form

$$\frac{(\lambda + 2\mu)\pi}{a^2} \int_0^{t_0 - \frac{\varepsilon}{a}} \frac{K\varepsilon^2}{\sqrt{z\left(z + \frac{2\varepsilon}{a}\right)}} dz.$$

The latter integral clearly tends to zero as $\varepsilon \to 0$. Similarly it can be shown that

$$\lim_{\varepsilon\to 0} \iint_M [u_1(X_x \cos\nu x + X_y \cos\nu y) + v_1(X_y \cos\nu x + Y_y \cos\nu y)]\, dt\, dl$$

$$= -2\pi(\lambda + 2\mu) \int_0^{t_0} (t_0 - t)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{x_0,\, y_0} dt.$$

At last,

$$\lim_{\varepsilon\to 0} \iint_M \mathfrak{G}_1 dS = -2\pi(\lambda + 2\mu) \int_0^{t_0} (t_0 - t)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{x_0,\, y_0} dt. \qquad (71)$$

Finally, we obtain

$$2\pi(\lambda+2\mu)\int_0^{t_0}(t_0-t)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dt=\iint_{S_1}\mathfrak{G}_1 dS+\iint_{\Sigma}\mathfrak{G}_1 dS. \tag{72}$$

For our purposes it is necessary to express the right side of equality (72) in terms of known quantities. Therefore, we must eliminate $\iint_{\Sigma}\mathfrak{G}_1 dS$. To do this, we consider additionally the domains Ω_2, Ω_3 and apply formula (5) to the required solution as well as to the solutions (u_2, v_2) and (u_3, v_3) constructed earlier. We obtain

$$-\iint_{\Sigma}\mathfrak{G}_2 dS-\iint_{L_2}\mathfrak{G}_2 dS-\iint_{S_2}\mathfrak{G}_2 dS=0, \tag{73}$$

or, since the integral over L_2 is zero,

$$-\iint_{\Sigma}\mathfrak{G}_2 dS-\iint_{S_2}\mathfrak{G}_2 dS=0,$$

similarly

$$-\iint_{\Sigma}\mathfrak{G}_3 dS-\iint_{S_3}\mathfrak{G}_3 dS=0. \tag{74}$$

Taking into account that

$$\mathfrak{G}_1+\mathfrak{G}_2+\mathfrak{G}_3=0 \quad \text{over} \quad \Sigma, \tag{75}$$

and using (2) and (61), we obtain

$$\begin{aligned}2\pi(\lambda+2\mu)\int_0^{t_0}(t_0-t)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_{x_0,\,y_0}dt\\ =\iint_{S_1}\mathfrak{G}_1 dS+\iint_{S_2}\mathfrak{G}_2 dS+\iint_{S_3}\mathfrak{G}_3 dS.\end{aligned} \tag{76}$$

In the same way we can apply formula (5) to the required solution and to the particular solution (u_4, v_4) in the domain Ω_4 with a cylinder M_1 of radius ε removed from it. This gives us

$$\iint_{S_4}\mathfrak{G}_4 dS+\iint_{L_4}\mathfrak{G}_4 dS+\iint_{M_1}\mathfrak{G}_4 dS+\iint_{\Sigma'}\mathfrak{G}_4 dS=0. \tag{77}$$

Calculating the integral over M_1, we obtain

$$-\lim_{\varepsilon\to 0}\iint\limits_{M_1}\mathfrak{G}_4 dS$$

$$= \lim_{\varepsilon\to 0}\iint\limits_{M_1}[u(X_{x,4}\cos\nu x + X_{y,4}\cos\nu y) + v(X_{y,4}\cos\nu x + Y_{y,4}\cos\nu y)]\,dS$$

$$-\lim_{\varepsilon\to 0}\iint\limits_{M_1}[u_4(X_x\cos\nu x + X_y\cos\nu y) + v_4(X_y\cos\nu x + Y_y\cos\nu y)]\,dS$$

$$= \lim_{\varepsilon\to 0}\int\limits_0^{t_0-\frac{\varepsilon}{b}}\left\{\left[2\mu\sqrt{(t_0-t)^2-\frac{\varepsilon^2}{b^2}} + \frac{\mu\varepsilon^2}{b^2\sqrt{(t_0-t)^2-\frac{\varepsilon^2}{b^2}}}\right]\frac{1}{\varepsilon^2}\right.$$

$$\left.\times\int\limits_C(u\cos\nu y - v\cos\nu x)\,dl\right\}dt - \lim_{\varepsilon\to 0}\iint\limits_{M_1}[(X_x u_4 + X_y v_4)\cos\nu x$$

$$+(X_y u_4 + Y_y v_4)\cos\nu y]\,dS,$$

and hence by arguments similar to the previous case we obtain

$$-\lim_{\varepsilon\to 0}\iint\limits_{M_1}\mathfrak{G}_4 dS = 2\pi\mu\int\limits_0^{t_0}(t_0-t)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)_{x_0,\,y_0}dt. \tag{78}$$

Applying again the previous arguments, we obtain the second fundamental formula

$$2\pi\mu\int\limits_0^{t_0}(t_0-t)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)_{x_0,\,y_0}dt = \iint\limits_{S_4}\mathfrak{G}_4 dS + \iint\limits_{S_5}\mathfrak{G}_5 dS + \iint\limits_{S_6}\mathfrak{G}_6 dS. \tag{79}$$

Denoting the right sides of equalities (76) and (79) by M and N, respectively, we rewrite these equations in the form

$$\int\limits_0^{t_0}(\lambda+2\mu)\left(\frac{\partial u}{\partial x_0}+\frac{\partial v}{\partial y_0}\right)(t_0-t)dt = \frac{1}{2\pi}M(x_0,y_0,t_0), \tag{76.1}$$

$$\int\limits_0^{t_0}\mu\left(\frac{\partial u}{\partial y_0}-\frac{\partial v}{\partial x_0}\right)(t_0-t)dt = \frac{1}{2\pi}N(x_0,y_0,t_0), \tag{79.1}$$

where $M(x_0,y_0,t_0)$ and $N(x_0,y_0,t_0)$ are known.

These formulas yield

$$\int_0^{t_0}\left[(\lambda+2\mu)\frac{\partial}{\partial x_0}\left(\frac{\partial u}{\partial x_0}+\frac{\partial v}{\partial y_0}\right)+\mu\frac{\partial}{\partial y_0}\left(\frac{\partial u}{\partial y_0}-\frac{\partial v}{\partial x_0}\right)\right](t_0-t)dt$$
$$=\frac{1}{2\pi}\left(\frac{\partial M}{\partial x_0}+\frac{\partial N}{\partial y_0}\right),$$
$$\int_0^{t_0}\left[(\lambda+2\mu)\frac{\partial}{\partial y_0}\left(\frac{\partial u}{\partial x_0}+\frac{\partial v}{\partial y_0}\right)-\mu\frac{\partial}{\partial x_0}\left(\frac{\partial u}{\partial y_0}-\frac{\partial v}{\partial x_0}\right)\right](t_0-t)dt$$
$$=\frac{1}{2\pi}\left(\frac{\partial M}{\partial y_0}-\frac{\partial N}{\partial x_0}\right),$$

and hence, using equations (1), we obtain

$$\int_0^{t_0}\varrho\frac{\partial^2(u,v)}{\partial t^2}(t_0-t)dt=\frac{1}{2\pi}\left[\left(\frac{\partial M}{\partial x_0},\frac{\partial M}{\partial y_0}\right)+\left(\frac{\partial N}{\partial y_0},-\frac{\partial N}{\partial x_0}\right)\right]. \tag{80}$$

Integrating by parts, we arrive at the relation

$$(u,v)|_{x_0,y_0,t_0}=(u,v)|_{x_0,y_0,0}+t_0\left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)\bigg|_{x_0,y_0,0}$$
$$+\frac{1}{2\pi\varrho}\left[\left(\frac{\partial M}{\partial x_0},\frac{\partial M}{\partial y_0}\right)+\left(\frac{\partial N}{\partial y_0},-\frac{\partial N}{\partial x_0}\right)\right]. \tag{81}$$

Formula (81) gives the solution of our problem.

The assumption that $T=0$, $X=0$, $Y=0$ is inessential for our arguments, since the same particular solutions with the same reflection principle permit us to obtain an analogous result even in the general case.

The functions $M(x_0,y_0,t_0)$ and $N(x_0,y_0,t_0)$ are expressed as sums of integrals:

$$M(x_0,y_0,t_0)=\iint\limits_{S_1}\mathfrak{G}_1dS+\iint\limits_{S_2}\mathfrak{G}_2dS+\iint\limits_{S_3}\mathfrak{G}_3dS,$$
$$N(x_0,y_0,t_0)=\iint\limits_{S_4}\mathfrak{G}_4dS+\iint\limits_{S_5}\mathfrak{G}_5dS+\iint\limits_{S_6}\mathfrak{G}_6dS.$$

The integrands contain the functions (u_k,v_k), $k=2,3,5,6$, and are somewhat unwieldy, because the corresponding functions U_k, V_k, as is evident from formulas (36) and (48), are elliptic integrals.

The right sides of formula (81) contain derivatives of the functions M and N. Differentiating the integrands, we obtain $\dfrac{\partial U}{\partial\theta}$ and $\dfrac{\partial V}{\partial\theta}$ instead of U and

V, thus obtaining a computationally more convenient expression for solution (81).

To simplify the notation we introduce[5]:

$$
\begin{gathered}
I_1 = \iint\limits_{S_1} \left\{ u \frac{\partial u_1}{\partial t} + v \frac{\partial v_1}{\partial t} \right\} dx\, dy = - \iint\limits_{S_1} \left\{ u \mathrm{Re} \left[U_1' \frac{1}{\delta_1'} \right] + v \mathrm{Re} \left[V_1' \frac{1}{\delta_1'} \right] \right\} dx\, dy, \\
I_2 = - \iint\limits_{S_2} \left\{ u \mathrm{Re} \left[U_2' \frac{1}{\delta_2'} \right] + v \mathrm{Re} \left[V_2' \frac{1}{\delta_2'} \right] \right\} dx\, dy, \\
I_3 = - \iint\limits_{S_3} \left\{ u \mathrm{Re} \left[U_3' \frac{1}{\delta_3'} \right] + v \mathrm{Re} \left[V_3' \frac{1}{\delta_3'} \right] \right\} dx\, dy, \\
I_4 = - \iint\limits_{S_4} \left\{ u \mathrm{Re} \left[U_4' \frac{1}{\delta_4'} \right] + v \mathrm{Re} \left[V_4' \frac{1}{\delta_4'} \right] \right\} dx\, dy, \\
I_5 = - \iint\limits_{S_5} \left\{ u \mathrm{Re} \left[U_5' \frac{1}{\delta_5'} \right] + v \mathrm{Re} \left[V_5' \frac{1}{\delta_5'} \right] \right\} dx\, dy, \\
I_6 = - \iint\limits_{S_6} \left\{ u \mathrm{Re} \left[U_6' \frac{1}{\delta_6'} \right] + v \mathrm{Re} \left[V_6' \frac{1}{\delta_6'} \right] \right\} dx\, dy,
\end{gathered}
\tag{82}
$$

where

$$
U_k' \equiv \frac{\partial U_k}{\partial \theta_k}, \qquad \delta_k' \equiv \frac{\partial \delta_k}{\partial \theta_k},
$$

$$
\begin{aligned}
J_1 &= \iint\limits_{S_1} \left\{ u_1 \frac{\partial u}{\partial t} + v_1 \frac{\partial v}{\partial t} \right\} dx\, dy, &\quad J_2 &= \iint\limits_{S_2} \left\{ u_2 \frac{\partial u}{\partial t} + v_2 \frac{\partial v}{\partial t} \right\} dx\, dy, \\
J_3 &= \iint\limits_{S_3} \left\{ u_3 \frac{\partial u}{\partial t} + v_3 \frac{\partial v}{\partial t} \right\} dx\, dy, &\quad J_4 &= \iint\limits_{S_4} \left\{ u_4 \frac{\partial u}{\partial t} + v_4 \frac{\partial v}{\partial t} \right\} dx\, dy, \\
J_5 &= \iint\limits_{S_5} \left\{ u_5 \frac{\partial u}{\partial t} + v_5 \frac{\partial v}{\partial t} \right\} dx\, dy, &\quad J_6 &= \iint\limits_{S_6} \left\{ u_6 \frac{\partial u}{\partial t} + v_6 \frac{\partial v}{\partial t} \right\} dx\, dy.
\end{aligned}
\tag{83}
$$

The functions I_k are more manageable, because they contain only $U_k'(\theta_k)$ and $V_k'(\theta_k)$.

However, the functions J_k are expressed in somewhat complicated form and for the following computations we have to consider their derivatives with respect to x_0 and y_0.

The range of integration S_k varies with x_0 and y_0. Its boundary has fixed portions, and portions that vary with x_0 and y_0. It is easy to see that the

[5] For differentiation S. L. Sobolev uses formulas (18.1), (27), (29)–(32), (38), (40), (42)–(45). – *Ed.*

integrands vanish on the variable parts of the boundary. Using this fact and applying the formula for differentiating an integral with a variable domain of integration, we obtain

$$\frac{\partial J_2}{\partial x_0} = \iint\limits_{S_2} \left\{ \frac{\partial \mathrm{Re}\,(U_2)}{\partial x_0} \frac{\partial u}{\partial t} + \frac{\partial \mathrm{Re}\,(V_2)}{\partial x_0} \frac{\partial v}{\partial t} \right\} dx\,dy$$

$$= \iint\limits_{S_2} \left\{ \mathrm{Re}\left[U_2' \frac{-\theta_2}{\delta_2'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_2' \frac{-\theta_2}{\delta_2'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy,$$

$$\begin{aligned}
\frac{\partial J_2}{\partial y_0} &= \iint\limits_{S_2} \left\{ \mathrm{Re}\left[U_2' \frac{\sqrt{\frac{1}{a^2} - \theta_2^2}}{-\delta_2'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_2' \frac{\sqrt{\frac{1}{a^2} - \theta_2^2}}{-\delta_2'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_3}{\partial x_0} &= \iint\limits_{S_3} \left\{ \mathrm{Re}\left[U_3' \frac{-\theta_3}{\delta_3'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_3' \frac{-\theta_3}{\delta_3'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_3}{\partial y_0} &= \iint\limits_{S_3} \left\{ \mathrm{Re}\left[U_3' \frac{\sqrt{\frac{1}{a^2} - \theta_3^2}}{-\delta_3'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_3' \frac{\sqrt{\frac{1}{a^2} - \theta_3^2}}{-\delta_3'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_5}{\partial x_0} &= \iint\limits_{S_5} \left\{ \mathrm{Re}\left[U_5' \frac{-\theta_5}{\delta_5'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_5' \frac{-\theta_5}{\delta_5'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_5}{\partial y_0} &= \iint\limits_{S_5} \left\{ \mathrm{Re}\left[U_5' \frac{\sqrt{\frac{1}{b^2} - \theta_5^2}}{-\delta_5'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_5' \frac{\sqrt{\frac{1}{b^2} - \theta_5^2}}{-\delta_5'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_6}{\partial x_0} &= \iint\limits_{S_6} \left\{ \mathrm{Re}\left[U_6' \frac{-\theta_6}{\delta_6'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_6' \frac{-\theta_6}{\delta_6'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy, \\
\frac{\partial J_6}{\partial y_0} &= \iint\limits_{S_6} \left\{ \mathrm{Re}\left[U_6' \frac{\sqrt{\frac{1}{b^2} - \theta_6^2}}{-\delta_6'} \right] \frac{\partial u}{\partial t} + \mathrm{Re}\left[V_6' \frac{\sqrt{\frac{1}{b^2} - \theta_6^2}}{-\delta_6'} \right] \frac{\partial v}{\partial t} \right\} dx\,dy.
\end{aligned} \tag{84}$$

We now have to compute the following two quantities:

$$\frac{\partial J_1}{\partial x_0} + \frac{\partial J_4}{\partial y_0} \quad \text{and} \quad \frac{\partial J_1}{\partial y_0} - \frac{\partial J_4}{\partial x_0},$$

expressing them as integrals containing $\dfrac{\partial U_k}{\partial \theta}$ and $\dfrac{\partial V_k}{\partial \theta}$.

We cannot carry out the differentiations of these expressions by the usual rules of differentiation, since the integrals so obtained do not converge in the ordinary sense. This is because the integrands become infinite at the point (x_0, y_0).

In a neighbourhood of this point

$$u_1 = t_0 \frac{x_0 - x}{r^2} + \tau_1, \qquad v_1 = t_0 \frac{y_0 - y}{r^2} + \sigma_1,$$
$$u_4 = t_0 \frac{y_0 - y}{r^2} + \tau_4, \qquad v_4 = -t_0 \frac{x_0 - x}{r^2} + \sigma_4, \tag{85}$$

where

$$\tau_1 = \frac{t_0 - \sqrt{t_0^2 - \frac{r^2}{a^2}}}{r^2}(x - x_0), \qquad \sigma_1 = \frac{t_0 - \sqrt{t_0^2 - \frac{r^2}{a^2}}}{r^2}(y - y_0),$$
$$\tau_4 = \frac{t_0 - \sqrt{t_0^2 - \frac{r^2}{b^2}}}{r^2}(y - y_0), \qquad \sigma_4 = -\frac{t_0 - \sqrt{t_0^2 - \frac{r^2}{b^2}}}{r^2}(x - x_0) \tag{86}$$

are the regular functions in a neighborhood of the point (x_0, y_0).

We decompose each integral J_1 and J_4 into the sum of two integrals by cutting out in the domains S_1 and S_4 a small disc centered at the point (x_0, y_0) and radius independent of x_0, y_0.

Denoting the disc by σ and the complements of S_1 and S_4 by $\overline{S}_1$ and $\overline{S}_4$, respectively, we have

$$J_1 = \iint\limits_{\overline{S}_1} \left(u_1 \frac{\partial u}{\partial t} + v_1 \frac{\partial v}{\partial t} \right) dx\, dy$$
$$+ \iint\limits_{\sigma} t_0 \left(\frac{\partial u}{\partial t} \frac{x_0 - x}{r^2} + \frac{\partial v}{\partial t} \frac{y_0 - y}{r^2} \right) dx\, dy + \iint\limits_{\sigma} \left(\tau_1 \frac{\partial u}{\partial t} + \sigma_1 \frac{\partial v}{\partial t} \right) dx\, dy,$$
$$J_4 = \iint\limits_{\overline{S}_4} \left(u_4 \frac{\partial u}{\partial t} + v_4 \frac{\partial v}{\partial t} \right) dx\, dy \tag{87}$$
$$+ \iint\limits_{\sigma} t_0 \left(\frac{\partial u}{\partial t} \frac{y_0 - y}{r^2} - \frac{\partial v}{\partial t} \frac{x_0 - x}{r^2} \right) dx\, dy + \iint\limits_{\sigma} \left(\tau_4 \frac{\partial u}{\partial t} + \sigma_4 \frac{\partial v}{\partial t} \right) dx\, dy.$$

The first and third terms on the right sides of these expressions can be differentiated under the integral sign. Thus, we obtain

$$\left(\frac{\partial J_1}{\partial x_0} + \frac{\partial J_4}{\partial y_0} \right) = \iint\limits_{\overline{S}_1} \left\{ \mathrm{Re} \left[U_1' \frac{-\theta_1}{\delta_1'} \right] \frac{\partial u}{\partial t} + \mathrm{Re} \left[V_1' \frac{-\theta_1}{\delta_1'} \right] \frac{\partial v}{\partial t} \right\} dx\, dy$$
$$+ \iint\limits_{\overline{S}_4} \left\{ \mathrm{Re} \left[U_4' \frac{\sqrt{\frac{1}{b^2} - \theta_4^2}}{\delta_4'} \right] \frac{\partial u}{\partial t} + \mathrm{Re} \left[V_4' \frac{\sqrt{\frac{1}{b^2} - \theta_4^2}}{\delta_4'} \right] \frac{\partial v}{\partial t} \right\} dx\, dy$$

$$+\frac{\partial}{\partial x_0}\iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\frac{x_0-x}{r^2}dx\,dy+\frac{\partial}{\partial y_0}\iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\frac{y_0-y}{r^2}dx\,dy$$
$$+\frac{\partial}{\partial x_0}\iint\limits_{\sigma} t_0\frac{\partial v}{\partial t}\frac{y_0-y}{r^2}dx\,dy-\frac{\partial}{\partial y_0}\iint\limits_{\sigma} t_0\frac{\partial v}{\partial t}\frac{x_0-x}{r^2}dx\,dy$$
$$+\iint\limits_{\sigma}\left(\frac{\partial A}{\partial x_0}+\frac{\partial B}{\partial y_0}\right)dx\,dy, \tag{88}$$

where A and B are certain regular functions. It is obvious that

$$\begin{aligned}\iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\frac{x_0-x}{r^2}dx\,dy &= -\frac{\partial}{\partial x_0}\iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\ln\frac{1}{r}dx\,dy,\\ \iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\frac{y_0-y}{r^2}dx\,dy &= -\frac{\partial}{\partial y_0}\iint\limits_{\sigma} t_0\frac{\partial u}{\partial t}\ln\frac{1}{r}dx\,dy.\end{aligned} \tag{89}$$

Then it follows from the classical theory of the logarithmic potential that the sum of the third and fourth terms on the right side of expression (88) tends, as the disc σ contracts to the point (x_0, y_0), to the limit $2\pi t_0\frac{\partial u}{\partial t}(x_0, y_0, 0)$[6]. We now transform expressions (88).

We break the domain of integration in the first term into two parts: $\overline{S}_1 = (S_1 - S_4) + \overline{S}_4$.

The first of the integrals so obtained does not depend on the disc σ. Combining the second integral with the second term of expression (88), it is not difficult to see that the integrand is regular at the point (x_0, y_0). Letting the disc σ contract to the point (x_0, y_0), we obtain the formulas:

$$\frac{\partial J_1}{\partial x_0}+\frac{\partial J_4}{\partial y_0}-2\pi t_0\frac{\partial u}{\partial t}(x_0, y_0, 0)$$
$$=\iint\limits_{S_1-S_4}\left\{\mathrm{Re}\left[U_1'\frac{-\theta_1}{\delta_1'}\right]\frac{\partial u}{\partial t}+\mathrm{Re}\left[V_1'\frac{-\theta_1}{\delta_1'}\right]\frac{\partial v}{\partial t}\right\}dx\,dy$$
$$+\iint\limits_{S_4}\left\{\left[\mathrm{Re}\left(U_4'\frac{\sqrt{\frac{1}{b^2}-\theta_4^2}}{\delta_4'}\right)+\mathrm{Re}\left(U_1'\frac{-\theta_1}{\delta_1'}\right)\right]\frac{\partial u}{\partial t}\right.$$
$$\left.+\left[\mathrm{Re}\left(V_4'\frac{\sqrt{\frac{1}{b^2}-\theta_4^2}}{\delta_4'}\right)+\mathrm{Re}\left(V_1'\frac{-\theta_1}{\delta_1'}\right)\right]\frac{\partial v}{\partial t}\right\}dx\,dy \tag{90.1}$$

and, in exactly the same way,

[6] The last three terms tend to zero. – *Ed.*

$$\frac{\partial J_1}{\partial y_0} - \frac{\partial J_4}{\partial x_0} - 2\pi t_0 \frac{\partial v}{\partial t}(x_0, y_0, 0)$$

$$= \iint\limits_{S_1 - S_4} \left\{ \operatorname{Re}\left[U_1' \frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} \right] \frac{\partial u}{\partial t} + \operatorname{Re}\left[V_1' \frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} \right] \frac{\partial v}{\partial t} \right\} dx\, dy$$

$$+ \iint\limits_{S_4} \left\{ \left[\operatorname{Re}\left(U_1' \frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} \right) + \operatorname{Re}\left(U_4' \frac{\theta_4}{\delta_4'} \right) \right] \frac{\partial v}{\partial t} \right.$$

$$\left. + \left[\operatorname{Re}\left(V_1' \frac{\sqrt{\frac{1}{a^2} - \theta_1^2}}{\delta_1'} \right) + \operatorname{Re}\left(V_4' \frac{\theta_4}{\delta_4'} \right) \right] \frac{\partial v}{\partial t} \right\} dx\, dy. \qquad (90.2)$$

If now in formula (81) the derivatives of M and N are replaced by the formulas we have obtained for them, the expression for the solution takes the following form [7]:

$$u(x_0, y_0, t_0) = u(x_0, y_0, 0) + \frac{1}{2\pi}\frac{\partial I_1}{\partial x_0} + \frac{1}{2\pi}\frac{\partial I_4}{\partial y_0} + \frac{1}{2\pi}\frac{\partial I_2}{\partial x_0}$$

$$+ \frac{1}{2\pi}\frac{\partial I_5}{\partial y_0} + \frac{1}{2\pi}\frac{\partial I_3}{\partial x_0} + \frac{1}{2\pi}\frac{\partial I_6}{\partial y_0} - \frac{1}{2\pi}\left(\frac{\partial J_1}{\partial x_0} + \frac{\partial J_4}{\partial y_0} - 2\pi t_0 \frac{\partial u}{\partial t}(x_0, y_0, 0) \right)$$

$$- \frac{1}{2\pi}\frac{\partial J_2}{\partial x_0} - \frac{1}{2\pi}\frac{\partial J_5}{\partial y_0} - \frac{1}{2\pi}\frac{\partial J_3}{\partial x_0} - \frac{1}{2\pi}\frac{\partial J_6}{\partial y_0}, \qquad (91)$$

$$v(x_0, y_0, t_0) = v(x_0, y_0, 0) + \frac{1}{2\pi}\frac{\partial I_1}{\partial y_0} - \frac{1}{2\pi}\frac{\partial I_4}{\partial x_0} + \frac{1}{2\pi}\frac{\partial I_2}{\partial y_0}$$

$$- \frac{1}{2\pi}\frac{\partial I_5}{\partial x_0} + \frac{1}{2\pi}\frac{\partial I_3}{\partial y_0} - \frac{1}{2\pi}\frac{\partial I_6}{\partial x_0} - \frac{1}{2\pi}\left(\frac{\partial J_1}{\partial y_0} - \frac{\partial J_4}{\partial x_0} - 2\pi t_0 \frac{\partial v}{\partial t}(x_0, y_0, 0) \right)$$

$$- \frac{1}{2\pi}\frac{\partial J_2}{\partial y_0} + \frac{1}{2\pi}\frac{\partial J_5}{\partial x_0} - \frac{1}{2\pi}\frac{\partial J_3}{\partial y_0} + \frac{1}{2\pi}\frac{\partial J_6}{\partial x_0}.$$

The derivatives of the functions I and J in these formulas can be evaluated by means of (82), (84), and (90). Thus, we have brought the formulas for the solution to a more calculable form.

8. Our formulas allow us to explain rigorously the phenomenon known as surface waves or the Rayleigh waves.

In the paper by V. I. Smirnov and S. L. Sobolev mentioned earlier, the problem of surface waves was studied in some particular cases. We can now study the problem in full generality.

Suppose that at the time $t = 0$ all the initial disturbance is concentrated in a bounded domain ω; we apply formulas (91) to the case when t_0 and the coordinate x_0 tend to infinity.

[7] From now on $\varrho = 1$. – *Ed.*

It is easy to see from these assumptions that u and v tend to zero when $\dfrac{x_0}{t_0}$ tends to a finite limit, except in one case which we will now discuss; this will lead to the concept of the Rayleigh surface waves.

Indeed, according to our assumptions, the quantities $\dfrac{1}{\delta'}$ appearing in the expressions for the derivatives of I and J tend to zero, because they contain $x - x_0$ in the denominator.

Consequently, if U', V' and θ all remain finite, the derivatives of I and J also tend to zero. However, as is easy to see, under our assumptions θ converges to a finite limit and U_1' and V_1' are also finite. Therefore, the derivatives of I and J always tend to zero, except the case when the coefficients in formulas (36) and (48) tend to infinity. This occurs only when $\theta \to \pm\dfrac{1}{c}$, where $\dfrac{1}{c}$ is a root of equation (49). In this case the ratio $\dfrac{x_0}{t_0}$ converges to the finite limit $\pm c$.

For the discussion of this case it is convenient to replace $\dfrac{x_0}{t_0}$ by a new variable ξ, where $x_0 - x = \xi \mp ct_0$.

Let us determine the asymptotic behavior of our integrals for finite ξ, as t_0 tends to infinity. For the sake of definiteness we take the plus sign in the last equation. Using the Laurent expansion, we obtain

$$\begin{aligned} \theta_2 &= a_0 + \frac{a_1}{t_0} + \cdots, & \theta_5 &= g_0 + \frac{g_1}{t_0} + \cdots, \\ \theta_3 &= b_0 + \frac{b_1}{t_0} + \cdots, & \theta_6 &= h_0 + \frac{h_1}{t_0} + \cdots \end{aligned} \tag{92}$$

and substituting these expressions into the equations

$$\begin{gathered} (t_0 - t) - \theta_2(x - x_0) - \sqrt{\frac{1}{a^2} - \theta_2^2}(y + y_0) = 0, \\ (t_0 - t) - \theta_3(x - x_0) - \sqrt{\frac{1}{b^2} - \theta_3^2}\,y - \sqrt{\frac{1}{a^2} - \theta_3^2}\,y_0 = 0, \\ (t_0 - t) - \theta_5(x - x_0) - \sqrt{\frac{1}{a^2} - \theta_5^2}\,y - \sqrt{\frac{1}{b^2} - \theta_5^2}\,y_0 = 0, \\ (t_0 - t) - \theta_6(x - x_0) - \sqrt{\frac{1}{b^2} - \theta_6^2}(y + y_0) = 0, \end{gathered} \tag{93}$$

which define θ_k, we obtain for $t = 0$

$$\theta_2 = -\frac{1}{c} + \frac{\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)}{t_0} + \cdots,$$

$$\theta_3 = -\frac{1}{c} + \frac{\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y_0}{t_0} + \cdots ,$$
$$\theta_5 = -\frac{1}{c} + \frac{\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y_0}{t_0} + \cdots , \tag{94}$$
$$\theta_6 = -\frac{1}{c} + \frac{\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,(y + y_0)}{t_0} + \cdots .$$

We also have

$$F(-\frac{1}{c}) = 0 \quad \text{and} \quad F(\theta) = \left(\theta + \frac{1}{c}\right) F'\left(-\frac{1}{c}\right) + \cdots .$$

Replacing θ_k in formulas (36) and (48) by expressions (94), we obtain

$$U_2'(\theta_2) = -\frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right) c\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}}\, t_0 + \cdots ,$$
$$V_2'(\theta_2) = \frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots , \tag{95.1}$$

$$U_3'(\theta_3) = \frac{\left[-4\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 \sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)\frac{1}{c}}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y_0\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots ,$$
$$V_3'(\theta_3) = \frac{\left[-\frac{4}{c^2}\left(\frac{2}{c^2} - \frac{1}{b^2}\right)\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y_0\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots , \tag{95.2}$$

$$U_5'(\theta_5) = \frac{\left[\frac{4}{c^2}\left(\frac{2}{c^2} - \frac{1}{b^2}\right)\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y_0\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots ,$$
$$V_5'(\theta_5) = \frac{4\left(\frac{2}{c^2} - \frac{1}{b^2}\right)\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,\frac{1}{c}i}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\,y + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\,y_0\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots , \tag{96.1}$$

$$U_6'(\theta_6) = \frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right)}\, t_0 + \cdots ,$$
$$V_6'(\theta_6) = \frac{-\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right] i\frac{1}{c}}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right)\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}}\, t_0 + \cdots . \tag{96.2}$$

We now compute the derivatives $\dfrac{\partial\theta}{\partial t}, \dfrac{\partial\theta}{\partial x_0}, \dfrac{\partial\theta}{\partial y_0}$ for $t = 0$:

$$\frac{\partial\theta_2}{\partial t} = -\frac{1}{\delta_2'} = \frac{1}{ct_0} + \cdots, \quad \frac{\partial\theta_2}{\partial x_0} = \frac{1}{c^2 t_0} + \cdots, \quad \frac{\partial\theta_2}{\partial y_0} = \frac{\frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}}{t_0} + \cdots,$$

$$\begin{aligned}
&\frac{\partial\theta_3}{\partial t} = \frac{1}{ct_0} + \cdots, \quad \frac{\partial\theta_3}{\partial x_0} = \frac{1}{c^2 t_0} + \cdots, \quad \frac{\partial\theta_3}{\partial y_0} = \frac{\frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}}{t_0} + \cdots, \\
&\frac{\partial\theta_5}{\partial t} = \frac{1}{ct_0} + \cdots, \quad \frac{\partial\theta_5}{\partial x_0} = \frac{1}{c^2 t_0} + \cdots, \quad \frac{\partial\theta_5}{\partial y_0} = \frac{\frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}}{t_0} + \cdots, \\
&\frac{\partial\theta_6}{\partial t} = \frac{1}{ct_0} + \cdots, \quad \frac{\partial\theta_6}{\partial x_0} = \frac{1}{c^2 t_0} + \cdots, \quad \frac{\partial\theta_6}{\partial y_0} = \frac{\frac{1}{c}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}}{t_0} + \cdots.
\end{aligned} \tag{97}$$

Replacing the derivatives in formulas (82) and (84) by the above expressions, we obtain

$$\begin{aligned}
I_2 = \iint\limits_{S_2} \Bigg\{ & u\mathrm{Re} \left[\frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right) c^2 \sqrt{\frac{1}{a^2} - \frac{1}{c^2}}} \right] \\
&+ v\mathrm{Re} \left[\frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right) c^2} \right] \Bigg\} dx\, dy + \cdots,
\end{aligned} \tag{98}$$

and similarly

$$\begin{aligned}
\frac{\partial J_2}{\partial x_0} = \iint\limits_{S_2} \Bigg\{ & \mathrm{Re} \left[-\frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)\frac{1}{c^2}}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right) c \sqrt{\frac{1}{a^2} - \frac{1}{c^2}}} \right] \frac{\partial u}{\partial t} \\
&+ \mathrm{Re} \left[-\frac{\left[\left(\frac{2}{c^2} - \frac{1}{b^2}\right)^2 - \frac{4}{c^2}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}\sqrt{\frac{1}{b^2} - \frac{1}{c^2}}\right](-i)\frac{1}{c^2}}{\left[\frac{\xi}{c^2} + \frac{1}{c}\sqrt{\frac{1}{a^2} - \frac{1}{c^2}}(y + y_0)\right] F'\left(-\frac{1}{c}\right)} \right] \frac{\partial v}{\partial t} \Bigg\} dx\, dy + \cdots.
\end{aligned} \tag{99}$$

Substituting expressions (98) and (99) in formula (91), we obtain under our assumption the solution in the form of the Laurent series

$$\begin{aligned}
u(\xi, y_0, t_0) &= u_0(\xi, y_0) + \frac{u_1(\xi, y_0)}{t_0} + \cdots, \\
v(\xi, y_0, t_0) &= v_0(\xi, y_0) + \frac{v_1(\xi, y_0)}{t_0} + \cdots.
\end{aligned} \tag{100}$$

The terms u_0 and v_0 in these series are nonzero; they give us the particular solution of the elasticity equations that has the form of the generalized Rayleigh waves. These waves were studied by the author in the paper [2]. So, we see that if the disturbance is concentrated in a finite domain ω, then, generally speaking, as t_0 and x_0 increase to infinity, the displacements become damped; however, there exists a unique velocity – the velocity of the Rayleigh waves, for which the wave is propagated with finite displacement. An observer moving with this velocity parallel to the axis $y = 0$ will see as the final result one and the same displacement for sufficiently large t_0. This wave had been investigated by the author in the article cited.

In conclusion I express my profound gratitude to V. I. Smirnov, the Head of the Theoretical Department of the Seismological Institute of the USSR Academy of Sciences, for his helpful advice, and also to all those employees of the Institute who have assisted the author in his work.

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[8] Paper [2] of Part I of this book. – *Ed.*
[9] Paper [1] of Part I of this book. – *Ed.*

5. On a New Method of Solving Problems about Propagation of Vibrations*

S. L. Sobolev

1. In this article we present in outline a new method of solving a certain class of problems on propagation of vibrations. This approach was developed and applied to a number of problems in the Theoretical Department of the Seismological Institute of the USSR Academy of Sciences.

The basis of the new method is to apply analytic functions of complex variables to solving dynamic problems in the case of one wave equation or the equations of the theory of elasticity. The theory of analytic functions gives a possibility to construct a certain class of solutions of the wave equation. In particular, this class contains fundamental solutions of two- and three-dimensional problems, usually used in the theory of characteristics and making it possible to construct the general solutions of the problems on vibrations of a half-space or a medium consisting of a number of parallel layers. The essential property of solutions of the mentioned class is that we can construct solutions in this class by means of reflection of a given solution from the rectilinear boundary. Some many-valued solutions also belong to this mentioned class. They are used in the problems of diffraction and, in particular, give a possibility to construct the general theory of diffraction of plane waves and diffraction of any perturbation with respect to an angle or a logarithmic point of the Riemann surface in the two-dimensional case.

In this review we follow partly the chronological order of the problems solved by the new method. First, we present one special problem, namely, the general theory of plane waves in an elastic half-space with a free boundary. In this problem we apply the new method only partially. Then we move on to the investigation of mathematical foundations of the new method and its application for solving problems on vibrations of an elastic half-space or a layered medium under the action of forces of special type. The next question is to construct fundamental solutions of the equations of the theory of elasticity and the general solutions of the problems on natural vibrations of the half-space or the layered medium for arbitrary initial conditions. These problems,

* Prikl. Mat. Mekh., **1**, 290–309 (1933)

as well as the previous one, can be solved in both two-dimensional and three-dimensional cases. Finally, we move on to the construction of some many-valued solutions of the wave equation connected with diffraction problems. We begin with the problem of diffraction of plane waves for two and three dimensions, and then move on to the general problem of the diffraction for the two-dimensional case.

2. The standard and most known application of analytic functions of a complex variable to partial differential equations is the application of these functions to the Laplace equation with two variables

$$\varDelta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

Let us briefly recall the basic points connecting equation (1) with the theory of analytic functions. Any real solution of equation (1) is the real part of an analytic function; on the other hand, both real and imaginary parts of any analytic function are solutions of equation (1). We can formally construct solutions of (1). Namely, using the well-known formula of d'Alembert, we obtain the general solution of equation (1) in the form

$$u = f_1(x + iy) + f_2(x - iy).$$

Obviously, the necessity to differentiate the written functions of the complex arguments leads us to the theory of analytic functions of complex variables.

Let us move on to the study of the wave equation

$$\varDelta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{2}$$

This equation has a real solution of the form

$$u = f(t - lx - my - nz), \tag{3}$$

where the constants l, m, and n determining the direction of motion of plane wave (3) must be connected by the equality

$$a^2(l^2 + m^2 + n^2) = 1, \tag{4}$$

and f is an arbitrary function. If we assign to these constants the complex values

$$l = l' + il'', \quad m = m' + im'', \quad n = n' + in'' \tag{5}$$

satisfying equality (4), and assume that the function f is an analytic function of the complex variable, then formula (3) again gives us a solution of equation (2). This solution can be called the complex plane wave. Its real and imaginary parts are arbitrary conjugate harmonic functions of the arguments

$$t - l'x - m'y - n'z, \quad l''x + m''y + n''z. \tag{6}$$

The problem on reflection of plane waves from a plane boundary in an elastic medium was the first problem to be completely solved by using complex plane waves [1–3].

As is known, the fundamental equations of the theory of elasticity can be reduced to the wave equations

$$\Delta\varphi - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad \Delta\overrightarrow{\psi} - \frac{1}{b^2}\frac{\partial^2\overrightarrow{\psi}}{\partial t^2} = 0, \quad a^2 > b^2, \tag{7}$$

where φ is the scalar potential, and the vector $\overrightarrow{\psi}$ is the vector potential. The displacement vector is defined by the formula

$$\mathbf{u} = \operatorname{grad}\varphi + \operatorname{rot}\overrightarrow{\psi}. \tag{8}$$

Such a displacement field defined by formula (8) is called the *plane elastic wave*, if the potentials φ and $\overrightarrow{\psi}$, being the solutions of the wave equations, are given by formulas of form (3).

Choosing an appropriate coordinate system, we can reduce the problem on plane waves to the plane problem of the theory of elasticity when the displacements do not depend on one of the coordinates, for instance, on the y-coordinate; for this it suffices to consider only the case when the displacement vector is located in the (x, z)-plane. In this case instead of the vector $\overrightarrow{\psi}$ we have the scalar ψ equal to the length of the vector $\overrightarrow{\psi}$; the last vector has the y-axis direction. Instead of the formulas given above we obtain two wave equations on scalar functions

$$\begin{aligned} &\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0,\\ &\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} = 0. \end{aligned} \tag{9}$$

The displacement components on the x-axis and the z-axis are defined by the formulas[1]

$$\begin{aligned} u &= \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial z},\\ w &= \frac{\partial\varphi}{\partial z} - \frac{\partial\psi}{\partial x}. \end{aligned} \tag{10}$$

We now move on to the presentation of the general theory of plane waves in the half-space $z > 0$. Boundary conditions on the surface $z = 0$ can be various. We have the greatest interest in the two simplest cases, namely, the

[1] For details see the article of S. L. Sobolev "Some questions of the theory of propagation of vibrations" in the book: Frank, F., Mises, R.: Differential and Integral Equations of Mathematical Physics. Vol. 2. ONTI, Leningrad – Moscow (1937). – *Ed.*

case when this boundary is free of stresses, or the case when this boundary is fixed, i.e., the displacement vector is zero on the boundary. For the sake of definiteness, we discuss the case of a free boundary[2].

Referring to formula (3), we see that in the case of the real parameters l, m, n the vector with the components l, m, n determines the direction of motion of the plane wave. Of course a single plane wave does not satisfy the limiting conditions; to satisfy them we have to add new plane waves. Only the combination of such waves will satisfy not only the differential equations, but the limiting conditions as well. We call the wave *incident* if the angle between the direction of its motion and the z-axis is obtuse, and *reflected* if this angle is acute. Notice that for the given real incident wave we can obtain both the real and complex reflected waves.

Consider first the case when the incident wave is purely longitudinal, i.e., the function ψ in formulas (10) is equal to zero. In this case the functions φ and ψ, determining the entire combination of plane waves, both incident and reflected, are determined by the formulas

$$\begin{gathered}\varphi=\mathrm{Re}\left\{\left[\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}\right]f_1\left(t-\theta x-\sqrt{\frac{1}{a^2}-\theta^2}\,z\right)\right.\\ \left.-\left[\left(2\theta^2-\frac{1}{b^2}\right)^2-4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}\right]f_1\left(t-\theta x+\sqrt{\frac{1}{a^2}-\theta^2}\,z\right)\right\},\\ \psi=\mathrm{Re}\left\{4\theta\sqrt{\frac{1}{a^2}-\theta^2}\left(2\theta^2-\frac{1}{b^2}\right)f_1\left(t-\theta x+\sqrt{\frac{1}{b^2}-\theta^2}\,z\right)\right\}.\end{gathered}\tag{11}$$

In the case of the incident pure transverse wave we obtain the formulas

$$\begin{gathered}\varphi=\mathrm{Re}\left\{-4\theta\sqrt{\frac{1}{b^2}-\theta^2}\left(2\theta^2-\frac{1}{b^2}\right)f_2\left(t-\theta x+\sqrt{\frac{1}{a^2}-\theta^2}\,z\right)\right\},\\ \psi=\mathrm{Re}\left\{\left[\left(2\theta^2-\frac{1}{b^2}\right)^2+4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}\right]f_2\left(t-\theta x-\sqrt{\frac{1}{b^2}-\theta^2}\,z\right)\right.\\ \left.-\left[\left(2\theta^2-\frac{1}{b^2}\right)^2-4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}\right]f_2\left(t-\theta x+\sqrt{\frac{1}{b^2}-\theta^2}\,z\right)\right\}.\end{gathered}\tag{12}$$

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[2] In this case the boundary conditions have the form

$$\left(a^2\frac{\partial^2\varphi}{\partial z^2}+(a^2-2b^2)\frac{\partial^2\varphi}{\partial x^2}-2b^2\frac{\partial^2\psi}{\partial x\partial z}\right)\bigg|_{z=0}=0,$$

$$\left(2\frac{\partial^2\varphi}{\partial x\partial z}+\frac{\partial^2\psi}{\partial z^2}-\frac{\partial^2\psi}{\partial x^2}\right)\bigg|_{z=0}=0. \quad \text{– } \textit{Ed.}$$

In these formulas the symbol Re denotes the real part, and all radicals in the formulas are considered negative when the radicand is positive, and positive imaginary when the radicand is negative. Next, f_1 and f_2 are arbitrary analytic functions, and, finally, θ is a real parameter. If this parameter satisfies the inequality

$$|\theta| < \frac{1}{a},$$

then we obtain the real plane waves; in other cases we have the complex plane waves. These formulas contain both incident and reflected waves; moreover, an incident wave is defined by the term where the argument of the function contains the radical with the plus sign.

Since we deal with an unbounded half-space, it is natural to impose on solutions certain restrictions at infinity, namely, to require that these solutions remain bounded in the entire half-space. For imaginary values of the radicals $\sqrt{\frac{1}{a^2} - \theta^2}$ and $\sqrt{\frac{1}{b^2} - \theta^2}$ we need to deal with values of the functions f_1 and f_2 on the entire plane of the complex variable. Recalling the famous Liouville theorem that the regular function bounded on the entire plane is constant, we have to conclude that in these cases either one of the functions mentioned above or both of them must be constant. Moreover, for this constant we can take zero.

These general arguments lead us to the following conclusions.

If θ satisfies the inequality $|\theta| < \frac{1}{a}$, then for any incident longitudinal or transverse wave we can construct the reflected waves. It is not difficult to obtain from the formulas presented the known rules for the angles of incidence and reflection.

If the parameter θ satisfies the condition $\frac{1}{a} < |\theta| < \frac{1}{b}$, then the argument of the function f_1 is from the entire plane of the complex variable, and formulas (11) are inapplicable. However, formulas (12) remain valid even in this case; it is not difficult to see that in the combination of waves presented by formulas (12) the transverse waves determined by the function f_2 are real and consist of two parts: incident and reflected. Let us point out one essential difference with the previous case. The analytic function f_2 in formulas (12) has both real and imaginary parts. In the present case the coefficients of f_2 in the expression for ψ in formulas (12) are complex and, isolating the real part, we have for ψ the expression containing both real and imaginary parts of f_2, where the incident and reflected transverse waves are given by the different real functions. The longitudinal wave is complex in this case. Here we have the case when the angle of incidence of the transverse wave is greater than the limiting angle of the full inner reflection.

Finally, assume that the parameter θ satisfies the condition $|\theta| > \frac{1}{b}$; in this case, generally speaking, neither formulas (11) nor formulas (12) have any meaning; however, there exists a unique positive value of θ for which we obtain a bounded solution. This value is a root of the equation

$$\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2} = 0. \tag{13}$$

In this case the terms of formulas (11) and (12) containing the argument from the lower half-plane of complex variables vanish, and we can take for the functions f_1 and f_2 any functions regular and bounded in the upper half-plane. The solutions obtained in such a way are the general case of plane surface waves or *Rayleigh waves*. Lord Rayleigh himself presented these formulas for a very particular case of harmonic vibrations, damping with z. The only positive root of equation (13) gives the reciprocal of the propagation speed of the Rayleigh waves.

The general theory of plane waves can be applied also to more general cases of vibration propagations; in particular, it was applied to solve the famous Lamb problem on vibrations of an elastic half-space under influence of a concentrated force applied at a point of a surface. The solution given by H. Lamb for surface points was possible to express as a sum of plane waves spread over the continuous spectrum of θ,

$$-\frac{1}{b} < \theta < +\frac{1}{b}, \tag{14}$$

adding the plane waves corresponding to $\theta = \pm\frac{1}{c}$[3]. Such representation of the solution for $z = 0$ led directly to the possibility to extend it into the half-space, i.e., gave the way to obtain formulas for the displacements also for $z > 0$.

3. We now move on to the study of the mathematical basis of our new method in its generality[4]. Consider the wave equation on the plane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{15}$$

and set a problem of the construction of a solution of this equation having the form

$$u = f(\theta), \tag{16}$$

where f is an arbitrary function, and θ is a certain determined, generally speaking, complex function of the variables x, z, t. The direct substitution into equation (16) gives

$$f''(\theta)\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 - \frac{1}{a^2}\left(\frac{\partial\theta}{\partial t}\right)^2\right] + f'(\theta)\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2\theta}{\partial t^2}\right] = 0,$$

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[3] $\frac{1}{c}$ is the root of the Rayleigh equation (13). – *Ed.*

[4] This method is presented in the works of V. I. Smirnov and S. L. Sobolev [4–7]. – *Ed.*

and since the analytic function f is chosen arbitrary, we obtain for θ the system of two equations

$$\begin{gathered}\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial z}\right)^2-\frac{1}{a^2}\left(\frac{\partial\theta}{\partial t}\right)^2=0,\\ \frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}-\frac{1}{a^2}\frac{\partial^2\theta}{\partial t^2}=0.\end{gathered}\tag{17}$$

The analysis of these equations brings us to the fact that the function θ must appear as the solution of the equation of the form

$$\delta\equiv l(\theta)t+m(\theta)x+n(\theta)z-\chi(\theta)=0,\tag{18}$$

where l, m, n, χ are any analytic functions satisfying only the condition

$$a^2(m^2+n^2)=l^2.\tag{19}$$

Any analytic function $f(\theta)$ is certainly a function of the variables x, z, t, and its derivatives with respect to these variables are expressed via the following formulas:

$$\begin{gathered}\frac{\partial f}{\partial x}=-\frac{m(\theta)}{\delta'}f'(\theta),\qquad\frac{\partial f}{\partial z}=-\frac{n(\theta)}{\delta'}f'(\theta),\qquad\frac{\partial f}{\partial t}=-\frac{l(\theta)}{\delta'}f'(\theta),\\ \frac{\partial^2 f}{\partial x^2}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{m^2}{\delta'}f'(\theta)\right),\qquad\frac{\partial^2 f}{\partial z^2}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{n^2}{\delta'}f'(\theta)\right),\\ \frac{\partial^2 f}{\partial t^2}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{l^2}{\delta'}f'(\theta)\right),\qquad\frac{\partial^2 f}{\partial x\partial z}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{mn}{\delta'}f'(\theta)\right),\\ \frac{\partial^2 f}{\partial x\partial t}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{ml}{\delta'}f'(\theta)\right),\qquad\frac{\partial^2 f}{\partial z\partial t}=\frac{1}{\delta'}\frac{\partial}{\partial\theta}\left(\frac{ln}{\delta'}f'(\theta)\right),\end{gathered}\tag{20}$$

where δ' is the derivative of the left side of equation (18) with respect to θ.

Consider the three-dimensional space S with coordinates x, z, t. If in some domain B of this space equation (18) gives us for θ a certain domain of the plane of the complex variable, then the function f in formula (16) must be analytic in this domain; however, if for the points x, z, t in some domain equation (18) gives for θ values generating a certain line (for instance, if θ is real), then in the formula we can take for $f(\theta)$ any real function defined on the mentioned above line and differentiable. Of course, we can consider such real function as a limiting value of a real part of some analytic function on this line. In the general case we can take, of course, only the real part of an analytic function $f(\theta)$ of the complex variable θ, and thus we obtain real solutions of equation (15).

Consider now equation (18) in detail. Without loss of generality we can assume that the coefficient of $l(\theta)$ is equal to 1, and set $-m(\theta)$ as the new complex variable. Moreover, by the condition we reduce equation (18) to the form

$$t - \theta x \pm \sqrt{\frac{1}{a^2} - \theta^2}\, z - \chi(\theta) = 0. \tag{21}$$

We can obtain another form of this equation convenient for applications, if we assume

$$l(\theta) = a, \qquad m(\theta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right),$$

at the same time our main equation is reduced to the form

$$at - \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) x \pm \frac{i}{2}\left(\zeta - \frac{1}{\zeta}\right) z - \chi(\zeta) = 0. \tag{22}$$

The preceding arguments define a certain class of solutions of equation (15). We select from this class some interesting solutions, namely, assume that equation (21) has the special form

$$(t - t_0) - \theta(x - x_0) \pm \sqrt{\frac{1}{a^2} - \theta^2}\,(z - z_0) = 0 \tag{23}$$

or that the same equation (22) has the form

$$a(t - t_0) - \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)(x - x_0) \pm \frac{i}{2}\left(\zeta - \frac{1}{\zeta}\right)(z - z_0) = 0. \tag{24}$$

An arbitrary analytic function ζ, where ζ is the solution of (24), gives us the solution of equation (15) depending on two arguments

$$\xi = \frac{x - x_0}{a(t - t_0)}, \qquad \eta = \frac{z - z_0}{a(t - t_0)}, \tag{25}$$

i.e., gives us the homogeneous solutions of zero order of the arguments $x - x_0$, $z - z_0$, $t - t_0$. One also can prove the inverse statement, namely: any homogeneous solution of equation (15) can be obtained in the above way.

4. Let us study now in detail the homogeneous solutions mentioned above, where for simplicity's sake we assume that $x_0 = z_0 = t_0 = 0$. Introduce instead of x, z the polar coordinates

$$x = r\cos\vartheta, \qquad z = r\sin\vartheta. \tag{26}$$

Then equation (24) becomes

$$at - \frac{1}{2} r\left(\zeta e^{-i\vartheta} + \frac{1}{\zeta} e^{i\vartheta}\right) = 0, \tag{27}$$

from where for ζ we obtain

$$\zeta = \left(\frac{at}{r} \pm \sqrt{\frac{a^2t^2}{r^2} - 1} \right) e^{i\vartheta}. \tag{28}$$

Choosing the "$-$" sign of the radical, we obtain the following law of correspondence. The disk $\xi^2 + \eta^2 < 1$ on the (ξ, η)-plane corresponds to the unit disk $|\zeta| < 1$ on the plane of the variable ζ, and each radius of the first disk corresponds to the radius of the second disk with the same central angle. Thus, inside the disk

$$\xi^2 + \eta^2 < 1 \tag{29}$$

the solution of the wave equation is given by the formula

$$u = \mathrm{Re}\,\{f(\zeta)\}, \tag{30}$$

where f is an analytic function in the unit disk. On the boundary of disk (29), equation (27) gives for ζ a double root with module equal to 1. Finally, moving to the exterior of disk (29), we obtain for ζ two distinct roots with module equal to 1; these roots remain constant on tangents to the boundary of disk (29). Thus, we see that the structure of the solution is completely different inside and outside the disk.

Let us explain now in general the possible ways of continuing the solution from disk (29) into the exterior of this disk. Suppose that we have inside the disk the homogeneous solution given by formula (30).

On the boundary of disk (29) this solution has a certain real value. We can present arbitrarily these real values in the form of a sum of two real terms and correspondingly we can present the analytic function f in the form of a sum of two analytic functions,

$$f(\zeta) = f_1(\zeta) + f_2(\zeta).$$

Let us draw now two systems of half-tangents to the boundary of disk (29), as in Fig. 1, and assume that outside the disk the real function u_1 is defined in the following way. On each of the half-tangents of the first family it remains constant, namely, it is equal to the value of the real part of f_1 at the tangency point. Similarly, we define the function u_2 on the second family of half-tangents, by using the values of the real part of f_2 on the circle.

Fig. 1.

The sum $u = u_1 + u_2$ gives us one of the possible continuations of solution (30) defined inside disk (29). Obviously, with such continuation the continuity is kept on the move over the circle, however, from above we see that such continuation is not single valued. In concrete problems the choice of continuation is related to the physical conditions of the problem, namely, to the well-known Fermat principle on the propagation of the disturbance front.

If instead of ξ, η we consider the three-dimensional space S with the coordinates x, z, t, then any point ξ, η corresponds to the certain half-line emitting from the coordinate origin (we assume $\theta > 0$), the inner part of disk (29) corresponds to the conic beam of half-lines with an apex at the origin and an apex angle $\arctan a$. We can say that this conic beam gives that part of the half-space, where the disturbance has propagated, appearing at $t = 0$ at the point $(0, 0)$ and propagating with the speed a. Inside of this conic beam equations (23) and (24) give for ζ and θ complex values filling up some domain, and the solution of the wave equation is obtained as a real part of a certain analytic function of one of these variables. For the variable ζ the above mentioned domain is the unit disk, and for the variable θ it is the entire plane with the cut $-\dfrac{1}{a} < \theta < +\dfrac{1}{a}$ along the real axis. In the exterior of this conic beam we can obtain a solution assuming that u remains constant on the half-planes tangent to the surface of the conic beam.

5. Let us apply the previous results on the homogeneous solutions to the Lamb problem of vibrations in the half-plane $z > 0$ under the action of a concentrated force at the moment $t = 0$ at the point $x = 0$, $z = 0$ along the direction parallel to the z-axis. We can show that in the presence of this force the potentials φ and ψ of longitudinal and transverse waves should be homogeneous solutions of the corresponding wave equations

$$\varDelta\varphi - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad \varDelta\psi - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{31}$$

Therefore, we seek these potentials in the form of the real parts of some analytic functions

$$\varphi = \mathrm{Re}\,\{\varPhi(\theta_1)\}, \qquad \psi = \mathrm{Re}\,\{\varPsi(\theta_2)\}, \tag{32}$$

where the complex variables θ_1 and θ_2 are determined from the equations

$$t - \theta_1 x + \sqrt{\frac{1}{a^2} - \theta_1^2}\, z = 0, \quad t - \theta_2 x + \sqrt{\frac{1}{b^2} - \theta_2^2}\, z = 0. \tag{33}$$

To define functions $\varPhi$ and $\varPsi$ we have limiting conditions expressing the absence of stresses on the boundary $z = 0$. The fact is essential that for $z = 0$ equations (33) coincide, and hence for each point of the boundary the variables θ_1 and θ_2 are equal. It gives us the possibility to express the limiting conditions via the same complex variable θ. Using differential formulas (20) and equating to zero both components of the stress vector acting at the boundary $z = 0$, we obtain two equations for determining the unknown functions Φ and Ψ. Adding the natural conditions at infinity to these equations, we obtain the following expressions for the unknown functions:

$$\begin{aligned} \Phi'(\theta_1) &= iC\frac{2\theta_1^2 - \frac{1}{b^2}}{\left(2\theta_1^2 - \frac{1}{b^2}\right)^2 + 4\theta_1^2\sqrt{\frac{1}{a^2} - \theta_1^2}\sqrt{\frac{1}{b^2} - \theta_1^2}}, \\ \Psi'(\theta_2) &= iC\frac{2\theta_2\sqrt{\frac{1}{b^2} - \theta_2^2}}{\left(2\theta_2^2 - \frac{1}{b^2}\right)^2 + 4\theta_2^2\sqrt{\frac{1}{a^2} - \theta_2^2}\sqrt{\frac{1}{b^2} - \theta_2^2}}, \end{aligned} \tag{34}$$

where C is a real constant proportional to the impulse of an instantaneous force.

Let us analyze in general the formulas obtained. For the complex variable θ_1, we have the entire plane with the cut $-\frac{1}{a} < \theta < \frac{1}{a}$, where the points of this cut correspond to the generators of the surface of the conic beam mentioned in the previous section. For the variable θ_2 we need to change $\frac{1}{a}$ for $\frac{1}{b}$, where a is the speed of propagation of longitudinal waves, and b is the speed of transverse waves. Only the half-plane $z > 0$ has a physical meaning, and in this respect we have to consider not an entire conical beam, but rather only its half, and for the complex variables θ_1 and θ_2 we have only the upper half-plane with the upper lip of the corresponding cut. For the complex variable θ_1 we have on this cut $-\frac{1}{a} < \theta < \frac{1}{a}$, and the first formula in (34) gives us for Φ' pure imaginary values. This leads us to the fact that the longitudinal potential φ vanishes on the surface of the conic beam, and we assume it is zero outside the beam as well. A somewhat different circumstance holds for the transverse potential. Here we have on the cut the condition $-\frac{1}{b} < \theta < \frac{1}{b}$, and the second formula in (34) gives us the complex value of Ψ' for $\frac{1}{a} < |\theta| < \frac{1}{b}$. Thus, on certain generators of the surface of the conic beam the potential ψ is not zero. Using elementary arguments, we get convinced that outside the beam it has to be continued in such a way so it remains constant on the half-planes tangent to the surface of the conic beam and going to the boundary $z = 0$. This special property of the potentials φ and ψ is connected to the fact that the longitudinal waves propagating along the boundary $z = 0$ more rapidly than the transverse waves generate (in view of the limiting conditions) also transverse waves, which pass ahead of waves propagating with the usual speed b.

Let us point out one more fact concerning formulas (34). If θ coincides with a root of the equation

$$\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2} = 0, \tag{35}$$

then formulas (34) give infinite values of the potentials. We have already encountered the real root of equation (35) in the theory of plane waves. As before, this leads to the appearance of surface waves.

6. We now move on to the study of the general laws of reflections of special elastic vibrations from a rectilinear boundary. Here, as in the Lamb problem, we deal with the two-dimensional case [8, 9]. Assume that in the half-plane $z > 0$ we have a longitudinal disturbance with the potential determined by the formula

$$\varphi = \mathrm{Re}\,\{\Phi(\theta)\}, \tag{36}$$

where the complex variable θ satisfies the usual equation of the form

$$\delta \equiv t - \theta x + \sqrt{\frac{1}{a^2} - \theta^2}\, z - \chi(\theta) = 0. \tag{37}$$

In the general case the given solution can be defined both in the domain of the (x, z, t)-space, where equation (37) is transformed to the complex values of θ, filling up some domain, and in the domain of this space where equation (37) gives, for instance, the real value for θ. Suppose that the analytic function $\Phi(\theta)$ and continuation of the solution into the domain of real values of θ are defined such that, in a neighbourhood of the boundary $z = 0$, there are no disturbances up to some point of time. Beginning at the moment when the given disturbances reach the surface, we have to add to the given disturbance, which we call the incident wave, also the reflected longitudinal wave and the reflected transverse wave. We seek for these reflections of the wave in the usual form

$$\varphi_1 = \mathrm{Re}\,\{\Phi_1(\theta_1)\}, \quad \psi_2 = \mathrm{Re}\,\{\Psi_2(\theta_2)\}, \tag{38}$$

where the complex variables θ_1 and θ_2 must satisfy the equations of the form

$$\begin{aligned} \delta_1 &\equiv t - \theta_1 x - \sqrt{\frac{1}{a^2} - \theta_1^2}\, z - \chi(\theta_1) = 0, \\ \delta_2 &\equiv t - \theta_2 x - \sqrt{\frac{1}{b^2} - \theta_2^2}\, z - \chi(\theta_2) = 0. \end{aligned} \tag{39}$$

These equations are written in such a way that values of the variables θ_1 and θ_2 coincide with θ for $z = 0$. Let us point out as well that in equations (39) we choose the radical sign opposite to the sign that we have in equation (37). This guarantees us that the reflected disturbances do not change the movement picture at the time preceding the reflection. Substituting into the limiting conditions the longitudinal potential $\varphi + \varphi_1$ and the transverse potential ψ_2 and using the corresponding differential formulas (20), we obtain two equations for determining the unknown functions Φ_1' and Ψ_2'. The essential point here is that the complex variables θ, θ_1, θ_2 coincide on the boundary $z = 0$.

The formulas presenting the final answer have the form

$$
\begin{aligned}
\Phi_1'(\theta_1) &= \frac{-\left(2\theta_1^2 - \frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2} - \theta_1^2}\sqrt{\frac{1}{b^2} - \theta_1^2}}{\left(2\theta_1^2 - \frac{1}{b^2}\right)^2 + 4\theta_1^2\sqrt{\frac{1}{a^2} - \theta_1^2}\sqrt{\frac{1}{b^2} - \theta_1^2}}\Phi'(\theta_1), \\
\Psi_2'(\theta_2) &= \frac{-4\theta_2\sqrt{\frac{1}{a^2} - \theta_2^2}\left(2\theta_2^2 - \frac{1}{b^2}\right)}{\left(2\theta_2^2 - \frac{1}{b^2}\right)^2 + 4\theta_2^2\sqrt{\frac{1}{a^2} - \theta_2^2}\sqrt{\frac{1}{b^2} - \theta_2^2}}\Phi'(\theta_2).
\end{aligned}
\tag{40}
$$

Notice also that they do not contain at all the function $\chi(\theta)$, which appeared in equation (37). In exactly the same way we solve the general problem of reflection from a rectilinear boundary, also in the case when the given incident wave is the disturbance of the purely transverse type

$$\psi = \mathrm{Re}\,\{\Psi(\theta)\}, \tag{41}$$

where θ is defined from the equation

$$\delta \equiv t - \theta x + \sqrt{\frac{1}{b^2} - \theta^2}\, z - \chi(\theta) = 0. \tag{42}$$

In all these cases, using physical conditions of the problem, we have to continue solutions from the domain of complex values of θ into the domain where this variable has a real value, as we did, for example, in the Lamb problem, continuing the values of θ into the exterior of the conic beam.

For an example we consider the case when the incident wave is a disturbance propagating from the source located at the point $x = 0$, $z = z_0$ and issuing at the moment $t = 0$ such vibrations for which $\varphi = 0$, and ψ are the solutions of the wave equation, homogeneous with respect to the arguments x, $z - z_0$, t.

In this case the incident wave is determined from the formula

$$\psi = \mathrm{Re}\,\{\Psi(\theta)\}, \tag{43}$$

where θ is the solution of the equation

$$\delta \equiv t - \theta x + \sqrt{\frac{1}{b^2} - \theta^2}\,(z - z_0) = 0. \tag{44}$$

The function $\Psi(\theta)$ is the given function of the complex variable, regular on the plane θ with the cut $-\dfrac{1}{b} < \theta < \dfrac{1}{b}$. This function has a singular point at infinity corresponding to the source of vibrations, and its real part must vanish on the cut corresponding to the front of the propagating disturbance. In this case the reflected transverse wave is also defined by the homogeneous solution

$$
\begin{aligned}
\psi_1 &= \mathrm{Re}\,\{\Psi_1(\theta_1)\}, \\
\delta_1 &\equiv t - \theta_1 x - \sqrt{\frac{1}{b^2} - \theta_1^2}\,(z + z_0) = 0,
\end{aligned}
\tag{45}
$$

with the center at the point $x = 0$, $z = -z_0$. The reflected longitudinal wave has more complex structure

$$\begin{aligned} &\psi_2 = \operatorname{Re}\{\Psi_2(\theta_2)\}, \\ &\delta_2 \equiv t - \theta_2 x - \sqrt{\frac{1}{a^2} - \theta_2^2}\, z - \sqrt{\frac{1}{b^2} - \theta_2^2}\, z_0 = 0, \end{aligned} \tag{46}$$

and its front is a certain algebraic curve. On the attached picture (see Fig. 2) we see the geometric picture of the front of disturbance at a point of time after the reflection.

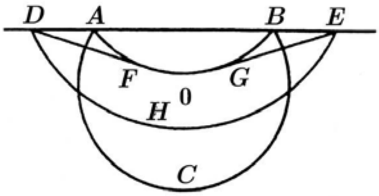

Fig. 2.

The part of the disk ACB is the domain occupied by the incident transverse wave. The algebraic curve DHE limits the domain of reflected longitudinal disturbance. The reflected transverse disturbance occupies particularly the segment $AFGB$, inside which the corresponding complex variable takes a complex value, and particularly this disturbance is located in the triangles DAF and BGE, where the value of this variable is real and belongs to the intervals

$$-\frac{1}{b} < \theta < -\frac{1}{a} \quad \text{and} \quad \frac{1}{a} < \theta < \frac{1}{b}. \tag{47}$$

Here, as in the Lamb problem, we have the phenomenon of surface waves.

7. We now move on to the description of the new method in three-dimensional space. The main point is the principle of superposition of plane waves. Choose the rotating system of coordinates

$$X = r\cos(\vartheta - \lambda), \quad Y = r\sin(\vartheta - \lambda), \tag{48}$$

where r, ϑ, z are cylindrical coordinates and λ is an arbitrary parameter. Let us take a solution of the plane problem in the unbounded space of coordinates X, z, depending on the parameter λ. According to the general rule, we have, for instance, for the potential of longitudinal waves

$$\begin{aligned} &\varphi = \operatorname{Re}\{\Phi(\theta_{\vartheta-\lambda}, \lambda)\}, \\ &\delta \equiv t - \theta_{\vartheta-\lambda} r\cos(\vartheta - \lambda) + \sqrt{\frac{1}{a^2} - \theta_{\vartheta-\lambda}^2}\, z - \chi(\theta_{\vartheta-\lambda}) = 0. \end{aligned} \tag{49}$$

Integrating with respect to the parameter λ, we have an expression for the potential of longitudinal waves in three-dimensional space,

$$\varphi(r,\vartheta,z) = \mathrm{Re}\int\limits_0^{2\pi} \Phi(\theta_{\vartheta-\lambda},\lambda)\,d\lambda. \tag{50}$$

To obtain a potential of transverse disturbances we need to apply the same process. In the solution of the plane problem

$$\begin{aligned} &\psi = \mathrm{Re}\,\{\Psi(\theta_{\vartheta-\lambda},\lambda)\},\\ &\delta \equiv t - \theta_{\vartheta-\lambda} r\cos(\vartheta-\lambda) + \sqrt{\frac{1}{b^2} - \theta^2_{\vartheta-\lambda}}\,z - \chi(\theta_{\vartheta-\lambda}) = 0, \end{aligned} \tag{51}$$

we should take Ψ as the length of a certain vector parallel to the y-axis, and then we add geometrically these vectors corresponding to different values of λ. This would lead us to the vector potential with components on the r-axis and the ϑ-axis of the cylindrical system of coordinates determined from the formulas

$$\begin{aligned} \psi_r &= \mathrm{Re}\int\limits_0^{2\pi} \Psi(\theta_{\vartheta-\lambda},\lambda)\sin(\vartheta-\lambda)\,d\lambda,\\ \psi_\vartheta &= \mathrm{Re}\int\limits_0^{2\pi} \Psi(\theta_{\vartheta-\lambda},\lambda)\cos(\vartheta-\lambda)\,d\lambda. \end{aligned} \tag{52}$$

This process leads us again to the construction of a certain class of solutions of the equations of the elasticity theory in three dimensions. With a help of this class of solutions we can consider, for the case of three dimensions, problems similar to those mentioned above in the case of two dimensions. Together with this class of solutions it is useful to introduce some new solutions as well, which are obtained by rotation of the transverse waves for which the displacement vector is parallel to the z-axis. Studying the solutions which could be expressed in complex form, we obtain in the plane case solutions of the form

$$\begin{aligned} &V = -\frac{\partial\psi_z}{\partial x}, \quad \psi_z = \mathrm{Re}\,\{\Psi_z(\theta'_{\vartheta-\lambda},\lambda)\},\\ &\delta' \equiv t - \theta'_{\vartheta-\lambda} r\cos(\vartheta-\lambda) + \sqrt{\frac{1}{b^2} - \theta'^2_{\vartheta-\lambda}}\,z - \chi(\theta'_{\vartheta-\lambda}) = 0. \end{aligned} \tag{53}$$

Integrating with respect to λ, we obtain the vector potential determined by a vector parallel to the z-axis, and the length of this vector is found from the formula

$$\psi_z = \mathrm{Re}\int\limits_0^{2\pi} \Psi_z(\theta'_{\vartheta-\lambda},\lambda)\,d\lambda. \tag{54}$$

It can be proved among other facts that all solutions of the equations of elasticity theory, for which the potentials are homogeneous functions, are contained in the set of the solutions introduced.

Let us discuss in greater detail the case when the potential is a homogeneous function of the variables r, z, t (it does not depend on ϑ), and when the displacement vector is located in the meridian plane. In this case the disturbances have axial symmetry with respect to the z-axis. The scalar potential is a homogeneous solution of the wave equation, and the vector potential is directed along the z-axis of the cylindrical system of coordinates. For the scalar potential we have the formula

$$\varphi = \operatorname{Re} \int\limits_0^\pi \Phi(\theta_{\vartheta-\lambda})\, d\lambda, \tag{55}$$

where

$$\delta \equiv t - \theta_{\vartheta-\lambda} r \cos(\vartheta - \lambda) + \sqrt{\frac{1}{a^2} - \theta^2_{\vartheta-\lambda}}\, z = 0. \tag{56}$$

We can attain by the appropriate choice of an analytic function that, without changing the potential φ, we can exclude the symbol of the real part from the formula determining this potential, i.e., we have

$$\varphi = \int\limits_0^\pi \Phi(\theta_{\vartheta-\lambda})\, d\lambda. \tag{57}$$

One of the main problems is determining an analytic function $\Phi(\theta_{\vartheta-\lambda})$, given the potential $\varphi(\xi, \eta) = \varphi\left(\frac{r}{t}, \frac{z}{t}\right)$. As it occurs, it can be done by the final formula

$$\Phi(\theta) = \frac{1}{\pi} \varphi \left(0, \frac{1}{\sqrt{\frac{1}{a^2} - \theta^2}} \right). \tag{58}$$

Similarly, we can determine an integrand and an expression for the potential of transverse waves

$$\psi'(\theta) = \frac{2}{\pi} \frac{\frac{\partial \psi}{\partial \xi}\left(0, \frac{1}{\sqrt{\frac{1}{b^2} - \theta^2}}\right)}{\theta \sqrt{\frac{1}{b^2} - \theta^2}}. \tag{59}$$

The main property of the class of solutions introduced is the possibility to construct relatively easily the law of reflection from the rectilinear boundary to obtain the reflected potentials for solutions determined by formulas (50) and (54). It is simple enough to produce the reflection of those plane waves whose rotations determine the given potentials. Obviously, this reflection process leads us to solutions satisfying the limiting conditions; however, we need to verify that the additional terms determining the reflected waves do not change the movement picture before the reflection. We can show that it will be actually so, if we explain in more detail the connection between the potentials φ, ψ and the integrand in formulas (50) and (54). Moreover, we need here to continue appropriately the solution as we have done in the two-dimensional case.

Let us notice one essential difference related to mechanical properties of homogeneous solutions for two- and three-dimensional cases. In the case of two dimensions the homogeneous potentials of zero degree give a solution of the problem on the action of instantaneous impulse at some point, or, better to say, some line, taking into account the third dimension. The homogeneous potentials in the three-dimensional case no longer present a concentrated impulse, but a force concentrated at the assigned place and acting from the prescribed moment of time on (turned on force). Superposing such turned on forces, we can obtain any force.

8. One of the basic problems which can be solved by using the theory presented is the problem about natural vibrations of the half-space and layered media [9]. The problem consists of the following: at the initial moment $t = 0$, given the values of displacements and its velocities in the coordinate functions

$$\begin{gathered} u|_{t=0} = u_0(x,y,z), \qquad v|_{t=0} = v_0(x,y,z), \qquad w|_{t=0} = w_0(x,y,z), \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_0'(x,y,z), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = v_0'(x,y,z), \quad \left.\frac{\partial w}{\partial t}\right|_{t=0} = w_0'(x,y,z), \end{gathered} \tag{60}$$

we have to find the displacements at all moments of time. The method of solution of this problem for an unbounded medium was proposed by V. Volterra. It is based on application of the so-called method of characteristics. Let us discuss in detail this method in the case of two dimensions. If in the (x, z, t)-space there are given two solutions of the elasticity equations whose displacement vectors are u, w and u_1, w_1, and the components of stress tensor are X_x, X_z, Z_z and $X_x^{(1)}$, $X_z^{(1)}$, $Z_z^{(1)}$, then there exists the formula

$$\begin{aligned} \iint\limits_S \Big\{ & (X_x u_1 + X_z w_1 - X_x^{(1)} u - X_z^{(1)} w) \cos nx \\ & + (X_z u_1 + Z_z w_1 - X_z^{(1)} u - Z_z^{(1)} w) \cos nz \\ & - \varrho \left(\frac{\partial u}{\partial t} u_1 + \frac{\partial w}{\partial t} w_1 - \frac{\partial u_1}{\partial t} u - \frac{\partial w_1}{\partial t} w \right) \cos nt \Big\} \, dS = 0, \end{aligned} \tag{61}$$

where S denotes an arbitrary closed surface in the (x, z, t)-space, inside which the displacement vectors have continuous first derivatives. Applying this formula relatively to the chosen surfaces, and taking as one of the solutions the desired solution, and for another one a specially chosen solution, we can find the value of the unknown solution in an arbitrary point of the space. As for the surfaces for which we apply formula (61), it is necessary to take closed surfaces bounded by the conic surface

$$a^2(t - t_0)^2 = (x - x_0)^2 + (y - y_0)^2 \tag{62}$$

or

$$b^2(t - t_0)^2 = (x - x_0)^2 + (y - y_0)^2, \tag{63}$$

by the plane $t = 0$ and a small cylinder with the axis $x = x_0$, $y = y_0$.

The described method solves the problem completely in unbounded space.

In the framework of the present article we cannot discuss in more detail the specifics of applying this method in the case of half-space. It is essential that the special solutions used by V. Volterra belong to the class of solutions studied by us, and for which we have obtained the reflection. We should point out a small difference with the above discussion. While in our previous solutions of equations of the elasticity theory we assumed that the potentials are expressed in terms of functions of complex variable via formulas (16), here we deal with the case when the displacements rather than the potentials are expressed in such form. This circumstance, however, is not difficult in principle. Obviously, the components of the displacement vector satisfy the wave equation, and we can construct solutions where these components are expressed in form (16). The formulas giving values of the displacement vector in two particular solutions used by V. Volterra have the form

$$\begin{aligned} u_1 &= \operatorname{Re}\left\{-i\sqrt{\tfrac{1}{a^2} - \theta_1^2}\right\}, \qquad & w_1 &= \operatorname{Re}\{-i\theta_1\}, \\ u_2 &= \operatorname{Re}\{-i\theta_2\}, & w_2 &= \operatorname{Re}\left\{i\sqrt{\tfrac{1}{b^2} - \theta_2^2}\right\}, \end{aligned} \tag{64}$$

where θ_1 is given by formula (23), and θ_2 differs from θ_1 by substituting b for a. In view of the fact that the theory of reflections from a boundary of such type solutions has been developed, it is possible to employ the Volterra method in the case of bounded space.

A detailed analysis allows us to construct the theory of surface waves also for this problem. If at the initial moment $t = 0$ the disturbance was concentrated in some bounded domain of the half-space, then the wave from the class of the surface waves studied in the theory of complex plane waves will propagate from this domain with Rayleigh velocity c.

Besides the problem of vibrations in the half-space, we can solve in the same way the problem about the vibrations of an arbitrary medium consisting of parallel layers with different physical properties.

The problem of free vibrations in three dimensions is solved by the same method in principle. Instead of formula (61) we have to write a generalization of this formula in the case of four-dimensional (x, y, z, t)-space. The form of particular solutions used here is also somewhat different. It is the most convenient to use for this purpose solutions not possessing an axial symmetry. We are not going to discuss the form of these particular solutions itself. The essential feature is that here, as in the case of two dimensions, the integrals of type (50) and (52) express not the potentials, but the displacements themselves. For these particular solutions it is possible to find their finite representation and, building the theory of reflections, proceed as in the two-dimensional case.

9. We have presented how to solve the basic questions of the theory of reflection of elastic waves from a plane boundary for cases of free and forced vibrations. Without discussing how to solve the questions of diffractions, which analysis is based on the same principle, we move on to another application of our method of complex variables. Here we have in mind the theory of the wave diffraction [10, 11].

One of the main problems of diffraction is the problem of diffraction of the wave around an obstacle in the form of an angle or a screen. Mathematically this problem is stated as follows. We have to find a solution of the wave equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{65}$$

for the domain in the space bounded by two planes passing through the z-axis, i.e., the domain

$$0 < \vartheta < \alpha, \tag{66}$$

where ϑ is the cylindrical coordinate of the space with the boundary conditions

$$u|_{\vartheta=0} = 0, \qquad u|_{\vartheta=\alpha} = 0 \tag{67}$$

or

$$\left.\frac{\partial u}{\partial n}\right|_{\vartheta=0} = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\vartheta=\alpha} = 0. \tag{68}$$

Usually in this case $\pi < \alpha < 2\pi$. Already A. Sommerfeld[5] proved the equivalence of these problems with the problem of solving the wave equation in a many-sheeted space with a branching line on the bending edge $z = 0$. This proof resembles the known arguments from the theory of reflections of waves from the ends of free and fixed string. For example, consider such vibration where besides boundary conditions (67) there are also given the initial conditions

$$\begin{aligned} u|_{t=0} &= u_0(r, \vartheta, z), \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= u_0'(r, \vartheta, z). \end{aligned} \tag{69}$$

Simultaneously with domain (66) let us consider the domain $-\alpha < \vartheta < 0$, and define additionally in this domain the initial conditions as follows:

$$\begin{aligned} u_0(r, \vartheta, z) &= u_0(r, -\vartheta, z), \\ u_0'(r, \vartheta, z) &= u_0'(r, -\vartheta, z). \end{aligned} \tag{70}$$

[5] See Frank, F., Mises, R.: Differential and Integral Equations of Mathematical Physics. Vol. 2. ONTI, Leningrad – Moscow (1937). – *Ed.*

Obviously, because of the complete symmetry of the problem, the value of displacements on the neutral plane ϑ is zero. Repeating many times the reflection process from both arms of the angle, we arrive at the desired results. Thus, if we continue respectively the initial conditions, then the problem can be reduced to the problem of vibration of the domain $-\infty < \vartheta < +\infty$ with arbitrary initial conditions. However, this domain is not our usual space. It is the Riemann space with the logarithmic branching line $r = 0$, since all functions of the coordinates which we study, generally speaking, are not periodic functions of ϑ anymore with the period 2π. They have different period. Therefore we have to place points with coordinates ϑ different by $2k\pi$ on distinct sheets of the Riemann space. When we consider the problem of forced vibrations, the method of its reduction to the many-sheeted problem coincides with the above one. Instead of the initial conditions we have to place in the Riemann space the reflected sources in corresponding points.

The solution of the problem by the method of complex variables described below does not impose any restrictions like periodicity. For the particular example let us consider first the plane problem. If we construct the complex variable ζ by the formula

$$\zeta = \left(\frac{at}{r} - \sqrt{\frac{a^2t^2}{r^2} - 1} \right) e^{i\vartheta}, \tag{71}$$

then any function of this variable with branching points at $\zeta = 0$ produces a branching solution of the wave equation with the branching point $r = 0$. Some of these solutions immediately give the answer to the specific physical problems. Consider one of such problems, namely, the problem of diffraction of the plane wave. The essence of this problem is in the following. In the (x, y, t)-space with the logarithmic branching point at the coordinate origin the elementary plane wave propagates over one sheet. For $t < 0$ this wave is given by the formula

$$\begin{aligned} &u = 0 \quad \text{for} \quad r\cos\vartheta < -at \quad \text{or} \quad |\vartheta| > \frac{\pi}{2}, \\ &u = 1 \quad \text{for} \quad r\cos\vartheta > -at, \quad |\vartheta| < \frac{\pi}{2}. \end{aligned} \tag{72}$$

This wave is a plane disturbance with the front parallel to the y-axis moving with velocity a toward the origin. Before a certain point the disturbance $u = 0$, and after $u = 1$.

Let us examine the same motion after the wave has passed through the logarithmic point. It can be proved that this solution after diffraction must be a homogeneous function of zero order with respect to the coordinates and time, i.e., inside the disk $r < at$ it must be expressed as the real part of a function of complex variable ζ, and in the exterior of the disk it has to continue via one of the laws described above (see Fig. 3). The interior of the disk is the domain where the phenomenon of diffraction has influence. It is not difficult to establish the value of the function of complex variable $f(\zeta)$ on the contour of the domain from physical arguments. These values must be such that u is continuous on the passage from the interior of the disk $r < at$ into the exterior of this disk, where the phenomenon of diffraction cannot have influence. Because of the established law of correspondence of the points of the disk $r < at$ with the points of the disk $|\zeta| < 1$, we can obtain the answer to the question.

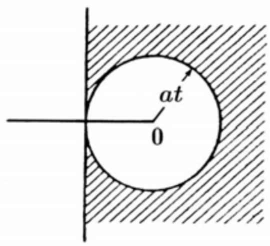

Fig. 3.

For convenience let us enumerate the sheets of the Riemann surface from $-\infty$ to $+\infty$, making a cut from the origin along the negative axis and letting the sheet where $-\pi < \vartheta < +\pi$ be the zero sheet. On the zero sheet, as we can see from the picture, the diffraction disk everywhere borders with the disturbed domain shaded on the picture and where, consequently, $u = 1$. On all other sheets it borders with the domain where $u = 0$. Thus, we have to construct such a harmonic function which on the zero sheet is equal to 1 on the contour, and to 0 on other sheets. Such a harmonic function is known [12]. It is given by the formula

$$u = \mathrm{Re}\left\{\frac{1}{\pi i}\ln\left[\frac{1}{i}\ln\zeta - \pi\right] - \frac{1}{\pi i}\ln\left[\frac{1}{i}\ln\zeta + \pi\right]\right\}, \tag{73}$$

where $\ln\zeta$ denotes its principal value, i.e., $i\arg\zeta$. Formula (73) allows studying the general phenomenon of diffraction of plane waves on the logarithmic surface. Any plane wave which is determined before diffraction by the formula

$$u = f(x + at) \quad \text{for} \quad t < 0, \tag{74}$$

where $f(s) = 0$ for $s < 0$, can be decomposed into a sum of plane waves by using the formula

$$u = \int_0^\infty u_0(x + at - h) f'(h)\, dh, \tag{75}$$

where u_0 is the discontinuous function we have studied. It is equal either to 0 or 1. We obtain the diffraction of the general form of plane waves by adding separate terms, i.e., studying diffraction of the integrand, and then integrating with respect to the parameter. The transition from diffraction on the logarithmic surface to diffraction on an angle can be obtained by using the theory of reflection.

However, we can avoid these arguments if from the very beginning we introduce periodic elementary plane waves. We will not discuss it here. The diffraction of plane waves in three-dimensional space is studied by using elementary solutions equal to 0 in an undisturbed domain and 1 in a disturbed domain. To study an incident elementary plane wave with a given angle to the branching axis, we introduce the special moving system of coordinates. Let the angle between the normal to the wave front and the z-axis be ω, then from geometric arguments we have that the apparent speed of the motion of the wave front along the z-axis is equal to $\dfrac{a}{\cos\omega}$. Here we can verify that the dependence of our solution on the coordinates z and t is such that u in all parts of the space must be a function only of

$$t - \frac{z\cos\omega}{a} = \tau. \tag{76}$$

If in the wave equation we make the corresponding substitution of variables, then we obtain an equation on u of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\sin^2\omega}{a^2}\frac{\partial^2 u}{\partial \tau^2}. \tag{77}$$

Thus, the problem has been reduced to the problem in two dimensions. If in the previous solution we replace a by $\dfrac{a}{\sin\omega}$ and t by τ, then it provides the answer to our question. We obtain the following physical picture of the phenomenon (see Fig. 4). The plane $ABCD$, the plane of the wave front, intersects the branching axis at the point E. The diffraction disturbance is concentrated on all sheets of the Riemann surface inside the cone $EFGHI$, with an apex at the point E, tangent to the plane of the wave along the line EF. The values $u = 1$ are obtained on the zero sheet of the Riemann space between the cone and the plane $ABCD$. The exterior of this plane on the zero sheet and the exterior of the cone on all other sheets is the unperturbed domain. The integration of such a plane wave, as in the case of two dimensions, allows us to obtain the diffraction of a plane wave of general type.

The last problem that has to be discussed is the question of diffraction for arbitrary initial conditions (the diffraction of free vibrations). So far this problem has been solved only for the case of wave propagation in two-dimensional space.

A method of solution is based on integration in the plane of a certain complex parameter of solutions depending on this parameter.

In view of the lack of space we cannot discuss this method in detail. We only present the final result.

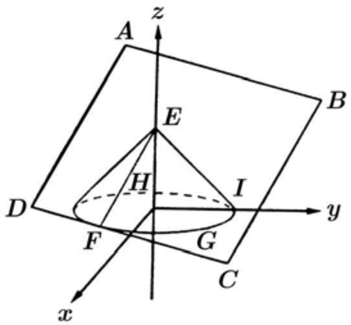

Fig. 4.

The essence of the approach is that, using contour integrals, we look for a representation of such solution of the wave equation on the Riemann surface which is equal on the first sheet for $a^2t^2 < (x_0^2 + y_0^2)$ to the known Volterra solution

$$\ln\left(\frac{at}{r_1} - \sqrt{\frac{a^2t^2}{r_1^2} - 1}\right), \tag{78}$$

where r_1 denotes the distance from the varying point of the domain to the fixed point with coordinates x_0, y_0, t_0, and vanishes on the other sheets. As is known, this solution plays the major role in solving the problem of free vibrations via the characteristics method. If we know its diffraction, then, applying techniques similar to the ones used in the theory of free elastic vibrations, which we do not have time to discuss, we can solve the general problem here as well. Particular solution (78) can be obtained superimposing the elementary plane waves discussed by us; the formulas presenting the answer to the question have the form

$$W = \frac{1}{2\pi i}\int_c v(\lambda)\,d\lambda + \psi(x, y), \tag{79}$$

where c is the contour chosen respectively, $v(\lambda)$ is the solution of the type discussed given by the formula

$$v(\lambda) = \frac{1}{\pi i}\ln\left[\frac{1}{i}\ln\zeta - \lambda\right], \tag{80}$$

where

$$\zeta = \left(\frac{at_1}{r} - \sqrt{\frac{a^2t_1^2}{r^2} - 1}\right)e^{i\varphi},$$

$$t_1 = t + \frac{r_0}{2ar}\left(e^{i(\varphi_0 - \lambda)} + e^{i(\lambda - \varphi_0)}\right).$$

Here r_0, φ_0 denote the polar coordinates of the point (x_0, y_0). Integral (79) gives a representation of the unknown function both before and after the diffraction. In such way the stated problem is completely solved.

Our essay would be incomplete if we say nothing about the problems standing at the present for this new method and about the perspectives for its further development.

First of all we note that certain results concerning, for example, the propagation of waves in a layer are not deeply enough studied. In spite of the fact that the method of complex variables, speaking theoretically, has presented the complete solution of the problem, which could not be obtained until now in any other way, some of the formulas obtained do not allow qualitative analysis at present. The problem is that for the late moments of time, i.e., for multiple reflections from a boundary, we represent displacements as a sum that, though finite, has a very large number of terms. Therefore we lose the qualitative character of the phenomenon. At the present this question is being studied and some results have been already obtained.

The second question that has to be looked at is the question of studying the mechanical properties of those special solutions, which are characterized by homogeneous potentials of zero order, and construction of homogeneous solutions of other orders. The problem is that we have only the mechanical characteristics of a few sources of the particular type so far, and the question about the mechanical behavior of all discussed sources remains open.

Finally, a very broad set of problems remains unsolved in the diffraction theory. Here, besides the naturally appearing question about diffraction in three dimensions, we have such extremely difficult questions as the question about the diffraction of elastic waves. An elementary inspection of this question immediately leads to quite complex boundary value problems of the theory of functions of a complex variable. These problems belong to the so-called class of mixed problems for which the theory has not been developed at all.

Further research in this direction will be devoted to the resolution of these questions.

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6. Functionally Invariant Solutions of the Wave Equation*

S. L. Sobolev

It is well-known that the elementary complex solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

in the real domain of x and y,

$$x + iy, \tag{2}$$

possesses the property which is called the property of functional invariance. We recall the definition of this property. Let $u(x, y)$ be a solution of equation (1). It is called the *functionally invariant solution* in a domain D of the real variables x, y, if an arbitrary function $\chi(u)$, differentiated with respect to u twice in the range of values of u, corresponding to real variables x and y from the domain D, is a solution of equation (1). Obviously, solution (2) is the functionally invariant solution of equation (1) in the entire domain, i.e., in the entire real (x, y)-plane.

The notion of the functional invariant can be transferred on other types of equations.

For the wave equation in the two-dimensional space

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \tag{3}$$

V. I. Smirnov and the author constructed a special class of the functionally invariant solutions. The definition of this class is contained, for example, in the paper by the author [1].

First these solutions were obtained by V. I. Smirnov and the author in [2] and in the note of the same authors [3].

Let us recall how to construct such solutions.

Construct a linear algebraic equation in the variables x, y, t, with coefficients depending on the unknown Ω,

* Tr. Fiz.-Mat. Inst. Steklova, **5**, 259–264 (1934)

$$l(\Omega)t + m(\Omega)x + n(\Omega)y - k(\Omega) = 0. \tag{4}$$

Let the coefficients of this equation be analytic functions of Ω, let them satisfy the condition

$$[l(\Omega)]^2 = a^2\{[m(\Omega)]^2 + [n(\Omega)]^2\}.$$

Suppose that equation (4) can be solved for Ω and the solution is a function

$$\Omega(x, y, t) \tag{5}$$

real or complex.

Then function (5) is the functionally invariant solution of the wave equation (3).

The class of solutions (5) was used by V. I. Smirnov and the author together with other members of the Theoretical Department of the Seismological Institute of the USSR Academy of Sciences when solving problems of different types. All its properties were studied in detail. The references on this question can be found in the paper by the author [4].

In the present note we prove that the class of solutions constructed in this way is unique.

The result that we prove can be formulated as a theorem.

Theorem. *Any function $\Omega(x, y, t)$, with continuous derivatives of the first and second orders, which is a functionally invariant solution of equation* (3), *can be obtained by solving an equation of type* (4).

Proof. Let us consider an arbitrary function $f(\Omega)$ differentiable twice as mentioned above.

Since the function is functionally invariant, we have

$$\frac{\partial^2 f(\Omega)}{\partial x^2} + \frac{\partial^2 f(\Omega)}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 f(\Omega)}{\partial t^2} = 0$$

or, after minor transformations,

$$f''(\Omega)\left[\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 - \frac{1}{a^2}\left(\frac{\partial \Omega}{\partial t}\right)^2\right]$$

$$+f'(\Omega)\left[\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 \Omega}{\partial t^2}\right] = 0.$$

To satisfy this condition, obviously, it is necessary and sufficient that the function Ω satisfies simultaneously two partial differential equations

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 \Omega}{\partial t^2} = 0 \tag{6}$$

and

$$\left(\frac{\partial\Omega}{\partial x}\right)^2+\left(\frac{\partial\Omega}{\partial y}\right)^2-\frac{1}{a^2}\left(\frac{\partial\Omega}{\partial t}\right)^2=0. \tag{7}$$

Our task is to integrate the system of equations (6) and (7).

Differentiating equation (7) with respect to all three independent variables, we have

$$\begin{aligned}
&\frac{\partial\Omega}{\partial x}\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial\Omega}{\partial y}\frac{\partial^2\Omega}{\partial x\partial y}-\frac{1}{a^2}\frac{\partial\Omega}{\partial t}\frac{\partial^2\Omega}{\partial x\partial t}=0,\\
&\frac{\partial\Omega}{\partial x}\frac{\partial^2\Omega}{\partial x\partial y}+\frac{\partial\Omega}{\partial y}\frac{\partial^2\Omega}{\partial^2 y}-\frac{1}{a^2}\frac{\partial\Omega}{\partial t}\frac{\partial^2\Omega}{\partial y\partial t}=0,\\
&\frac{\partial\Omega}{\partial x}\frac{\partial^2\Omega}{\partial x\partial t}+\frac{\partial\Omega}{\partial y}\frac{\partial^2\Omega}{\partial y\partial t}-\frac{1}{a^2}\frac{\partial\Omega}{\partial t}\frac{\partial^2\Omega}{\partial t^2}=0.
\end{aligned} \tag{8}$$

Solving equations (8) for the mixed derivatives of Ω and using (6) and (7), we obtain

$$\begin{aligned}
&2\frac{\partial^2\Omega}{\partial x\partial y}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial y}=\frac{\partial^2\Omega}{\partial x^2}\left(\frac{\partial\Omega}{\partial y}\right)^2+\frac{\partial^2\Omega}{\partial y^2}\left(\frac{\partial\Omega}{\partial x}\right)^2,\\
&2\frac{\partial^2\Omega}{\partial x\partial t}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial t}=\frac{\partial^2\Omega}{\partial x^2}\left(\frac{\partial\Omega}{\partial t}\right)^2+\frac{\partial^2\Omega}{\partial t^2}\left(\frac{\partial\Omega}{\partial x}\right)^2,\\
&2\frac{\partial^2\Omega}{\partial y\partial t}\frac{\partial\Omega}{\partial y}\frac{\partial\Omega}{\partial t}=\frac{\partial^2\Omega}{\partial y^2}\left(\frac{\partial\Omega}{\partial t}\right)^2+\frac{\partial^2\Omega}{\partial t^2}\left(\frac{\partial\Omega}{\partial y}\right)^2.
\end{aligned} \tag{9}$$

Besides equations (9) we can obtain similarly three more equations. However, these equations are not independent of (9).

The new equations have the form

$$\begin{aligned}
&\frac{\partial^2\Omega}{\partial t^2}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial y}+\frac{\partial^2\Omega}{\partial x\partial y}\left(\frac{\partial\Omega}{\partial t}\right)^2=\frac{\partial^2\Omega}{\partial x\partial t}\frac{\partial\Omega}{\partial y}\frac{\partial\Omega}{\partial t}+\frac{\partial^2\Omega}{\partial y\partial t}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial t},\\
&\frac{\partial^2\Omega}{\partial x^2}\frac{\partial\Omega}{\partial y}\frac{\partial\Omega}{\partial t}+\frac{\partial^2\Omega}{\partial y\partial t}\left(\frac{\partial\Omega}{\partial x}\right)^2=\frac{\partial^2\Omega}{\partial x\partial y}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial t}+\frac{\partial^2\Omega}{\partial x\partial t}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial y},\\
&\frac{\partial^2\Omega}{\partial y^2}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial t}+\frac{\partial^2\Omega}{\partial x\partial t}\left(\frac{\partial\Omega}{\partial y}\right)^2=\frac{\partial^2\Omega}{\partial x\partial y}\frac{\partial\Omega}{\partial y}\frac{\partial\Omega}{\partial t}+\frac{\partial^2\Omega}{\partial y\partial t}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial y}.
\end{aligned} \tag{10}$$

If we add several elementary identities to equations (9) and (10), we can write these equations in an invariant form. In the usual symbolism of tensor analysis the obtained system of equations can be written in the form

$$\begin{aligned}
&\nabla_\alpha\nabla_\beta\Omega\nabla_\gamma\Omega\nabla_\delta\Omega+\nabla_\gamma\nabla_\delta\Omega\nabla_\alpha\Omega\nabla_\beta\Omega\\
&=\nabla_\alpha\nabla_\gamma\Omega\nabla_\beta\Omega\nabla_\delta\Omega+\nabla_\beta\nabla_\delta\Omega\nabla_\alpha\Omega\nabla_\gamma\Omega.
\end{aligned}$$

Consider an open set E_x of the points of the domain D, where the derivative $\dfrac{\partial \Omega}{\partial x}$ is nonzero. Construct two functions

$$a_1 = \frac{\partial \Omega}{\partial y}\left(\frac{\partial \Omega}{\partial x}\right)^{-1}, \qquad a_2 = \frac{\partial \Omega}{\partial t}\left(\frac{\partial \Omega}{\partial x}\right)^{-1}.$$

Differentiating these functions, by (9) and (10), we have

$$\frac{\partial a_i}{\partial x}\frac{\partial \Omega}{\partial y} - \frac{\partial a_i}{\partial y}\frac{\partial \Omega}{\partial x} = 0,$$

$$\frac{\partial a_i}{\partial x}\frac{\partial \Omega}{\partial t} - \frac{\partial a_i}{\partial t}\frac{\partial \Omega}{\partial x} = 0,$$

$$\frac{\partial a_i}{\partial y}\frac{\partial \Omega}{\partial t} - \frac{\partial a_i}{\partial t}\frac{\partial \Omega}{\partial y} = 0, \qquad i = 1, 2.$$

From here it is easy to see that a_1 and a_2 are functions depending only on Ω, i.e., they remain constant on any connected part of the surface $\Omega = \text{const}$ belonging to E_x. Let us form the expression

$$x + a_1(\Omega)y + a_2(\Omega)t = g$$

and show that g also depends only on Ω. Indeed, an elementary computation gives

$$\frac{\partial g}{\partial x}\frac{\partial \Omega}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial \Omega}{\partial x} = 0,$$

$$\frac{\partial g}{\partial x}\frac{\partial \Omega}{\partial t} - \frac{\partial g}{\partial t}\frac{\partial \Omega}{\partial x} = 0,$$

$$\frac{\partial g}{\partial y}\frac{\partial \Omega}{\partial t} - \frac{\partial g}{\partial t}\frac{\partial \Omega}{\partial y} = 0.$$

Thus, Ω must satisfy the equation

$$\delta_1 \equiv x + a_1(\Omega)y + a_2(\Omega)t - g(\Omega) = 0.$$

It is easy to show that the function Ω can be determined from this equation. Indeed, forming the derivative of δ_1 with respect to x, we obtain

$$1 + \delta_1'(\Omega)\frac{\partial \Omega}{\partial x} = 0, \tag{11}$$

where $\delta_1'(\Omega)$ denotes the partial derivative of δ_1 with respect to Ω. Obviously, equality (11) can happen only when $\delta_1'(\Omega) \neq 0$, which proves our assertion.

Since systems (9) and (10) are invariant, we can claim that for the points of the open set E, where at least one of the partial derivatives of Ω is nonzero, Ω can be expressed as the solution of the linear equation

$$\delta(\Omega) \equiv l(\Omega)t + m(\Omega)x + n(\Omega)y - k(\Omega) = 0. \tag{12}$$

For the derivatives of Ω with respect to the coordinates a direct computation gives the expressions

$$\frac{\partial \Omega}{\partial t} = \frac{-l(\Omega)}{\delta'}, \quad \frac{\partial \Omega}{\partial x} = \frac{-m(\Omega)}{\delta'}, \quad \frac{\partial \Omega}{\partial y} = \frac{-n(\Omega)}{\delta'},$$

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\delta'} \frac{\partial}{\partial \Omega} \left(\frac{l^2(\Omega)}{\delta'} \right), \qquad \frac{\partial^2 \Omega}{\partial x^2} = \frac{1}{\delta'} \frac{\partial}{\partial \Omega} \left(\frac{m^2(\Omega)}{\delta'} \right),$$

$$\frac{\partial^2 \Omega}{\partial y^2} = \frac{1}{\delta'} \frac{\partial}{\partial \Omega} \left(\frac{n^2(\Omega)}{\delta'} \right).$$

Substituting these expressions in (6) and (7), we see that both of these equations will hold if the coefficients l, m, and n satisfy the condition

$$[l(\Omega)]^2 = a^2\{[m(\Omega)]^2 + [n(\Omega)]^2\}.$$

At those limiting points of E, that do not belong to E, the value of the function Ω can be defined by continuity.

In the remaining open set M consisting of the union of domains Ω is obviously constant along each such domain and can be presented in the form of a solution of the equation of type (12).

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[1] Paper [4] of Part I of this book. – *Ed.*
[2] Paper [2] of Part I of this book. – *Ed.*

7. General Theory of Diffraction of Waves on Riemann Surfaces*

S. L. Sobolev

Chapter 1 Weak Solutions of the Wave Equation

In the theory of integration of the wave equation

$$\square u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

in the two-dimensional space, so-called discontinuous solutions of this equation are often used in one or another form with discontinuous derivatives on different surfaces of the three-dimensional space.

Which of these solutions have a physical sense, and which can be used in different parts of mathematical apparatus applied to particular problems? This question has been studied many times already. There is a large number of studies devoted to the so-called kinematic and dynamic conditions of compatibility, i.e., to the conditions satisfied by the first derivatives of an unknown function on the discontinuity surface.

As we will see further, the research of N. M. Gunter devoted to equations of the potential theory and the heat equation, are very close to this set of ideas. There, N. M. Gunter shows that, for these problems of mathematical physics, it is quite useful to turn from a differential equation in the classical form to the study of certain integral equalities containing derivatives of orders lower than the main differential equation.

For solving of the diffraction problem on logarithmic surfaces, which we undertake in the second part of our work, we need to use certain functions, which are solutions of the wave equation in a certain generalized sense. These solutions not only can be nondifferentiable, but also be unbounded themselves. In the first part of the work we study certain properties of such solutions.

We give a definition of these weak solutions and show that the solutions with certain continuity properties are solutions satisfying dynamic and kine-

* Tr. Fiz.-Mat. Inst. Steklova, **9**, 39–105 (1935)

matic properties of compatibility. Conversely, we establish that discontinuous solutions, satisfying the compatibility conditions, are the weak solutions.

Moreover, we establish that the integration with respect to a parameter of weak solutions depending on this parameter under some additional conditions leads us again to weak solutions. Finally, we state and solve the Cauchy problem for these solutions.

Besides these general theoretical considerations, we study one particular class of such solutions, which are constructed by using the theory of functions of a complex variable.

This class was constructed by V. I. Smirnov and the author in the paper [1].

In Chap. 2 of the present work we integrate certain solutions of this type with respect to a parameter, and construct a solution of the diffraction problem by using this approach.

1. Let us consider a certain integrable function $u(x, y, t)$ and also the sequence of integrable functions

$$u_n(x, y, t). \tag{2}$$

Let us write the integral over a certain domain of the three-dimensional space

$$\iiint\limits_D |u_n(x, y, t) - u(x, y, t)|\, d\tau,$$

where $d\tau$ is the volume element.

If this integral tends to zero,

$$\lim_{n\to\infty} \iiint\limits_D |u_n(x, y, t) - u(x, y, t)|\, d\tau = 0, \tag{3}$$

then we say that the sequence converges in the mean of order 1 to the function $u(x, y, t)$ in the domain D, or, in other words, converges in L_1.

A function $u(x, y, t)$ is called a weak or a limiting solution of the wave equation (1)*, or, for brevity, an L_1-solution in a domain D, if a sequence of functions $u_n(x, y, t)$ with continuous second derivatives such that it satisfies the wave equation and converges in L_1 to the function $u(x, y, t)$ in this domain.*

Let us explain a necessary and sufficient condition for a given function $u(x, y, t)$ to be a limiting solution of equation (1)[1].

Suppose that a function $u(x, y, t)$ is an L_1-solution. Let us construct a sequence of continuous solutions $u_n(x, y, t)$ convergent to $u(x, y, t)$ as above.

In the (x, y, t)-space we consider a domain G located entirely inside D and bounded by the surface S compounded from a finite number of pieces with continuously changing tangent planes.

[1] In what follows, S. L. Sobolev assumes a weaker convergence in the definition of the weak solution, namely, $u_n \to u$ in L_{loc} – *Ed.*

Let us construct in this domain a function $v(x, y, t)$ that is continuous up to the surface S together with its first and second derivatives and vanishes on this surface together with the first-order derivatives. Apply the classical Green formula to this constructed function $v(x, y, t)$ and to the solutions $u_n(x, y, t)$. We obtain

$$\iiint_G (u_n \Box v - v \Box u_n)\, d\tau = \iint_S \left[v \frac{\partial u_n}{\partial \mu} - u_n \frac{\partial v}{\partial \mu} \right] dS, \tag{4}$$

where $d\tau$ is the volume element of the (x, y, t)-space, dS is the element of the surface S, and $\dfrac{\partial}{\partial \mu}$ denotes formally the operation

$$\frac{\partial}{\partial \mu} = \cos \nu x \frac{\partial}{\partial x} + \cos \nu y \frac{\partial}{\partial y} - \frac{1}{a^2} \cos \nu t \frac{\partial}{\partial t},$$

where ν is the direction of the inward normal to S.

Since v vanishes on S together with $\dfrac{\partial v}{\partial \mu}$, the right side of formula (4) is equal to zero. Taking into account that u_n satisfies the wave equation, we obtain

$$\iiint_G u_n \Box v \, d\tau = 0. \tag{5}$$

Let us introduce the integral

$$\iiint_G u \Box v \, d\tau \tag{6}$$

and prove that it is equal to zero.

For this purpose, we write the difference

$$\iiint_G u_n \Box v \, d\tau - \iiint_G u \Box v \, d\tau = \iiint_G (u_n - u) \Box v \, d\tau$$

and prove that it tends to zero.

Indeed,

$$\left| \iiint_G (u_n - u) \Box v \, d\tau \right|$$
$$\leq \iiint_G |u_n - u| |\Box v| \, d\tau \leq M \iiint_G |u_n - u| \, d\tau,$$

where M is the maximum of $|\Box v|$ in G.

Consequently,

$$\lim_{n\to\infty}\left|\iiint\limits_G (u_n - u)\Box v\, d\tau\right| = 0.$$

Since integral (5) is equal to zero, and the difference between the integral and (6) tends to zero, then integral (6) is equal to zero.

As we will show further, the condition that integral (6) vanishes and any domain G and an arbitrary function v, vanishing with its first derivatives on the boundary and continuous with its second derivatives up to the boundary

$$\iiint\limits_G u\Box v\, d\tau = 0, \tag{7}$$

is not only necessary, but also sufficient for the function $u(x, y, t)$ to be an L_1-solution of the wave equation.

For brevity, we call it Condition A.

2. Moving on to the proof of sufficiency of Condition A, let us prove several elementary statements on the general properties of integrable functions.

Let us consider in a domain D of the (x, y, t)-space an arbitrary function $f(x, y, t)$ integrable in this domain. Consider an inner domain D_1 such that the distance between its inner points and points of the boundary of D is greater than a number η_1.

For any point x_0, y_0, t_0 of the domain D_1 let us take a countable system of balls with radii η_1, η_2, …, where

$$0 < \eta_n < \ldots < \eta_2 < \eta_1, \qquad \lim_{n\to\infty} \eta_n = 0.$$

Inside of each ball of radius η_n, circumscribed around x_0, y_0, t_0, we construct a function

$$\omega_n(x, y, t; x_0, y_0, t_0) \tag{8}$$

of the variables (x, y, t) satisfying the following conditions:

1) all functions $\omega_n(x, y, t; x_0, y_0, t_0)$ are uniformly bounded for any n and for any point $(x_0,\ y_0,\ t_0)$ from D_1;

2) the functions $\omega_n(x, y, t; x_0, y_0, t_0)$ are measurable as functions of six variables $(x, y, t; x_0, y_0, t_0)$;

3) for all (x_0, y_0, t_0) and for any n the integrals

$$I_n(x_0, y_0, t_0) = \iiint\limits_{(x-x_0)^2+(y-y_0)^2+(t-t_0)^2=r^2\le\eta_n^2} \omega_n(x, y, t; x_0, y_0, t_0)\, d\tau \tag{9}$$

satisfy the inequality

$$I_n(x_0, y_0, t_0) > \gamma V_n, \tag{10}$$

where

$$V_n = \frac{4\pi\eta_n^3}{3},$$

and γ is a fixed positive number;

4) we require that the function ω_n be zero outside the ball of radius η_n:

$$\omega_n(x, y, t; x_0, y_0, t_0) = 0 \quad \text{for} \quad (x-x_0)^2+(y-y_0)^2+(t-t_0)^2 = r^2 > \eta_n^2. \tag{11}$$

The system of the functions ω_n satisfying these conditions is called the *regular system of nuclei*.

Let us construct now the system of functions

$$f_n(x_0, y_0, t_0) = \frac{\iiint\limits_{r^2 \le \eta_n^2} \omega_n(x, y, t; x_0, y_0, t_0) f(x, y, t)\, dx\, dy\, dt}{I_n(x_0, y_0, t_0)} \tag{12}$$

corresponding to a function $f(x, y, t)$.

The sequence of $f_n(x_0, y_0, t_0)$ is called the sequence of *average functions* corresponding to the given regular system of nuclei $\omega_n(x, y, t; x_0, y_0, t_0)$.

We prove several elementary properties of sequences of average functions.

Property 1. The sequence of average functions corresponding to the regular system of nuclei ω_n converges almost everywhere to $f(x, y, t)$.

Property 2. If a function $f(x, y, t)$ is bounded, then the sequence of average functions $f_n(x, y, t)$ is also bounded uniformly.

Property 3. The convergence $f_n(x, y, t)$ to $f(x, y, t)$ is the mean convergence with order 1.

Before moving on to the proof of these statements, we note that in the proof we can assume that the function $f(x, y, t)$ is nonnegative; moreover,

$$f > \varkappa > 0. \tag{13}$$

Indeed, from the algorithm of construction of average functions it follows that the average function f_n of the sum of two terms $f = f^{(1)} + f^{(2)}$ is equal to the sum of the average functions

$$f_n = f_n^{(1)} + f_n^{(2)}.$$

Any integrable function f can be decomposed into two terms satisfying the above stated condition. Obviously, in this case from Properties 1–3 for the sequences $f_n^{(1)}$ and $f_n^{(2)}$ the same properties for the sequence of f_n follows.

Furthermore, it is not difficult to see that the function ω_n can be considered bounded by positive constants

$$0 < m < \omega_n < M. \tag{14}$$

If ω_n does not satisfy this condition, we can always express it by the difference of two functions satisfying this condition

$$\omega_n = \overline{\omega}_n - \overline{\overline{\omega}}_n.$$

Let us denote by $\overline{f}_n$ and $\overline{\overline{f}}_n$ the average functions corresponding to $\overline{\omega}_n$ and $\overline{\overline{\omega}}_n$, respectively. Let $\overline{I}_n$ and $\overline{\overline{I}}_n$ be values of the integrals

$$\overline{I}_n = \iiint\limits_{r^2<\eta_n^2} \overline{\omega}_n \, d\tau, \qquad \overline{\overline{I}}_n = \iiint\limits_{r^2<\eta_n^2} \overline{\overline{\omega}}_n \, d\tau.$$

We have the obvious identities

$$I_n f_n = \overline{I}_n \overline{f}_n - \overline{\overline{I}}_n \overline{\overline{f}}_n, \quad I_n = \overline{I}_n - \overline{\overline{I}}_n.$$

Hence,

$$f_n = \frac{\overline{I}_n \overline{f}_n - \overline{\overline{I}}_n \overline{\overline{f}}_n}{\overline{I}_n - \overline{\overline{I}}_n}.$$

Dividing the numerator and the denominator of this fraction by V_n, we obtain

$$f_n = \frac{\dfrac{\overline{I}_n}{V_n}\overline{f}_n - \dfrac{\overline{\overline{I}}_n}{V_n}\overline{\overline{f}}_n}{\dfrac{\overline{I}_n}{V_n} - \dfrac{\overline{\overline{I}}_n}{V_n}} = \overline{f}_n + \frac{\dfrac{\overline{\overline{I}}_n}{V_n}(\overline{f}_n - \overline{\overline{f}}_n)}{\dfrac{\overline{I}_n}{V_n} - \dfrac{\overline{\overline{I}}_n}{V_n}} = \overline{f}_n + \frac{\dfrac{\overline{\overline{I}}_n}{V_n}(\overline{f}_n - \overline{\overline{f}}_n)}{\dfrac{I_n}{V_n}}.$$

Obviously, the second term on the right side almost everywhere tends to zero, because its denominator is larger than a fixed constant γ, $\dfrac{\overline{\overline{I}}_n}{V_n} < M$, and the difference $\overline{f}_n - \overline{\overline{f}}_n$ almost everywhere tends to zero. From the boundedness of $\overline{f}_n$ and $\overline{\overline{f}}_n$ the uniform boundedness of this second term follows. Finally, if $\overline{f}_n$ and $\overline{\overline{f}}_n$ converge to f in L_1, then the second term converges to zero in L_1 too.

Therefore, in the proof we assume that inequalities (13) and (14) are automatically fulfilled.

A particular case of Property 1 is the well-known Lebesgue theorem about differentiation of indefinite integrals.

Indeed, if we define a function $\omega_n(x, y, t; x_0, y_0, t_0)$ equal to one at some points of a set $\mathcal{E}_n$ and zero outside this set, then $f_n(x_0, y_0, t_0)$ has the form $\dfrac{F(\mathcal{E}_n)}{m\mathcal{E}_n}$, where $F(\mathcal{E}_n)$ is the Lebesgue integral of f over $\mathcal{E}_n$.

In such case this property is a reformulation of the fact that the given function is equal almost everywhere to the derivative of its indefinite integral.

Recall that points with such a property are called Lebesgue points.

Let us prove now Property 1 for sequences of average functions.

We establish that $f_n(x_0, y_0, t_0)$ converges to $f(x_0, y_0, t_0)$ at all the Lebesgue points.

Let M be the upper bound of the functions ω_n, and $m > 0$ be the lower bound. Let us decompose the ball of radius η_n into l collections of

$$\mathcal{E}_r = (K_{r-1}^{(l)} < \omega_n(x, y, t; x_0, y_0, t_0) < K_r^{(l)}), \quad r = 1, 2, \dots, l, \tag{15}$$

where

$$K_j^{(l)} = m + \frac{j(M-m)}{l}.$$

Obviously, all these sets are measurable.

It is clear that $f_n(x_0, y_0, t_0)$ can be rewritten in the form

$$f_n(x_0, y_0, t_0) = \frac{\sum\limits_{r=1}^{l} \iiint\limits_{\mathcal{E}_r} \omega_n(x, y, t; x_0, y_0, t_0) f(x, y, t)\, d\tau}{\sum\limits_{r=1}^{l} \iiint\limits_{\mathcal{E}_r} \omega_n(x, y, t; x_0, y_0, t_0)\, d\tau}. \tag{16}$$

We consider a Lebesgue point for the function f. Let us estimate separately the numerator and the denominator of expression (16).

First, we point out one important equality.

By the Lebesgue theorem,

$$\frac{1}{V_n} \iiint\limits_{r<\eta_n} f(x, y, t)\, d\tau = f(x_0, y_0, t_0) + \xi_1(n),$$

where the function $\xi_1(n)$ tends to zero as $n \to \infty$, or

$$\iiint\limits_{r<\eta_n} f(x, y, t)\, d\tau = V_n f(x_0, y_0, t_0) + V_n \xi_1(n) \le K V_n. \tag{17}$$

Let us write the expression

$$f_n^{(l)}(x_0, y_0, t_0) = \frac{\sum\limits_{r=1}^{l} K_{r-1}^{(l)} \iiint\limits_{\mathcal{E}_r} f(x, y, t)\, d\tau}{\sum\limits_{r=1}^{l} K_{r-1}^{(l)} m\mathcal{E}_r}$$

and show that the difference $f_n(x_0, y_0, t_0) - f_n^{(l)}(x_0, y_0, t_0)$ can be made less than any prior given number, if l is sufficiently large.

Indeed,

$$f_n(x_0, y_0, t_0) - f_n^{(l)}(x_0, y_0, t_0)$$
$$= \frac{\left[\sum\limits_{r=1}^{l} \iiint\limits_{\mathcal{E}_r} (\omega_n - K_{r-1}^{(l)}) f(x, y, t)\, d\tau\right] \sum\limits_{r=1}^{l} K_{r-1}^{(l)} m\mathcal{E}_r}{\left(\sum\limits_{r=1}^{l} K_{r-1}^{(l)} m\mathcal{E}_r\right) I_n}$$

$$-\frac{\left[\sum\limits_{r=1}^{l}\iiint\limits_{\mathcal{E}_r}(\omega_n - K_{r-1}^{(l)})\,d\tau\right]\sum\limits_{r=1}^{l}\iiint\limits_{\mathcal{E}_r}K_{r-1}^{(l)}f(x,y,t)\,d\tau}{\left(\sum\limits_{r=1}^{l}K_{r-1}^{(l)}m\mathcal{E}_r\right)I_n}.$$

Since $K_{r-1}^{(l)} > m$, the denominator of the fraction is greater than $m\gamma V_n^2$. It is not difficult to see that the numerators, in turn, are less than

$$\left(\sum_{r=1}^{l}\frac{M-m}{l}\iiint\limits_{\mathcal{E}_r}f(x,y,t)\,d\tau\right)MV_n$$

and

$$\frac{M-m}{l}V_n\sum_{r=1}^{l}\iiint\limits_{\mathcal{E}_r}Mf(x,y,t)\,d\tau,$$

respectively. Hence,

$$|f_n(x_0,y_0,t_0) - f_n^{(l)}(x_0,y_0,t_0)| < \frac{2MK(M-m)}{m\gamma l} = \xi_2(l). \tag{18}$$

Therefore, the difference $f_n(x_0,y_0,t_0) - f_n^{(l)}(x_0,y_0,t_0)$ is arbitrarily small for sufficiently large l and uniform with respect to n.

Let us fix l so large that $\xi_2(l)$ is less than $\dfrac{\varepsilon}{2}$, where ε is an arbitrary positive number.

By the Lebesgue theorem, for an arbitrary given number δ and all sets $\mathcal{E}_r'$ from $\mathcal{E}_r$ with measures greater than $\dfrac{\delta V_n}{l}$, we have

$$\iiint\limits_{\mathcal{E}_r'}f(x,y,t)\,d\tau = m\mathcal{E}_r'f(x_0,y_0,t_0) + m\mathcal{E}_r'\xi_r(\delta,n), \tag{19}$$

where $\xi_r(\delta,n) < \xi(\delta,n)$, $\xi_r(\delta,n)$ tends to zero as $n\to\infty$ for a fixed δ. In this case, for $\delta < 1$,

$$\xi_1(n) < \xi(\delta,n).$$

Let $\mathcal{E}_r''$ be sets of measures less than $\dfrac{\delta V_n}{l}$. Obviously,

$$\sum m\mathcal{E}_r'' < \delta V_n.$$

Comparing (17) and (19), we obtain

$$\sum\iiint\limits_{\mathcal{E}_r''}f\,d\tau + \sum\iiint\limits_{\mathcal{E}_r'}f\,d\tau = \sum\iiint\limits_{\mathcal{E}_r''}f\,d\tau + f(x_0,y_0,t_0)\sum m\mathcal{E}_r'$$

$$+\sum \xi_r(\delta, n) m\mathcal{E}'_r = f(x_0, y_0, t_0)V_n + \xi_1(n)V_n.$$

Solving this equation for $\sum \iiint\limits_{\mathcal{E}''_r} f\, d\tau$, we obtain

$$\sum \iiint\limits_{\mathcal{E}''_r} f\, d\tau = f(x_0, y_0, t_0)\left(\sum m\mathcal{E}''_r\right) - \sum \xi_r(\delta, n) m\mathcal{E}'_r + \xi_1(n)V_n.$$

Recalling that

$$\left|\sum \xi_r(\delta, n) m\mathcal{E}'_r\right| < V_n \xi(\delta, n),$$

we have

$$\left|\sum \iiint\limits_{\mathcal{E}''_r} f\, d\tau\right| \leq V_n \omega(\delta, n), \tag{20}$$

where $\omega(\delta, n)$ is arbitrarily small for sufficiently small δ and sufficiently large n.

Returning to $f_n^{(l)}(x_0, y_0, t_0)$, let us consider the difference

$$f_n^{(l)}(x_0, y_0, t_0) - f(x_0, y_0, t_0).$$

This difference can be written in the form

$$f_n^{(l)}(x_0, y_0, t_0) - f(x_0, y_0, t_0)$$

$$= \frac{\sum K_{r-1}^{(l)} \iiint\limits_{\mathcal{E}'_r} f\, d\tau + \sum K_{r-1}^{(l)} \iiint\limits_{\mathcal{E}''_r} f\, d\tau}{\sum K_{r-1}^{(l)} m\mathcal{E}'_r + \sum K_{r-1}^{(l)} m\mathcal{E}''_r} - \frac{\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right) f(x_0, y_0, t_0)}{\sum K_{r-1}^{(l)} m\mathcal{E}'_r}$$

$$= \frac{\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)\left(\sum K_{r-1}^{(l)} \xi_r(\delta, n) m\mathcal{E}'_r\right)}{\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r + \sum K_{r-1}^{(l)} m\mathcal{E}''_r\right)\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)}$$

$$+ \frac{\left(\sum K_{r-1}^{(l)} \iiint\limits_{\mathcal{E}''_r} f\, d\tau\right)\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)}{\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r + \sum K_{r-1}^{(l)} m\mathcal{E}''_r\right)\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)}$$

$$- \frac{\left(\sum K_{r-1}^{(l)} m\mathcal{E}''_r\right) f(x_0, y_0, t_0)\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)}{\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r + \sum K_{r-1}^{(l)} m\mathcal{E}''_r\right)\left(\sum K_{r-1}^{(l)} m\mathcal{E}'_r\right)}.$$

First, we fix sufficiently small δ, and then choose sufficiently large n. It is obvious that this difference is arbitrarily small, i.e.,

$$\left|f_n^{(l)}(x_0, y_0, t_0) - f(x_0, y_0, t_0)\right| < \frac{\varepsilon}{2}. \tag{21}$$

Comparing this inequality with (18), we see that

$$|f_n(x_0, y_0, t_0) - f(x_0, y_0, t_0)| < \varepsilon,$$

which is required.

For nuclei of constant signs, Property 2 for sequences of average functions is valid in the stronger form.

Namely, if $|f| \le M$, then the absolute values of all f_n satisfy the same inequality

$$|f_n| \le M.$$

The proof is completely obvious. For the same nuclei of constant signs, substituting $-f$ for f and comparing the obtained results, we have

$$\text{v.}\min f \le f_n \le \text{v.}\max f. \tag{22}$$

Here the symbols v. min and v. max denote the so-called essential minimum and maximum.

If the function $f(x, y, t)$ is bounded, then from Property 1 and Property 2 of average functions we have also Property 3. Indeed, in this case $|f_n(x_0, y_0, t_0) - f(x_0, y_0, t_0)|$ almost everywhere converges to zero. Since the difference is bounded, by the well-known Lebesgue theorem, we obtain

$$\lim_{n\to\infty} \iiint |f_n - f| \, d\tau = 0. \tag{23}$$

Going on to the study of unbounded functions $f(x, y, t)$, we prove a certain auxiliary statement.

Lemma 1. *For any measurable set $\mathcal{E}$ one can pass to the limit under the integral sign*

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} f_n(x, y, t) \, d\tau = \iiint\limits_{\mathcal{E}} f(x, y, t) \, d\tau. \tag{24}$$

The property formulated in the lemma is usually called the weak convergence.

Therefore, the second form of our lemma is the following: *the sequence of the functions $f_n(x, y, t)$ weakly converges to the function $f(x, y, t)$.*

Proof. We replace the function $f_n(x, y, t)$ by its representation. In this case we obtain

$$\iiint\limits_{\mathcal{E}} f_n(x, y, t) \, d\tau$$

$$= \iiint\limits_{\mathcal{E}} \left\{ \iiint\limits_{r^2 < \eta_n^2} \frac{\omega_n(x_1, y_1, t_1; x, y, t) \, f(x_1, y_1, t_1)}{I_n(x, y, t)} \, d\tau_1 \right\} d\tau. \tag{25}$$

It is not difficult to see that the product

$$\omega_n(x_1, y_1, t_1; x, y, t)\, f(x_1, y_1, t_1)$$

is a measurable summable function in the space of six variables (x_1, y_1, t_1; x, y, t), because it is the product of the summable function $f(x_1, y_1, t_1)$ and the measurable bounded function.

Hence, in the integral on the right side of (25), by the well-known Lebesgue theorem, we can change the order of integration. Recalling that $\omega_n = 0$ for $r > \eta_n$, we can replace the integration domain in the inner integral by a domain $\mathcal{E}'$ containing all points such that the distance from these points to $\mathcal{E}$ does not exceed η_n. After such rearrangement, we obtain

$$\iiint\limits_{\mathcal{E}} f_n(x, y, t)\, d\tau$$

$$= \iiint\limits_{\mathcal{E}'} f(x_1, y_1, t_1) \left\{ \iiint\limits_{\mathcal{E}} \frac{\omega_n(x_1, y_1, t_1; x, y, t)}{I_n(x, y, t)}\, d\tau \right\} d\tau_1. \tag{26}$$

The inner integral on the right side of formula (26) is a bounded function, because ω_n is bounded above, I_n satisfies condition (10), and the function ω_n for fixed (x_1, y_1, t_1) is nonzero only for $r \leq \eta_n$.

For brevity, we denote

$$\iiint\limits_{\mathcal{E}} \frac{\omega_n(x_1, y_1, t_1; x, y, t)}{I_n(x, y, t)}\, d\tau = K_n^{(\mathcal{E})}(x_1, y_1, t_1). \tag{27}$$

On the basis of the above, $K_n^{(\mathcal{E})}(x_1, y_1, t_1)$ is bounded.

We prove later that as $n \to \infty$ $K_n^{(\mathcal{E})}$ weakly converges to the so-called characteristic function of the set $\mathcal{E}$, which is defined as

$$\psi_{\mathcal{E}}(x_1, y_1, t_1) = \begin{cases} 1 & \text{for} \quad (x_1, y_1, t_1) \in \mathcal{E}, \\ 0 & \text{for} \quad (x_1, y_1, t_1) \in R^3 \backslash \mathcal{E}. \end{cases} \tag{28}$$

To establish our statement about the weak convergence of f_n to f, it suffices to prove the following lemma.

Lemma 2. *The product of a certain sequence ψ_n, weakly convergent to ψ and bounded, and any summable function f is itself weakly convergent to the limit equal to ψf.*

Whence it immediately follows that the integral on the right side of formula (26) converges to $\iiint\limits_{\mathcal{E}} f(x_1, y_1, t_1)\, d\tau_1$, and our statement is proved.

Proof. The proof of Lemma 2 is elementary.

First, we establish it for the case when $f = 1$ at points of a certain set $\mathcal{E}_1$ and equal to zero outside.

Here[2]

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} f(x_1,y_1,t_1)\psi_n(x_1,y_1,t_1)\,d\tau_1 = \lim_{n\to\infty} \iiint\limits_{\mathcal{E}\mathcal{E}_1} \psi_n(x_1,y_1,t_1)\,d\tau_1$$

$$= \iiint\limits_{\mathcal{E}\mathcal{E}_1} \psi(x_1,y_1,t_1)\,d\tau_1 = \iiint\limits_{\mathcal{E}} f(x_1,y_1,t_1)\psi(x_1,y_1,t_1)\,d\tau_1,$$

which is required.

From this particular case it is easy to move on to the case when f is a bounded function

$$|f| < N. \tag{29}$$

Let

$$|\psi_n| < M, \qquad |\psi| < M. \tag{30}$$

Let us choose the number K larger than $\dfrac{8NMm}{\varepsilon}$, where m is the measure of a domain D, and ε is a given positive number.

Let us construct a function $f_K(x,y,t)$ as follows:

$$f_K(x,y,t) = -N + \frac{2(l-1)N}{K}$$

at points of the set

$$\mathcal{E}_l = \left(-N + \frac{2(l-1)N}{K} < f < -N + \frac{2lN}{K}\right), \quad l = 1,2,\dots,K.$$

Obviously, we have

$$\iiint\limits_{\mathcal{E}} f(\psi_n - \psi)\,d\tau - \iiint\limits_{\mathcal{E}} f_K(\psi_n - \psi)\,d\tau = \iiint\limits_{\mathcal{E}} (f - f_K)(\psi_n - \psi)\,d\tau.$$

Since

$$|f - f_K| < \frac{2N}{K} < \frac{\varepsilon}{4Mm} \quad \text{and} \quad |\psi_n - \psi| \le 2M,$$

we obtain

$$\left| \iiint\limits_{\mathcal{E}} f(\psi_n - \psi)\,d\tau - \iiint\limits_{\mathcal{E}} f_K(\psi_n - \psi)\,d\tau \right| \le \frac{\varepsilon}{2}. \tag{31}$$

[2] S. L. Sobolev uses the notation $\mathcal{E}\mathcal{E}_1$ for the intersection of sets $\mathcal{E}$ and $\mathcal{E}_1$. – *Ed.*

However, $f_K(x, y, t)$ is a sum of a finite number of functions admitting only two values. Therefore, by the result proved above,

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} f_K \psi_n \, d\tau = \iiint\limits_{\mathcal{E}} f_K \psi \, d\tau.$$

Hence, for sufficiently large n we obtain

$$\left| \iiint\limits_{\mathcal{E}} f_K (\psi_n - \psi) \, d\tau \right| < \frac{\varepsilon}{2}. \tag{32}$$

From (31) and (32) it follows that

$$\left| \iiint\limits_{\mathcal{E}} f (\psi_n - \psi) \, d\tau \right| < \varepsilon, \tag{33}$$

which is required.

Finally, let us prove our lemma for an arbitrary summable function.

Let $f(x_1, y_1, t_1)$ be such a function.

Let us construct the function $\overline{f}(x_1, y_1, t_1)$ by the following rule:

$$\begin{aligned} &\overline{f}(x_1, y_1, t_1) = f(x_1, y_1, t_1), && \text{if} \quad |f(x_1, y_1, t_1)| < N, \\ &\overline{f}(x_1, y_1, t_1) = N, && \text{if} \quad f(x_1, y_1, t_1) > N, \\ &\overline{f}(x_1, y_1, t_1) = -N, && \text{if} \quad f(x_1, y_1, t_1) < -N. \end{aligned}$$

Denote by $\overline{\overline{f}}$ the difference $f - \overline{f}$.

Then, by the main integrability property, for a sufficiently large N,

$$\iiint\limits_{\mathcal{E}} |\overline{\overline{f}}| \, d\tau < \frac{\varepsilon}{4M}, \tag{34}$$

where ε is a given fixed number. Obviously,

$$\iiint\limits_{\mathcal{E}} f(\psi_n - \psi) \, d\tau = \iiint\limits_{\mathcal{E}} \overline{f}(\psi_n - \psi) \, d\tau + \iiint\limits_{\mathcal{E}} \overline{\overline{f}}(\psi_n - \psi) \, d\tau.$$

We choose N large enough for (34) to hold. Then,

$$\left| \iiint\limits_{\mathcal{E}} \overline{\overline{f}}(\psi_n - \psi) \, d\tau \right| \leq \iiint\limits_{\mathcal{E}} |\overline{\overline{f}}| \{|\psi_n| + |\psi|\} \, d\tau \leq 2M \iiint\limits_{\mathcal{E}} |\overline{\overline{f}}| \, d\tau \leq \frac{\varepsilon}{2}.$$

Next, for the bounded function $\overline{f}$ we can take n so large that

$$\left| \iiint\limits_{\mathcal{E}} \overline{f}(\psi_n - \psi)\, d\tau \right| < \frac{\varepsilon}{2}.$$

Then,

$$\left| \iiint\limits_{\mathcal{E}} f(\psi_n - \psi)\, d\tau \right| < \varepsilon, \tag{35}$$

and Lemma 2 is proved. □

To finish our proof, it only remains to establish the weak convergence of the function

$$K_n^{(\mathcal{E})} \quad \text{to} \quad \psi_{\mathcal{E}}.$$

Our statement will be proved when we establish that for any set E,

$$\lim_{n\to\infty} \iiint\limits_{E} K_n^{(\mathcal{E})}(x_1, y_1, t_1)\, d\tau_1 = m(\mathcal{E}E). \tag{36}$$

Let us prove the equality.

Let us consider an arbitrary set E and construct the characteristic function

$$\varphi(x, y, t) = \begin{cases} 1 & \text{for} \quad (x, y, t) \in E, \\ 0 & \text{for} \quad (x, y, t) \in R^3 \backslash E. \end{cases}$$

By the result proved above, for this function the sequence of average functions converges to φ in mean of order 1, hence

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}E} |1 - \varphi_n(x, y, t)|\, d\tau = 0.$$

Recalling that for the nucleus of constant signs the values of φ_n are between 0 and 1, and using this equality, we obtain

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}E} \left\{ \iiint\limits_{\mathcal{E}} \frac{\omega_n(x_1, y_1, t_1; x, y, t)}{I_n(x, y, t)}\, d\tau_1 \right\} d\tau = m(E\mathcal{E}).$$

Changing the order of integration, we have

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} \left\{ \iiint\limits_{E\mathcal{E}} \frac{\omega_n(x_1, y_1, t_1; x, y, t)}{I_n(x, y, t)}\, d\tau \right\} d\tau_1 = m(E\mathcal{E}). \tag{37}$$

Next, on the same basis, we obtain

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}-E} \varphi_n(x, y, t)\, d\tau = 0.$$

Using the positivity of φ_n and changing the order of integration as above, we obtain

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} \left\{ \iiint\limits_{\mathcal{E}-E} \frac{\omega_n(x_1, y_1, t_1; x, y, t)}{I_n(x, y, t)}\, d\tau \right\} d\tau_1 = 0. \tag{38}$$

Summing (37) and (38), we get (36).

Thus, Lemma 1 is proved. □

Let us begin the proof of Property 3 for the sequence of average functions.

Consider a summable function $f(x, y, t)$, which we again represent in the form of the sum

$$f = \overline{f} + \overline{\overline{f}}, \quad \text{where} \quad |\overline{f}| \le N,$$

and $\overline{\overline{f}}$ possesses the property

$$\iiint\limits_{D} |\overline{\overline{f}}|\, d\tau \le \frac{\varepsilon}{4}. \tag{39}$$

Obviously, any average function $f_n(x, y, t)$ also can be represented in the form of the sum

$$f_n(x, y, t) = \overline{f}_n(x, y, t) + \overline{\overline{f}}_n(x, y, t), \tag{40}$$

where

$$\overline{f}_n > 0, \qquad \overline{\overline{f}}_n > 0.$$

Let us consider the integral

$$\iiint\limits_{D} |f_n(x, y, t) - f(x, y, t)|\, d\tau. \tag{41}$$

We obtain

$$\iiint\limits_{D} |f_n(x, y, t) - f(x, y, t)|\, d\tau \le \iiint\limits_{D} |\overline{f}_n(x, y, t) - \overline{f}(x, y, t)|\, d\tau$$

$$+ \iiint\limits_{D} \overline{\overline{f}}_n(x, y, t)\, d\tau + \iiint\limits_{D} \overline{\overline{f}}(x, y, t)\, d\tau. \tag{42}$$

By the result proved above, for sufficiently large n, the integral $\iiint\limits_{D} \overline{\overline{f}}_n(x, y, t) d\tau$ differs from $\iiint\limits_{D} \overline{\overline{f}}(x, y, t)\, d\tau$ no more than by $\frac{\varepsilon}{4}$. Consequently,

$$\iiint\limits_{D} \overline{\overline{f}}_n(x, y, t)\, d\tau < \frac{\varepsilon}{2}. \tag{43}$$

Since Property 3 was proved for bounded functions, for sufficiently large n we have

$$\iiint\limits_D |\overline{\overline{f}}_n(x,y,t) - \overline{f}(x,y,t)|\,d\tau < \frac{\varepsilon}{4}. \tag{44}$$

Using (39), (43), (44), and (42), we obtain

$$\iiint\limits_D |f_n(x,y,t) - f(x,y,t)|\,d\tau < \varepsilon, \tag{45}$$

which is required.

3. We can return to the proof of the sufficiency of Condition A for a given function to be a limiting solution.

Let $u(x,y,t)$ be a given function satisfying this condition. We construct for it the sequence of average functions by using the regular system of

$$\omega_n(x,y,t;x_0,y_0,t_0) = \begin{cases} e^{\frac{r^2}{r^2-\eta_n^2}}, & r < \eta_n, \\ 0, & r \geq \eta_n, \end{cases} \tag{46}$$

where r is equal to $\sqrt{(x-x_0)^2+(y-y_0)^2+(t-t_0)^2}$. We obtain

$$u_n(x_0,y_0,t_0) = \frac{1}{I_n}\iiint\limits_{r<\eta_n} u(x,y,t)e^{\frac{r^2}{r^2-\eta_n^2}}\,d\tau = \frac{1}{I_n}\iiint\limits_D u\omega_n\,d\tau. \tag{47}$$

Let us prove that function (47) has continuous second derivatives and satisfies the wave equation.

By the general properties of average functions, it is known that

$$\lim_{n\to\infty}\iiint |u_n - u|\,d\tau = 0. \tag{48}$$

Thus, we will establish that the function $u(x,y,t)$ is a weak solution of the wave equation. For the proof, let us note that the function

$$u_n(x_0,y_0,t_0) = \frac{1}{I_n}\iiint\limits_D u(x,y,t)\omega_n(x,y,t;x_0,y_0,t_0)\,d\tau \tag{49}$$

can be infinitely differentiated under the integral sign.

Indeed, the functions

$$D_x\omega_n = \frac{\omega_n(x,y,t;x_0+h,y_0,t_0) - \omega_n(x,y,t;x_0,y_0,t_0)}{h},$$

$$D_{xx}\omega_n = \frac{D_x\omega_n(x,y,t;x_0+h,y_0,t_0) - D_x\omega_n(x,y,t;x_0,y_0,t_0)}{h},$$

and so on converge to the corresponding derivatives of ω_n as $h \to 0$, and the functions remain bounded. Therefore, by the Riesz theorem, one can pass to the limit under the integral sign in the expressions

$$\frac{u_n(x_0+h, y_0, t_0) - u_n(x_0, y_0, t_0)}{h}.$$

Taking into account that

$$\frac{\partial \omega_n}{\partial x_0} = -\frac{\partial \omega_n}{\partial x}, \quad \frac{\partial \omega_n}{\partial y_0} = -\frac{\partial \omega_n}{\partial y}, \quad \frac{\partial \omega_n}{\partial t_0} = -\frac{\partial \omega_n}{\partial t}, \ldots,$$

we obtain

$$\Box_0 u_n(x_0, y_0, t_0) = \iiint\limits_D u \Box_0 \omega_n \, d\tau = \iiint\limits_{r^2 < \eta_n^2} u \Box \omega_n \, d\tau;$$

however, by (7), since ω_n vanishes on the boundary together with its first derivatives, we obtain

$$\Box_0 u_n(x_0, y_0, t_0) = 0. \tag{50}$$

Our statement is proved.

The sequence of functions (47) has one more important property.

If a function u has absolutely continuous derivatives of order k and, therefore, summable derivatives of order $k+1$, then the $(k+1)$st derivatives of u_n almost everywhere converge to $(k+1)$st derivatives of u as $n \to \infty$; moreover, the convergence is the mean convergence of order 1*, and the derivatives of order k of u_n converge to the derivatives of order k of u uniformly.*[3]

For the proof, let us consider, for instance, the first derivative with respect to x_0,

$$\frac{\partial u_n}{\partial x_0} = \iiint\limits_{r<\eta_n} u \frac{\partial \omega_n}{\partial x_0} \, d\tau = -\iiint\limits_{r<\eta_n} u \frac{\partial \omega_n}{\partial x} \, d\tau.$$

If u is an absolutely continuous function, then we can integrate by parts the integral on the right side. Since the function ω_n vanishes on the boundary together with its derivatives, then we obtain

$$\frac{\partial u_n}{\partial x_0} = \iiint\limits_{r<\eta_n} \frac{\partial u}{\partial x} \omega_n \, d\tau. \tag{51}$$

Thus, the derivative of an average function is just the average function of the derivative, and our assertion follows. In the same way, we can prove our statement for derivatives of any order. We say that the sequence of average functions having all derivatives satisfying the wave equation and this property is "proper".

[3] All functions are considered in an inner domain D_1. – *Ed.*

4. To determine how usual solutions of the wave equation and limiting solutions defined in Sect. 1 are related, we study properties of those L_1-solutions, which possess known properties of continuity.

Their main property is the following one.

Let a function u have absolutely continuous first derivatives, therefore, summable second derivatives. If u is an L_1-solution, then it satisfies the wave equation almost everywhere.

For the proof, let us construct the regular sequence of average functions (47).

By the result proved above, $\Box u_n$ almost everywhere converges to $\Box u$ as $n \to \infty$. Furthermore, since $\Box u_n = 0$, we see that $\Box u = 0$ almost everywhere, whence our assertion follows.

Let us consider in the domain D of the (x, y, t)-space a certain surface S_1 with a continuously changing tangential plane. Let us take a domain Ω on this surface. In this domain we consider a certain function v continuous together with first-order and second-order derivatives up to the boundary and vanishing on the boundary of Ω.

Assume that a function u is continuous in the domain D.

Let us compose the integral

$$\iint\limits_{\Omega} v \frac{\partial u_n}{\partial \mu}\, dS, \tag{52}$$

where u_n is a certain family of "proper" average functions for the function u.

We prove now the main statement.

Integral (52) as $n \to \infty$ has a definite finite limit independent of construction of the family of average functions. We denote this limit by

$$\overline{\iint\limits_{\Omega} v \frac{\partial u}{\partial \mu}\, dS}. \tag{53}$$

In our space we construct a closed surface S consisting of a finite number of pieces with continuously changing tangential planes and piece-wise regular contours, such that it is a part of the surface S_1.

Then, we extrapolate the function v into the interior of this surface so it has continuous first-order and second-order derivatives up to S. Moreover, we require that the function v vanishes on S except for Ω. Applying to the domain G bounded by the surface S the Green formula (4) for the functions v and u_n, we obtain

$$\iiint\limits_{G} u_n \Box v\, d\tau = \iint\limits_{\Omega} \frac{\partial u_n}{\partial \mu} v\, dS - \iint\limits_{S} u_n \frac{\partial v}{\partial \mu}\, dS.$$

Hence,

$$\iint_{\Omega} v\frac{\partial u_n}{\partial\mu}\,dS = \iiint_G u_n\Box v\,d\tau + \iint_S u_n\frac{\partial v}{\partial\mu}\,dS. \tag{54}$$

Since the right side of this equality as $n\to\infty$ has the definite limit

$$\iiint_G u\Box v\,d\tau + \iint_S u\frac{\partial v}{\partial\mu}\,dS \tag{55}$$

independent of the choice of average functions, then the left side also does not depend on the choice. It is obvious that this limit cannot depend on the choice of the domain G.

Our assertion is proved.

By the definition of $\overline{\iint v\frac{\partial u}{\partial\mu}\,dS}$, for any closed surface S and for any v with continuous second-order derivatives, we have

$$\overline{\iint_S v\frac{\partial u}{\partial\mu}\,dS} = \iiint_G u\Box v\,d\tau + \iint_S u\frac{\partial v}{\partial\mu}\,dS. \tag{56}$$

This formula is a generalization of the classical Green formula.

The symbol $\overline{\iint v\frac{\partial u}{\partial\mu}\,dS}$ can have meaning not only for continuous solutions, but also for the general L_1-solutions of the wave equation.

We say that a domain Ω on a certain surface S with a continuously changing tangential plane is a *domain of full summability* for a given summable function u, if

a) u is summable on this surface,

b) an arbitrary sequence of average functions u_n converges to the function u in mean of order 1 on this surface.

We prove that the expression $\overline{\iint_{\Omega} v\frac{\partial u}{\partial\mu}\,dS}$ has meaning for any domain of full summability.

The proof coincides word by word with the one carried out for continuous functions.

Let us consider the domain G and the function v mentioned above. Applying the Green formula, we obtain as above

$$\iiint_G u_n\Box v\,d\tau = \iint_{\Omega} v\frac{\partial u_n}{\partial\mu}\,dS - \iint_{\Omega} u_n\frac{\partial v}{\partial\mu}\,dS.$$

From the existence of the limit for the integrals

$$\iiint_G u_n\Box v\,d\tau \quad \text{and} \quad \iint_{\Omega} u_n\frac{\partial v}{\partial\mu}\,dS$$

we can immediately conclude the existence of the limit for the integral

$$\iint_{\Omega} v \frac{\partial u_n}{\partial \mu}\, dS.$$

As a corollary, we also obtain a generalization of formula (56). Let us consider an arbitrary domain G bounded by a piece-wise regular surface S. Assume that Ω on S is a full summability domain of a weak solution. Then, for any function v with continuous derivatives up to the second order and vanishing everywhere on the boundary of S except for Ω, the following formula holds:

$$\iiint_{G} u \Box v\, d\tau = \overline{\iint_{\Omega} v \frac{\partial u}{\partial \mu}\, dS} - \iint_{\Omega} u \frac{\partial v}{\partial \mu}\, dS. \tag{56.1}$$

At first sight, formula (56.1) is an almost tautological consequence of the definition of $\overline{\iint v \frac{\partial u}{\partial \mu}\, dS}$; however, it should be noted that the introduced symbol does not depend on values of $\dfrac{\partial v}{\partial \mu}$ and the domain G.

It is not difficult to establish that the symbol $\overline{\iint_{\Omega} v \frac{\partial u}{\partial \mu}\, dS}$ is a linear operator, i.e., for any two functions v_1, v_2 satisfying the given conditions we have

$$\overline{\iint_{\Omega} v_1 \frac{\partial u}{\partial \mu}\, dS} + \overline{\iint_{\Omega} v_2 \frac{\partial u}{\partial \mu}\, dS} = \overline{\iint_{\Omega} (v_1 + v_2) \frac{\partial u}{\partial \mu}\, dS}. \tag{57}$$

Moreover, $\overline{\iint v \frac{\partial u}{\partial \mu}\, dS}$ is an additive function of the domain, i.e., if the domain Ω can be represented in the form of the sum of two domains Ω_1 and Ω_2 so that the function v satisfies the given conditions in each domain, then

$$\overline{\iint_{\Omega_1+\Omega_2} v \frac{\partial u}{\partial \mu}\, dS} = \overline{\iint_{\Omega_1} v \frac{\partial u}{\partial \mu}\, dS} + \overline{\iint_{\Omega_2} v \frac{\partial u}{\partial \mu}\, dS}. \tag{58}$$

Let $\mathcal{E}$ be a measurable set on Ω. Assume that, for average functions, the equality holds

$$\lim_{n\to\infty} \iint_{\mathcal{E}} \left(\frac{\partial u_n}{\partial \mu} - \frac{\overline{\partial u}}{\partial \mu} \right) dS = 0. \tag{59}$$

Then, we can write the operator $\overline{\iint_{\Omega} v \frac{\partial u}{\partial \mu}\, dS}$ in the form

$$\overline{\iint_{\Omega} v \frac{\partial u}{\partial \mu}\, dS} = \iint_{\Omega} v \frac{\overline{\partial u}}{\partial \mu}\, dS. \tag{60}$$

In this case, we say that the quantity

$$\overline{\frac{\partial u}{\partial \mu}} \tag{61}$$

is the average conormal derivative of u.

An example of functions having average conormal derivatives on any surface is a function satisfying the so-called kinematic and dynamic conditions of compatibility.

The definition of these solutions is the following.

We say that a weak solution of the wave equation u is *proper*, or it satisfies the *kinematic conditions of compatibility*, if the solution being a continuous function has first derivatives continuous everywhere except for a finite number of surfaces. Moreover, everywhere on these surfaces except possibly for a finite number of lines, there is a continuously changing tangential surface, and the function u is differentiable along any line completely lying on the surface of discontinuity. Next, we assume that derivatives of u at a certain point M along the direction tangent to the surface of discontinuity Σ (along the direction parallel to the tangential plane to this surface at the point that is the limit of M) tend to the derivative of u along the surface, when M tends to Σ along any nontangential path; moreover, in any closed set of points of the surface without some number of the mentioned lines, this convergence is uniform for any nontangential path inside each specific aperture angle smaller than π.

As regards normal derivatives at the point M, we also require that on each side of Σ they either converge uniformly to a limit in the sense given above or converge uniformly to infinity of a certain sign.

Let us establish the existence of $\overline{\frac{\partial u}{\partial \mu}}$ almost everywhere on the discontinuity surface and, in any case, at all those points that do not belong to the lines indicated above.

For this purpose, let us first prove an important property of discontinuity surfaces.

Let us take a part of such surface without singular lines or singular points indicated above, and construct a sphere σ with center M_0 on this part. We choose the radius of the sphere sufficiently small so that there is no singular line inside the sphere on the surface, and straight lines parallel to the normal at the point M_0 cross the discontinuity surface inside σ only at one point and do not cross each other inside σ.

The discontinuity surface Σ breaks our sphere into two parts: σ_1 and σ_2.

First, we make an assumption that on one side of Σ the normal derivative of u tends to infinity uniformly in the above-indicated sense. We now construct the "conormal vector" $\overrightarrow{\mu}$ with the components

$$\cos \nu x, \quad \cos \nu y, \quad \text{and} \quad -\frac{1}{a^2} \cos \nu t.$$

Let us decompose this vector into two terms, one of which goes along the normal to the surface, and another one belongs to the tangential plane to the surface

$$\vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2. \tag{62}$$

Construct now a surface Σ_1 close to Σ and parallel it. This surface possesses the known property that its normal coincides with the normal to Σ. The surface Σ_1 divides the domain bounded by σ and Σ into two pieces.

Let us denote by V_1 the piece separated off Σ. Apply the Green formula (56) to this domain, taking instead of the function v simply 1. We have

$$\iint\limits_{\sigma_1^1} \frac{\partial u}{\partial \mu}\, dS + \iint\limits_{\Sigma_1} \frac{\partial u}{\partial \mu}\, dS = 0. \tag{63}$$

Using (62), it is easy to transform the integral over the surface Σ_1,

$$\iint\limits_{\Sigma_1} \frac{\partial u}{\partial \mu}\, dS = \iint\limits_{\Sigma_1} \left[(\vec{\mu}_1 \cdot \operatorname{grad} u) + (\vec{\mu}_2 \cdot \operatorname{grad} u)\right] dS. \tag{64}$$

It is not difficult to see that $(\vec{\mu}_1 \cdot \operatorname{grad} u) = |\mu_1| \dfrac{\partial u}{\partial \nu}$.

We prove that the length of the vector $\vec{\mu}_1$ must be zero.

Assuming the contrary, let us consider the case when $\vec{\mu}_1$ is nonzero everywhere on Σ_1. In this case, in view of our assumptions, the integral $\iint\limits_{\Sigma_1} \dfrac{\partial u}{\partial \mu}\, dS$ has to increase unboundedly as Σ_1 approaches to Σ.

However, it is not difficult to verify that, according to our assumptions on the solution, the integral $\iint\limits_{\sigma_1^1} \dfrac{\partial u}{\partial \mu}\, dS$ has a finite limit. Then equality (63) leads us to the contradiction.

The existence of such a limit is completely obvious.

Indeed, $\dfrac{\partial u}{\partial \mu}$ can be again divided into two terms containing the tangential and normal derivatives on σ_1^1. Obviously, the tangential derivative is integrable, and the normal derivative coincides with the derivative on the surface σ_1 and can be integrated by parts.

Thus, the discontinuity surface, if the normal derivative is infinite on it, has to possess the property that its conormal belongs to the tangential plane.

We can also prove this statement in the case when the normal derivative on the discontinuity surface has a finite jump.

Let us choose a sufficiently small sphere σ such that this jump is of the same sign everywhere inside σ, and the conormal differs essentially from the tangent everywhere. First, we apply the Green formula to the domain bounded by the surface σ, and then we do it to two pieces of the domains separated by the surface Σ. Hence,

$$\iint\limits_{\sigma_1} \frac{\partial u}{\partial \mu}\, dS + \iint\limits_{\sigma_2} \frac{\partial u}{\partial \mu}\, dS = 0,$$

$$\iint\limits_{\sigma_1} \frac{\partial u}{\partial \mu}\, dS + \iint\limits_{\Sigma} \frac{\partial u^{(+)}}{\partial \mu}\, dS = 0,$$

$$\iint\limits_{\sigma_2} \frac{\partial u}{\partial \mu}\, dS + \iint\limits_{\Sigma} \frac{\partial u^{(-)}}{\partial \mu}\, dS = 0.$$

Consequently,

$$\iint\limits_{\Sigma} \frac{\partial u^{(+)}}{\partial \mu}\, dS + \iint\limits_{\Sigma} \frac{\partial u^{(-)}}{\partial \mu}\, dS = 0. \tag{65}$$

Taking into account the change of the direction of the normal on both sides, we come to an absurd conclusion that the integral of a quantity with constant sign is zero, which provides the inconsistency of the assumption that the conormal does not belong to the tangential plane.

From a mathematical standpoint, our result is easily formulated in the form of the equality

$$\cos^2 \nu x + \cos^2 \nu y - \frac{1}{a^2} \cos^2 \nu t = 0 \tag{66}$$

valid at all regular points of the discontinuity surface.

In the case when the discontinuity surface is implicitly given in the form

$$\varphi(x, y, t) = 0, \tag{67}$$

equation (66) becomes the known equation of characteristics

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 - \frac{1}{a^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 = 0. \tag{68}$$

This equation is sometimes called the *dynamic condition of strong discontinuities*.

It is interesting to note that condition (66) or (68) is sufficient for a given function, satisfying the wave equation everywhere except for the discontinuity surfaces subject to the requirements formulated above and the kinematic conditions of compatibility, to be a weak solution of the wave equation.

Using this property, it is easy to show that the average conormal derivative on the discontinuity surface is the derivative along a certain tangential direction. Therefore, the existence of $\overline{\dfrac{\partial u}{\partial \mu}}$ is proved.

5. For limiting solutions of the wave equation, as well as for usual solutions, we can state and solve the Cauchy problem. Moreover, it is easy to prove the uniqueness of such a solution.

Let us solve this problem.

First, we construct an auxiliary function

$$v_\varepsilon(x, y, t; x_0, y_0, t_0)$$

defined in the following way.

Let us consider the cone of characteristics

$$(x - x_0)^2 + (y - y_0)^2 = a^2(t - t_0)^2 \tag{69}$$

and its part, where

$$t < t_0. \tag{70}$$

Let us define a function

$$\psi(x - x_0, y - y_0, t - t_0) \tag{71}$$

in the domain

$$(x - x_0)^2 + (y - y_0)^2 \leq a^2(t - t_0)^2, \quad t_0 - 1 < t < t_0. \tag{72}$$

Assume that this function is positive and infinitely differentiable in this domain; moreover, it vanishes together with all derivatives on the boundary of this domain.

As an example of such a function, we can take

$$\psi = e^{-\frac{1}{[a^2(t-t_0)^2-(x-x_0)^2+(y-y_0)^2][t-t_0+1]}}. \tag{73}$$

Let us consider a function ψ_ε defined by the equality

$$\psi_\varepsilon = \psi\left(\frac{x - x_0}{\varepsilon}, \frac{y - y_0}{\varepsilon}, \frac{t - t_0}{\varepsilon}\right) \tag{74}$$

in a domain D_ε,

$$(x - x_0)^2 + (y - y_0)^2 \leq a^2(t - t_0)^2, \quad t_0 - \varepsilon < t < t_0, \tag{75}$$

and

$$\psi_\varepsilon = 0 \quad \text{for} \quad t < t_0 - \varepsilon.$$

In the domain

$$(x - x_0)^2 + (y - y_0)^2 \leq a^2(t - t_0)^2, \quad t < t_0, \tag{76}$$

we define the function v_ε as follows:

$$v_\varepsilon\big|_{(x-x_0)^2+(y-y_0)^2=a^2(t-t_0)^2} = 0 \tag{77}$$

and

$$\Box v_\varepsilon = \psi_\varepsilon. \tag{78}$$

As we know from a general course of mathematical physics, in this case the function v_ε is defined by the formula

$$v_\varepsilon(x,y,t;x_0,y_0,t_0)$$
$$= -\frac{1}{2\pi} \iiint\limits_{\substack{t<t_1<t_0 \\ (x-x_1)^2+(y-y_1)^2 \le a^2(t-t_1)^2 \\ (x_1-x_0)^2+(y_1-y_0)^2 \le a^2(t_1-t_0)^2}} \psi_\varepsilon(x_1-x_0, y_1-y_0, t_1-t_0)$$
$$\times \frac{d\tau_1}{\sqrt{a^2(t-t_1)^2-(x-x_1)^2-(y-y_1)^2}}, \tag{79}$$

and it vanishes together with all derivatives on surface (69).

We now move on to solving the Cauchy problem.

In the (x,y,t)-space we consider a surface S defined by the equation

$$t = T(x,y). \tag{80}$$

Assume that the surface has a continuous tangential plane and satisfies the condition

$$\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 < \frac{1}{a^2}. \tag{81}$$

The following problem is called the Cauchy problem.

Let a value of a weak solution u be given on surface (80):

$$u|_{t=T} = u_0. \tag{82}$$

Let the linear operator

$$\overline{\iint\limits_{\Omega} v \frac{\partial u}{\partial \mu}\, dS} \tag{83}$$

be defined.

The question is to find the weak solution u.

Solving this problem is elementary by using the Green formula derived above. We can obtain this solution at any point (x_0,y_0,t_0) possessing the property that surface (80) cuts off from the cone of characteristics (69) a closed domain G.

Applying (56.1) to the function v_ε and the unknown solution u, we obtain

$$\iiint\limits_{D_\varepsilon} u\psi_\varepsilon\, dx\, dy\, dt = -\iint\limits_{S_1} u \frac{\partial v_\varepsilon}{\partial \mu}\, dS + \overline{\iint\limits_{S_1} v_\varepsilon \frac{\partial u}{\partial \mu}\, dS}. \tag{84}$$

By condition, the right side is the known function $F_\varepsilon(x_0,y_0,t_0)$. However, the left side differs by a constant factor from an average function for u.

Using Sect. 2, we obtain

$$u(x_0,y_0,t_0) = \lim_{\varepsilon\to 0} \frac{F_\varepsilon(x_0,y_0,t_0)}{\iiint\limits_{D_\varepsilon} \psi_\varepsilon(x,y,t)\, d\tau}. \tag{85}$$

Obviously, the construction method gives us the uniqueness of the solution. It should be noted that F_ε, as quite obviously, is a solution of the equation $\Box u = 0$. Therefore, if the right side of (85) has a limit in L_1, then this limit is a weak solution. In this case, the initial conditions on u follow from the same conditions on F_ε.

Let us point out in conclusion one additional important property of weak solutions.

Theorem. *Let*

$$u(x,y,t,\lambda) \tag{86}$$

be a summable function of the variables x, y, t, λ in a domain of the four-dimensional space. Assume that $u(x,y,t,\lambda)$ is a limiting solution of (1)*.*

Then the Lebesgue integral

$$\int\limits_{\mathcal{E}} u(x,y,t,\lambda)\, d\lambda = U(x,y,t) \tag{87}$$

over any measurable set $\mathcal{E}$ is a weak solution of equation (1) *in the domain.*

Proof. To prove the theorem we use the necessary and sufficient condition[4] obtained by us. Let us note that

$$\iiint\limits_G U \Box v \, d\tau = \iiint\limits_G \left\{ \int\limits_{\mathcal{E}} u \, d\lambda \right\} \Box v \, d\tau.$$

By condition, $\Box v$ is bounded. Then, by the well-known Lebesgue theorem, we can change the order of integration.

Thus, we obtain

$$\iiint\limits_G U \Box v \, d\tau = \int\limits_{\mathcal{E}} \left\{ \iiint\limits_G u \Box v \, d\tau \right\} d\lambda, \tag{88}$$

which proves our assertion. □

6. We now move on to one important application of the theory of limiting solutions of the wave equation.

In 1930 V. I. Smirnov and the author developed the so-called class of complex solutions of the wave equation.

This class was introduced in [1].

A short review of results from this work was published in [2,3]. In [4] the author presented an axiomatic construction of this class of solutions. This construction is based on the main property of functional invariance, i.e., the

[4] Condition A, see equality (7). – *Ed.*

property that every function $f(u(x,y,t))$ of a solution $u(x,y,t)$ is, in turn, a solution.

This class of solutions was studied in detail and used in a series of articles on the theory of reflection, refraction, and wave diffraction[5].

Let us briefly recall certain theorems about this class of solutions.

Let us construct an equation linear with respect to x, y, t

$$\delta \equiv at - m(\Omega)x - n(\Omega)y - g(\Omega) = 0, \tag{89}$$

where coefficients m, n, and g are analytic functions of an unknown quantity Ω.

Let the coefficients m and n satisfy the equation

$$m^2(\Omega) + n^2(\Omega) = 1. \tag{90}$$

Solving equation (89), we obtain the quantity Ω, generally speaking, as a complex analytic function of real variables in a domain G.

In a part G_1 of the domain G this function can accept values filling a certain domain in the complex plane. In another part G_2 of this domain, the function can accept values forming a continuum.

The basic property of the function Ω is the following: in the first case an arbitrary analytic function of Ω is a solution of the wave equation; in the second case an arbitrary function twice differentiable with respect to Ω along the set of values admitted by Ω, is a solution of the wave equation.

The proof follows from the formulas of differentiation

$$\frac{\partial f(\Omega)}{\partial x} = \frac{m(\Omega)}{\delta'} f'(\Omega), \quad \frac{\partial f(\Omega)}{\partial y} = \frac{n(\Omega)}{\delta'} f'(\Omega), \quad \frac{\partial f(\Omega)}{\partial t} = \frac{-a}{\delta'} f'(\Omega), \tag{91}$$

and

$$\begin{aligned}
\frac{\partial^2 f(\Omega)}{\partial x^2} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{m^2(\Omega)}{\delta'} f'(\Omega)\right), \\
\frac{\partial^2 f(\Omega)}{\partial y^2} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{n^2(\Omega)}{\delta'} f'(\Omega)\right), \\
\frac{\partial^2 f(\Omega)}{\partial t^2} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{a^2}{\delta'} f'(\Omega)\right), \\
\frac{\partial^2 f(\Omega)}{\partial x \partial y} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{m(\Omega)n(\Omega)}{\delta'} f'(\Omega)\right), \\
\frac{\partial^2 f(\Omega)}{\partial x \partial t} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{-am(\Omega)}{\delta'} f'(\Omega)\right), \\
\frac{\partial^2 f(\Omega)}{\partial y \partial t} &= \frac{1}{\delta'}\frac{\partial}{\partial \Omega}\left(\frac{-an(\Omega)}{\delta'} f'(\Omega)\right),
\end{aligned} \tag{92}$$

[5] See [5] for references on this question. – *Ed.*

where, for brevity, δ' denotes the derivative of the left side of (89) with respect to Ω.

Formulas (91) and (92) are verified directly.

Both parts are separated one from another by a characteristic surface, which is an envelope of the family of planes $\Omega = \text{const}$ filling the domain G_2. The line of tangency divides such a plane into two parts. The obtained half-planes form two systems. Equation (89) has two roots inside G_2. Each of the roots corresponds to one system of such half-planes. Obviously, the manifolds $\Omega = \text{const}$ are rays in the domain G_1 (see, for instance, [5]).

In the plane of the variable Ω we denote by L the continuum of values attained by the function in G_2. Obviously, L is the boundary of the range of Ω, attained by this function in G_1.

In the literature cited, it was proved that if values of the function $f(\Omega)$ on L are limits of its values in B, then $f(\Omega)$ in G is a solution of the wave equation, satisfying the kinematic and dynamic conditions of compatibility.

Thus, the result proved there is essentially the fact that the function $f(\Omega)$ with the second derivative continuous up to the contour, is a solution of the wave equation.

Using weak solutions, one can expand somewhat this result.

Our main task is to find sufficient conditions such that $f(\Omega)$ is a weak solution in G.

We try to present these conditions in the widest form.

First, we simplify somewhat our problem by introducing instead of the general variable Ω a new variable ζ by the formula

$$m(\Omega) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right).$$

Then, by (90), equation (89) can be rewritten[6] in terms of the variable ζ,

$$\delta \equiv at - \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)x + \frac{i}{2}\left(\zeta - \frac{1}{\zeta}\right)y + \chi(\zeta) = 0. \tag{93}$$

Using the variable ζ, we can easily investigate shapes of the domain B and the contour L.

Indeed, the contour L must consist of such values of ζ for which equation (93) determines a real plane in the (x, y, t)-space. Obviously, only points of a unit circle in the plane ζ could be taken for such values; moreover, it could be only points such that $\chi(\zeta)$ is real. If this fact does not hold, i.e., the function $\chi(\zeta)$ attains no real values, then the question about weak solutions of desired type does not at all arise. However, if there exists an arc $\overset{\frown}{\alpha\beta}$ of the unit circle in the plane ζ such that $\chi(\zeta)$ is real, then, because of its analyticity, it can

[6] From (90) it follows that $n(\Omega) = \pm\frac{i}{2}\left(\zeta - \frac{1}{\zeta}\right)$. To be specific, S. L. Sobolev chooses the "–" sign. – *Ed.*

be extended over this arc. The extended function takes conjugate values at points symmetric with respect to our circle.

Let us substitute into (93) two symmetric values of ζ, i.e., two values ζ_1 and ζ_2 such that

$$\operatorname{Re}\big(\frac{1}{i}\ln\zeta_1\big) = \operatorname{Re}\big(\frac{1}{i}\ln\zeta_2\big), \qquad \operatorname{Im}\big(\frac{1}{i}\ln\zeta_1\big) = -\operatorname{Im}\big(\frac{1}{i}\ln\zeta_2\big).$$

As is not difficult to verify, in the real (x, y, t)-space we obtain two complex conjugate equations of planes, which give one straight line after selecting real and imaginary parts.

Thus, solving the equation for each ray, it is possible to always choose a value of ζ inside of the unit disk. Therefore, the domain B is a part of such a disk.

In the literature mentioned above we studied in detail the question on correspondence between the plane of the variable ζ and the direction of a ray in the (x, y, t)-space.

We established that to the cone of directions l,

$$\tan \overset{\frown}{lt} < a,$$

there corresponds the entire unit disk. To concentric circles $|\zeta| = \varrho$ there correspond directions $\tan \overset{\frown}{lt} = m$. The quantity m decreases as ϱ decreases. To radii $\arg\zeta = \vartheta$ there correspond directions such that $\vartheta = \overset{\frown}{lx}$.

In the case of $\chi(\zeta) = 0$, the cone of directions drawn from the origin coincides with the cone of rays $\zeta = \text{const}$.

Let us introduce the notion of the Riesz class: functions of a complex variable on the given arc.

Consider the function $f(\zeta)$, regular in the domain with a part of the boundary being the arc of the circle $\alpha < \vartheta < \beta$. We say that $f(\zeta)$ belongs to the Riesz class $H_1^{\alpha\beta}$ on the given arc, if integrals

$$\int\limits_{\alpha+\varepsilon}^{\beta-\varepsilon} |f(re^{i\vartheta})|\, d\vartheta$$

are uniformly bounded for $r < 1$, for any ε.

This notion is the natural generalization of the well-known definition of functions of the H_1 class on a circle.

Functions from $H_1^{\alpha\beta}$, belonging to the H_1 class on the arc, have the same property as the functions of this class on the circle:

1) they have almost everywhere on the arc $\overset{\frown}{\alpha\beta}$ summable limiting values along all nontangential paths,

2) in integrals

$$\int\limits_{\mathcal{E}(\alpha+\varepsilon<\vartheta<\beta-\varepsilon)} |f(\zeta)|\, d\zeta,$$

computed over any union $\mathcal{E}$ on the arc $\alpha+\varepsilon$, $\beta-\varepsilon$ of the concentric circle of radius r, we can pass to the limit by letting r go to 1.

The proof of these properties is completely elementary.

After these definitions, we can already formulate our conditions for the function $u = f(\zeta)$, where ζ is defined by equation (93), to be the limiting solution of the wave equation in the entire domain consisting of the part of G_1, when ζ belongs to the arc of the unit circle, and the part of G_2, when ζ belongs to the unit disk.

Theorem. *For $u = f(\zeta)$ to be the limiting solution of the wave equation in G, it is sufficient that the function $f(\zeta)$ belongs to the class $H_1^{\alpha\beta}$, where $\overset{\frown}{\alpha\beta}$ is the arc where ζ is changing in G_1.*

Proof. To prove this theorem, we construct a sequence of real numbers $g_1 < g_2 < \ldots < g_n < \ldots < 1$ convergent to 1, $\lim\limits_{n\to\infty} g_n = 1$, and consider the sequence of functions $f_n(\zeta) = f(g_n\zeta)$.

These functions could be possibly defined in the domain somewhat narrower than the domain where $f(\zeta)$ is defined, in the (x, y, t)-space; however, it is completely obvious that their domain can be defined arbitrarily close to the original one, and independently of the arbitrary closed union G', consisting of the inner points of the domain G, where $f(\zeta)$ is defined, we can choose g_1 so that all functions $f_n(\zeta)$ are defined in G'. Let us point out the following properties of functions $f_n(\zeta)$:

1) $f_n(\zeta)$ are the analytic solutions of the wave equation in the domains G_1' and G_2', where $G_1' = G' \cdot G_1$, $G_2' = G' \cdot G_2$, and the proper solutions in G;

2) the sequence of $f_n(\zeta)$ converges almost everywhere to the limit $f(\zeta)$;

3) in any closed union $\mathcal{E}$ in the (x, y, t)-space,

$$\lim_{n\to\infty} \iiint\limits_{\mathcal{E}} |f(\zeta) - f_n(\zeta)|\, d\tau = 0.$$

The first property is obvious. The second one follows from the fact that in G_1' we can pass to the limit in the usual way, and in G_2' the values of $f_n(\zeta)$ are the values on the concentric arc of the function $f(\zeta)$, from the Riesz class $H_1^{\alpha\beta}$.

To prove the third property, let us recall that because $f(\zeta)$ belongs to the class $H_1^{\alpha\beta}$ on the arc $\overset{\frown}{\alpha\beta}$,

$$\lim_{n\to\infty} \int\limits_{\alpha+\varepsilon}^{\beta-\varepsilon} |f_n(\zeta) - f(\zeta)|\, d\zeta = 0.$$

Let us transform in the domain G_2' the coordinates x, y, t into the new ones, introducing new variables ζ, η_1, and η_2, where η_1 and η_2 are chosen arbitrary.

With the appropriate selection of η_1 and η_2, the functional determinant

$$\frac{D(x,y,t)}{D(\zeta,\eta_1,\eta_2)},$$

as easy to verify, is bounded in the entire domain G_2', since the derivatives $\frac{\partial\zeta}{\partial x}$, $\frac{\partial\zeta}{\partial y}$, and $\frac{\partial\zeta}{\partial t}$ become infinite on the boundary and nowhere vanish. In this case as $n\to\infty$,

$$\iiint |f_n(\zeta)-f(\zeta)|\,d\tau = \iiint |f_n(\zeta)-f(\zeta)|\frac{D(x,y,t)}{D(\zeta,\eta_1,\eta_2)}\,d\eta_1\,d\eta_2\,d\zeta \to 0,$$

which provides our statements.

Thus, the function $f(\zeta)$ is the limit of the strongly convergent sequence of functions $f_n(\zeta)$, which are the solutions satisfying the kinematic and dynamic conditions of compatibility. These functions, being the weak solutions, are, in turn, limits of sequences of solutions with continuous derivatives

$$f_n(\zeta) = \lim_{n\to\infty} f_n^{(n)}(\zeta).$$

In this case, via the standard arguments, we can establish that, for example, the sequence $f_n^{(n)}(\zeta)$ converges almost everywhere to $f(\zeta)$ in a way that it is possible to pass to the limit in the integral

$$\lim_{n\to\infty}\iiint |f_n^{(n)}(\zeta)-f(\zeta)|\,d\tau = 0.$$

Therefore, $f(\zeta)$ is itself the weak solution. The theorem is proved. □

Chapter 2 The Problem of Integration of the Wave Equation on Riemann Surfaces

1. In the literature cited in Chap. 1, elementary solutions of the problem of diffraction of plane waves were studied in detail both on logarithmic surfaces and on surfaces with finite branching.

These solutions have the form

$$w = \frac{1}{\pi i}\ln\left(\frac{1}{\pi i}\ln\zeta\right) \tag{1}$$

or

$$w = \frac{1}{\pi i}\ln[(\zeta)^{\chi} - (\zeta)^{-\chi}], \tag{2}$$

where ζ is a complex variable defined by the equation

$$2at-\left(\zeta+\frac{1}{\zeta}\right)x+i\left(\zeta-\frac{1}{\zeta}\right)y=0. \tag{3}$$

There, in greater detail, the geometric picture of correspondence between the variables x, y, t and the variable ζ were studied[7].

Therefore, we do not discuss this question now, considering it is known. There we also touched on some questions of diffraction. For example, we established that the plane wave, propagating in the direction characterized by λ and arriving at the origin at the moment $t=0$, was presented on the logarithmic Riemann surface in the form of a linear combination of several values of the function

$$w=\frac{1}{\pi i}\ln\left[\frac{1}{\pi i}\ln\left(\frac{\zeta}{e^{i\lambda}}\right)\right]=\frac{1}{\pi i}\ln\left[\frac{1}{\pi i}\ln\left(\frac{\zeta}{\xi}\right)\right],\qquad \xi=e^{i\lambda}. \tag{4}$$

We consider now a union of plane waves propagating along different directions such that at the moment $t=0$ the front of every such plane wave passes a given fixed point ϱ_0, ϑ_0.

It is not difficult to see that in this case we must define the variable ζ by the equation

$$2at_1-\left(\zeta\xi+\frac{1}{\zeta\xi}\right)x+i\left(\zeta\xi-\frac{1}{\zeta\xi}\right)y=0, \tag{5}$$

where

$$t_1=t+\frac{\varrho_0}{a}\cos(\vartheta_0-\lambda)=t+\frac{\varrho_0}{2a}\left(\frac{\xi}{e^{i\vartheta_0}}+\frac{e^{i\vartheta_0}}{\xi}\right). \tag{6}$$

Our task is to construct a certain particular solution of the wave equation by means of integration with respect to the parameter ξ of the weak solution determined by (4) in the plane of complex variables.

For this purpose, first of all, we conduct a detailed study of the analytic character of this function.

Since, by the result proved earlier,

$$\zeta\xi=e^{i\left(\vartheta+\arccos\frac{at_1}{\varrho}\right)},$$

then

$$\frac{1}{i}\ln\zeta=\vartheta-\frac{1}{i}\ln\xi+\arccos\frac{at_1}{\varrho}.$$

As we see, the function $\ln\zeta$, as an analytic function of all its parameters, attains an infinite number of values. To systemize these values, let us, first, take for the principal value of $\ln\xi$ and denote it as $\ln_0\xi$ the branch where the argument is located between $\vartheta_0-\pi$ and $\vartheta_0+\pi$.

---

[7] It is not difficult to establish the correspondence by using the polar coordinates $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$. Equation (3) is written as $\zeta^2\varrho e^{-i\vartheta}-2at\zeta+\varrho e^{i\vartheta}=0$. Hence, $\zeta=e^{i\vartheta}\left(\frac{at}{\varrho}\pm\sqrt{\frac{a^2t^2}{\varrho^2}-1}\right)$ or $\zeta=e^{i(\vartheta\pm\arccos\frac{at}{\varrho})}$. – *Ed.*

In exactly the same way $\arccos_0 \dfrac{at_1}{\varrho}$ denotes the principal value of arccos, i.e., the value that it takes on the plane $\dfrac{at_1}{\varrho}$, if we make there a cut from $+1$ to $-\infty$ along the real axis and assume that on the upper lip of the cut $(1, -1)$ it is bounded between 0 and π, and on the lower lip, between 0 and $-\pi$. Obviously, in this case the values of $\dfrac{1}{i}\ln\zeta$ will be

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi \pm \arccos_0\frac{at_1}{\varrho} + 2\pi j. \tag{7}$$

The Riemann surface for the domain of ζ can be comprised of such sheets, where $\ln\xi$ and $\arccos\dfrac{at_1}{\varrho}$ take well-defined values.

We systematize these values by enumerating the sheets of this surface. Let us agree to denote by the symbol L_j those sheets where

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi + \arccos_0\frac{at_1}{\varrho} + 2\pi j, \tag{8}$$

and by the symbol M_j the sheets where

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi - \arccos_0\frac{at_1}{\varrho} + 2\pi j. \tag{9}$$

Obviously, on the interval of the real axis $\dfrac{at_1}{\varrho} > 1$ the value $\dfrac{1}{i}\ln\zeta$ on the sheets L_j is expressed in the form

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi + \frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right) + 2\pi j, \tag{10}$$

and on the sheets M_j,

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi - \frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right) + 2\pi j. \tag{11}$$

If we disregard mentally the branching of $\ln\xi$ and consider only the branching of $\arccos\dfrac{at_1}{\varrho}$, denoting by L_j^+ and M_j^+ upper lips of the cuts, and by L_j^- and M_j^- their lower lips, then on the interval of the cut $-1 < \dfrac{at_1}{\varrho} < +1$ the function $\dfrac{1}{i}\ln\zeta + \dfrac{1}{i}\ln_0\xi$ is a decreasing function on L_j^+ and M_j^-,

$$(2j+1)\pi > \frac{1}{i}\ln\zeta - \vartheta + \frac{1}{i}\ln_0\xi > 2\pi j \tag{12}$$

and increasing on the sheets L_j^- and M_j^+,

$$(2j-1)\pi < \frac{1}{i}\ln\zeta + \frac{1}{i}\ln_0\xi - \vartheta < 2\pi j. \tag{13}$$

On the interval $-\infty < \dfrac{at_1}{\varrho} < -1$ the function $\dfrac{1}{i}\ln\zeta$ is represented on the sheets $L_j^+ = L_{j+1}^-$ in the form

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi - \frac{1}{i}\ln\left[\left|\frac{at_1}{\varrho}\right| + \sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right] + (2j+1)\pi, \tag{14}$$

and on the sheets $M_j^+ = M_{j-1}^-$ in the form

$$\frac{1}{i}\ln\zeta = \vartheta - \frac{1}{i}\ln_0\xi + \frac{1}{i}\ln\left[\left|\frac{at_1}{\varrho}\right| + \sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right] + (2j-1)\pi. \tag{15}$$

Let us now study critical points of this function and the character of connection of the sheets at these points.

For this purpose, first, let us note that if $\arccos\dfrac{at_1}{\varrho}$ would not be a multi-valued function, then the Riemann surface for the function $\ln\zeta$ would have the only cut made along the line,

$$\arg\xi = \vartheta_0 + (2k+1)\pi, \quad k = \dots,-2,-1,0,1,2,\dots$$

If we agree to consider as the upper lip and denote by the "+" sign the side of the cut bordering with points of smaller arg, and by "−" sign the opposite side, then we obtain the following simple table of the correspondence of the sheets when moving through this cut:

+	−
L_j	L_{j-1}
M_j	M_{j-1}

.

We denote this cut by digit I.

Besides this cut, it is necessary to make several cuts separating the different values of $\arccos\dfrac{at_1}{\varrho}$.

We consider preliminary the function $\ln\zeta$ as a function only of the argument $\dfrac{at_1}{\varrho}$, assuming that the quantity $\dfrac{1}{i}\ln\xi$ on the right side is the independent parameter.

In this case, in the complex plane of the argument $\dfrac{at_1}{\varrho}$ we need to make two cuts: cut II$'$ along the real axis from $+1$ to -1, and cut III$'$ along the same real axis from -1 to $-\infty$. If we denote by "+" sign the upper lips of these cuts, i.e., the lips approachable from the upper half-plane of the corresponding sheet, where

$$\operatorname{Im}\left\{\frac{at_1}{\varrho}\right\} > 0,$$

and by the "−" sing the lower lips, then it is easy to comprise tables explaining the order of sewing of these sheets:

II′

+	−
L_j	M_j
M_j	L_j

III′

+	−
L_j	L_{j+1}
M_j	M_{j-1}

.

To explain the arrangement of the critical points on the surface of ξ, it is sufficient to note that $\dfrac{at_1}{\varrho}$ is the single-valued function of ξ.

Thus, we can simply arrange the cuts on ξ so that they are mapped on the corresponding cuts in the plane $\dfrac{at_1}{\varrho}$.

Obviously, $\dfrac{at_1}{\varrho}$ accepts real values where

$$\frac{\xi}{e^{i\vartheta_0}} + \frac{e^{i\vartheta_0}}{\xi}$$

is real. It requires that either

$$\arg\xi = \vartheta_0 + k\pi, \quad k = \dots, -2, -1, 0, 1, 2, \dots,$$

or $|\xi| = 1$. Obviously, the domains

$$|\xi| > 1, \qquad 2k\pi < \arg\xi - \vartheta_0 < (2k+1)\pi$$

and

$$|\xi| < 1, \qquad (2k-1)\pi < \arg\xi - \vartheta_0 < 2k\pi$$

are mapped to the upper half-plane $\dfrac{at_1}{\varrho}$, and the domains

$$|\xi| < 1, \qquad 2k\pi < \arg\xi - \vartheta_0 < (2k+1)\pi$$

and

$$|\xi| > 1, \qquad (2k-1)\pi < \arg\xi - \vartheta_0 < 2k\pi$$

are mapped to the lower half-plane $\dfrac{at_1}{\varrho}$ (see Fig. 1).

It is not difficult to see that intervals of the ray $\arg\xi = \vartheta_0 + 2k\pi$ from the circle $|\xi| = 1$ to infinity or to zero correspond to those points of the real axis $\dfrac{at_1}{\varrho}$, where $\dfrac{at_1}{\varrho} > \dfrac{at + \varrho_0}{\varrho}$, intervals of the ray $\arg\xi = \vartheta_0 + (2k+1)\pi$ from the

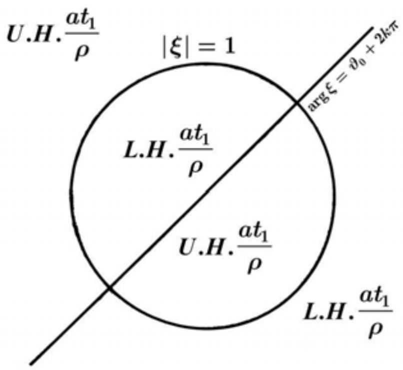

Fig. 1.

circle $|\xi| = 1$ to infinity or to zero correspond to those points of the real axis $\frac{at_1}{\varrho}$, where $\frac{at_1}{\varrho} < \frac{at - \varrho_0}{\varrho}$, and, finally, both semicircles $|\xi| = 1$ correspond to the points where $\frac{at - \varrho_0}{\varrho} < \frac{at_1}{\varrho} < \frac{at + \varrho_0}{\varrho}$.

Therefore, it is directly evident how on the plane ξ all cuts and all critical points of the function arccos $\frac{at_1}{\varrho}$ are located. First, we note that these critical points correspond to the values

$$\frac{at_1}{\varrho} = -1, 1, \infty.$$

The direct computation gives us six critical values:

$$\xi = 0, \quad \xi = \infty,$$

$$\begin{aligned}
\xi &= \left[-\frac{at - \varrho}{\varrho_0} + \sqrt{\frac{(at - \varrho)^2}{\varrho_0^2} - 1} \right] e^{i\vartheta_0}, \\
\xi &= \left[-\frac{at - \varrho}{\varrho_0} - \sqrt{\frac{(at - \varrho)^2}{\varrho_0^2} - 1} \right] e^{i\vartheta_0}, \\
\xi &= \left[-\frac{at + \varrho}{\varrho_0} + \sqrt{\frac{(at + \varrho)^2}{\varrho_0^2} - 1} \right] e^{i\vartheta_0}, \\
\xi &= \left[-\frac{at + \varrho}{\varrho_0} - \sqrt{\frac{(at + \varrho)^2}{\varrho_0^2} - 1} \right] e^{i\vartheta_0}.
\end{aligned} \tag{16}$$

Here we can distinguish four possible cases of arrangement of points on the plane ξ:

1) $\varrho > at + \varrho_0$,
2) $at + \varrho_0 > \varrho > |at - \varrho_0|$,
3) $at < \varrho_0$, $\varrho < \varrho_0 - at$,
4) $at > \varrho_0$, $\varrho < at - \varrho_0$.

In view of the fact that here we study only the positive values of ϱ, t, and ϱ_0, there are no other cases.

Geometrically all four considered cases admit very simple interpretation. If from the point ϱ_0, ϑ_0 in the (ϱ, ϑ)-plane we draw a circle C_1 of radius at, and then construct two circles centered at the origin and tangent to C_1, then the entire (ϱ, ϑ)-plane gets divided into three pieces:

a) exterior of the larger tangential disk, where case 1) holds;

b) annular domain between two tangential disks, where case 2) holds;

c) interior of the smaller disk, in which we have either case 3) or case 4) depending whether this smaller circle touches C_1 from without or from within (see Fig. 2).

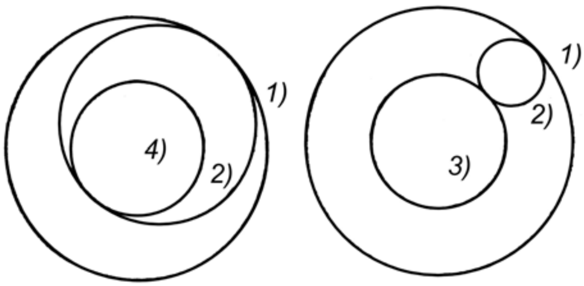

Fig. 2.

To make cuts more conveniently on the plane ξ we slightly deform on the plane $\dfrac{at_1}{\varrho}$ cuts II′ and III′ so that they would be entirely located in the upper half-plane.

Then, in correspondence to all four cases, we have the following picture (see Fig. 3):

The transition tables from one sheet to another are given above. The positive and negative lips of the cuts are denoted by "+" and "−" signs marked on the drawing.

The critical points located on the unit circle can be easily obtained by purely geometric construction.

In case 2) or 3) we construct on the (ϱ, ϑ)-plane a circle with radius ϱ centered at the origin, and the circle C_1 with radius at centered at the point ϱ_0, ϑ_0, and draw common tangents to both these circles.

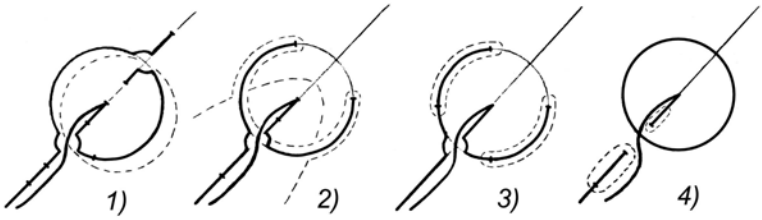

Fig. 3.

In case 2) there are two such tangents, and in case 3) we obtain two outer tangents and two inner tangents. Then, we draw the radii of the circle C_1 to the tangency points. In this case arg of critical points ξ_1 or ξ_2 is respectively equal to the angle composed by the radius connected with the tangency point of the outer tangent and the axis $\vartheta = 0$, and arg of critical points ξ_3 or ξ_4 in case 3) is respectively equal to the angle between the radius of C_1, connected with the tangency point of one of the inner tangents.

For the proof, it is sufficient to calculate the location of the tangency points. It can be done by using completely elementary computations, on which we cannot dwell now.

Besides the study of the character of multiformity of the function $\ln \zeta$, we should know for future reference how to compute its roots and determine the sheets of the Riemann surface where these roots are located.

For the computation of roots, let us exponentiate the equality

$$i\left(\vartheta - \frac{1}{i}\ln\xi\right) = -i\arccos\frac{at_1}{\varrho}.$$

Then,

$$\frac{e^{i\vartheta}}{\xi} = e^{-i\arccos\frac{at_1}{\varrho}} = e^{\ln\left(\frac{at_1}{\varrho}\mp\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)}.$$

Hence,

$$\frac{e^{i\vartheta}}{\xi} + \frac{\xi}{e^{i\vartheta}} = 2\frac{at_1}{\varrho}$$

or

$$\frac{at}{\varrho} + \frac{\varrho_0}{2\varrho}\left(\frac{e^{i\vartheta_0}}{\xi} + \frac{\xi}{e^{i\vartheta_0}}\right) = \frac{1}{2}\left(\frac{e^{i\vartheta}}{\xi} + \frac{\xi}{e^{i\vartheta}}\right),$$

i.e.,

$$\xi^2\left(\frac{\varrho_0}{e^{i\vartheta_0}} - \frac{\varrho}{e^{i\vartheta}}\right) + 2at\xi + (\varrho_0 e^{i\vartheta_0} - \varrho e^{i\vartheta}) = 0.$$

Keeping the letter ξ for the independent variable, and denoting the roots of this equation by ζ, we obtain

$$\zeta = \frac{-at \mp \sqrt{a^2t^2 - (\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0\cos(\vartheta - \vartheta_0))}}{\frac{\varrho_0}{e^{i\vartheta_0}} - \frac{\varrho}{e^{i\vartheta}}}$$

or

$$\zeta = \left\{ \frac{at}{\sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos(\vartheta - \vartheta_0)}} \right.$$

$$\left. \pm \sqrt{\frac{a^2t^2}{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos(\vartheta - \vartheta_0)} - 1} \right\} e^{i \arg(\varrho e^{-i\vartheta} - \varrho_0 e^{-i\vartheta_0})}. \tag{17}$$

Obviously, each of the roots calculated for fixed ϱ, ϑ, t, ϱ_0, ϑ_0 can lie only on one sheet of the Riemann surface, because the values of $\ln\zeta$ are different on all sheets. With a change in our parameters, the roots can sometimes approach each other and even coincide.

To clarify the question on the location of these roots, we give a simple geometric interpretation for them.

If in our (ϱ, ϑ)-plane we would move the origin to the point ϱ_0, ϑ_0 and calculate the value of the new complex variable $\zeta^{(1)}$ via the formula

$$2at - \left(\zeta^{(1)} + \frac{1}{\zeta^{(1)}}\right) x_1 + i\left(\zeta^{(1)} - \frac{1}{\zeta^{(1)}}\right) y_1 = 0, \tag{18}$$

where

$$x_1 = x\varrho\cos\vartheta_0, \quad y_1 = y - \varrho\sin\vartheta_0, \tag{19}$$

then we would see that our $\zeta^{(1)}$ is given exactly by formula (17).

Thus, we can repeat for $\zeta^{(1)}$ all claims made for the variable ζ. Inside the circle C_1 the variable $\zeta^{(1)}$ accepts values from the interior of a unit disk, and outside it, values on the boundary of this disk. The construction of these values can be done, as before, by drawing from the given point two tangents to the circle C_1.

We now move on to the study. From our geometric interpretation we see that in case 1) both roots of (17), which we denote by $\zeta^{(1)}$ and $\zeta^{(2)}$, are always located on a unit circle.

Let us study a character of motion of the roots in all possible cases, assuming that ϱ, ϑ_0, ϱ_0, t are fixed and varying only ϑ.

In the first case, for $\vartheta = \vartheta_0$, both roots are located symmetrically about the ray $\arg\xi = \vartheta_0$ on the circle $|\xi| = 1$. We denote the root with arg greater than ϑ_0 by $\zeta^{(1)}$, and another one by $\zeta^{(2)}$. From the geometrical interpretation of these roots it follows that they are all, remaining always distinct as ϑ grows, moving in the positive direction along the circle $|\xi| = 1$. With the increase of ϑ by 2π, they both describe the entire circle.

To determine on which sheets these roots are located, it is the simplest to use the fact that they are both located on cut II, where $\arccos\dfrac{at_1}{\varrho}$ is real.

Obviously, $\dfrac{1}{i}\ln\xi$ for $\vartheta = \vartheta_0$ is changing on a circle from $-\pi$ to π, and hence, $\pm\arccos\dfrac{at_1}{\varrho} + 2\pi j$ for the root $\zeta^{(2)}$, with arg less than ϑ_0, has to end up in the interval $(-\pi, 0)$, and for the root $\zeta^{(1)}$, with arg larger than ϑ_0, this

$\pm\arccos\dfrac{at_1}{\varrho}+2\pi j$ must lie in the interval $(0,\pi)$; it is clear from here that $\zeta^{(1)}$ is located on the cut $M_0^{(-)}L_0^{(+)}$.

From the motion character of the roots itself for this case, there follows their location on the sheets of the Riemann surface for all values of ϑ_0.

Obviously, for $\vartheta=(2k-1)\pi+\vartheta_0$, the root $\zeta^{(1)}$ is located on the cut between $L_{-k}^{(+)}$ and $M_{-k}^{(-)}$, and the root $\zeta^{(2)}$ is on the cut between $L_{-k+1}^{(-)}$ and $M_{-k+1}^{(+)}$.

For $\vartheta=2k\pi+\vartheta_0$, the root $\zeta^{(1)}$ is located on the cut between $L_{-k}^{(+)}$ and $M_{-k}^{(-)}$, and the root $\zeta^{(2)}$ is on the cut between $L_{-k}^{(-)}$ and $M_{-k}^{(+)}$.

On Fig. 3, the dotted curves indicate the paths of the roots in all cases in which we are interested.

To study case 2), first, we point out that the point $\xi=0$ is not, actually speaking, critical for the sheets M, and the point $\xi=\infty$ is not critical for the sheets L. Indeed, going around one of these points, we cross two of the cuts made by us. It is not difficult to see from the presented tables that going around zero we move from the sheet M_{-k} on the same one, and going around ∞ we get from the sheet L_{-k} on the same one.

Moving on to the study of case 2), we investigate again, first, the location of the roots for the value $\vartheta=\vartheta_0$. From the geometrical interpretation of the roots given above, it follows that one of them is located inside the disk $|\xi|\le 1$, and another one is outside it, and for $\varrho>\varrho_0$ they are both located on the ray $\arg\xi=\vartheta_0$, and for $\varrho<\varrho_0$ – on the ray $\arg\xi=\vartheta_0+\pi$. It is not difficult to verify that the inner root $\zeta^{(1)}$ is located on the sheet M_0, and the exterior one $\zeta^{(2)}$ is on the sheet L_0. (The notations are taken with no relations to case 1).) For $\varrho>\varrho_0$, this follows from the fact that separating in the equation

$$\vartheta_0-\frac{1}{i}\ln_0\xi\pm\arccos_0\frac{at_1}{\varrho}+2\pi j=0$$

the real and imaginary parts, we obtain

$$j=0,\quad \operatorname{Im}\left\{-\frac{1}{i}\ln_0\xi\pm\arccos_0\frac{at_1}{\varrho}\right\}=0.$$

The second equality can occur only when the sign in front of arccos is "$-$", since the imaginary part of $-\dfrac{1}{i}\ln_0\xi$ and the imaginary part of $\arccos_0\dfrac{at_1}{\varrho}$ are both negative.

Therefore, the inner root falls on the sheet M_0.

By the same arguments, we see that for the outer root $\zeta^{(2)}$ as well

$$\pm\arccos_0\frac{at_1}{\varrho}$$

must appear with the "+" sign, and, hence, this root must fall on the sheet L_0.

Observing further how the roots are changing, we use again their geometric interpretation. We see that as ϑ grows, both these roots have increasing argument, and their modules tend to 1. For a certain value of ϑ both these roots come together at a certain point on the contour on cut II between ξ_1 and ξ_3 and become a double root. Next, one of them begins to move immediately along the circle in the positive direction toward the critical point ξ_2, and turns backward after it reaches it, while another root, returning first in the negative direction to the point ξ_1, then turns into the positive direction. After this, both roots meet again at a certain point on the cut. Taking into account that for values of ϑ, neighboring with $\vartheta_0 + \pi$, case 2) must continuously transform into case 1), we get convinced that the root that moved first in the positive direction is the root $\zeta^{(1)}$ and must first move along the cut $L_0^{(+)}\ M_0^{(-)}$, and another root $\zeta^{(2)}$ must, reaching the critical point, go around it and move on the cut $L_0^{(-)}\ M_0^{(+)}$.

It is not difficult further, using very simple arguments, to establish that for the values $\vartheta = \vartheta_0 + 2\pi$ the root $\zeta^{(1)}$, with module smaller than 1, must lie on the sheet M_{-1}, and the root $\zeta^{(2)}$, with module greater than 1, is on the sheet L_{-1}.

It is clear from here that both these roots must merge into one multiple root on cut II between the points ξ_2 and ξ_4, where the sheets $L_{-1}^{(-)}$ and $M_{-1}^{(-)}$ merge.

Therefore, the root $\zeta^{(1)}$, that had been moving before toward the point ξ_2, must go around it before it moves back to merge with the root $\zeta^{(2)}$. With the further change of ϑ, the entire movement picture repeats; moreover, only the numbers of the sheets diminish by one with every cycle.

Case 3) is easily obtained from case 2), if we take into account the continuity. From the geometric interpretation, it is obvious that the roots move along cuts II, the root $\zeta^{(1)}$ is between the points ξ_2 and ξ_4, and the root $\zeta^{(2)}$ is between the points ξ_1 and ξ_3, moving from one point to another and backward.

For $\vartheta = \vartheta_0 + 2k\pi$, the root $\zeta^{(2)}$ lies on the seam between the sheets $L_{-k}^{(+)}$ and $M_{-k}^{(-)}$, and the root $\zeta^{(1)}$ is on the seam between the sheets $L_{-k}^{(-)}$ and $M_{-k}^{(+)}$.

Similarly, for $\vartheta = \vartheta_0 + (2k+1)\pi$ the root $\zeta^{(1)}$ ends up on the seam between $M_{-k}^{(+)}$ and $L_{-k}^{(-)}$, and the root $\zeta^{(2)}$ is between the sheets $M_{-k-1}^{(-)}$ and $L_{-k-1}^{(+)}$.

Finally, case 4) also can be obtained from case 2) by using simple ideas of continuity.

Here the root $\zeta^{(1)}$ moves always inside the disk $|\xi| < 1$, and the root $\zeta^{(2)}$ is outside this disk. Both these roots must get onto cut I for the value $\vartheta = \vartheta_0 + (2k+1)\pi$, and for $(2k-1)\pi < \vartheta - \vartheta_0 < (2k+1)\pi$ they lie, respectively, on the sheets M_{-k} and L_{-k}.

Finishing the study of analytic properties of the function $\dfrac{1}{i}\ln\zeta$, we also clarify its values at the point $\xi = 0$ on the sheets M and at the point $\xi = \infty$ on the sheets L, which will be important for us in the future.

Expanding the function

$$\frac{at_1}{\varrho} - \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}$$

in series of increasing powers of ξ, we obtain

$$\frac{at_1}{\varrho} - \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1} = \frac{\varrho}{\varrho_0 e^{i\vartheta_0}}\xi + \cdots , \tag{20}$$

where the second-order and higher-order terms are neglected.

Similarly,

$$\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1} = \frac{\varrho}{\varrho_0} e^{i\vartheta_0}\xi^{-1} + \cdots , \tag{21}$$

where the zero-order and higher-order terms are neglected.

Obviously, $\arccos_0 \dfrac{at_1}{\varrho}$ containing in $\ln\zeta$ takes the value

$$\frac{1}{i}\ln\left(\frac{at_1}{\varrho} - \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)$$

on the sheets M_k.

This follows from the fact that its imaginary part on the ray $\arg\xi = \vartheta_0$ in this case tends to $-\infty$.

By the same reason $\arccos_0 \dfrac{at_1}{\varrho}$ on the sheets L_k must be equal to

$$\frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right).$$

Taking this into account and substituting into $\dfrac{1}{i}\ln\zeta$ expression (20), we finally obtain on M_j,

$$\lim_{\xi\to 0}\frac{1}{i}\ln\zeta = \vartheta + \frac{1}{i}\ln\frac{\varrho}{\varrho_0 e^{i\vartheta_0}} + 2\pi j. \tag{22}$$

In absolutely the same way, we can establish that on L_j,

$$\lim_{\xi\to \infty}\frac{1}{i}\ln\zeta = \vartheta + \frac{1}{i}\ln\frac{\varrho_0}{\varrho e^{i\vartheta_0}} + 2\pi j. \tag{23}$$

After these preliminary calculations, we can already begin the construction of our fundamental solution.

We will seek it in the form of a contour integral of the function $\frac{1}{2\pi i}\frac{1}{\xi}\ln\frac{1}{i}\ln\zeta$, computed along the contour C, presented on the drawing (see Fig. 4).

This contour consists of the circle $|\xi| = 1$, of twice traveled radius of this circle $\arg\xi = \vartheta_0$, and twice traveled radius of this circle $\arg\xi = \vartheta_0 + \pi$.

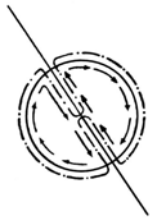

Fig. 4.

This contour is somehow located on sheets of the Riemann surface and is not necessarily closed on this surface.

For the function $w = \frac{1}{2\pi i}\int\limits_C \frac{1}{\xi}\ln\frac{1}{i}\ln\zeta\, d\xi$ to be the solution of the wave equation, it suffices that the values of the integrand $\frac{1}{\xi}\ln\frac{1}{i}\ln\zeta$ for any fixed ξ on the contour C be the weak solution of the wave equation and that the result of integration be the function possessing the required conditions of continuity.

We will analyze the requirements necessary for the choice of the integration contour.

First, we study the parts of the contour located inside our circle $|\xi| = 1$ on the radii. On the radius $\arg\xi = \vartheta_0$ we can deal either with cut II, or the cut is not present at all. If the cut is not present, then it is sufficient to assign for $\ln\zeta$ a value from the sheet L. Indeed, with this choice of the path, the function $\ln\zeta$ cannot vanish on the contour, and $\ln\ln\zeta$ is always finite. While the choice of the sheet M would threaten unpleasant consequences, limiting the possibility to define uniquely $\ln\ln\zeta$ in the entire space.

Absolutely the same arguments force us to choose also the sheets L on the path taken along another radius for the cases when the given value of ξ lies either on the joint cut I and III, or on cut I.

Let us point out now certain remarkable features of the solutions obtained for $|\xi| = 1$.

Obviously, a certain point ξ, for a different choice of ϱ, ϑ, and t, can end up either outside the cuts, or on cut II, or on cut III.

If the point is located outside the cuts, i.e., $\dfrac{at_1}{\varrho} > 1$ (this can occur for such ϱ when we obtain cases 2), 3) or 4)), then, giving to the angle ϑ the increment of 2π, we obviously increase $\dfrac{1}{i}\ln\zeta$ also by 2π.

If, in this case, the values of the function $\dfrac{1}{i}\ln\zeta$ are taken on the sheet M, then they will be located in the upper half-plane, and, therefore, its argument will decrease with the growth of the real part. If, however, these values are taken from the sheet L, then they will be located in the lower half-plane, and its argument will increase with the growth of the real part. As ϑ moves from $-\infty$ to $+\infty$, $\arg\dfrac{1}{i}\ln\zeta$ gets on the sheets M the increment of $-\pi$, and on the sheets L the increment of π. If we now move on to analysis of the case when the point $\xi = \text{const}$ gets on cut II, which is possible in cases 2) and 3), then, as is not difficult to verify here, $\dfrac{1}{i}\ln\zeta$ can be only a real number; moreover, it could be either positive, or negative at different moments depending on the location of the roots. If we require that for the large in module values of ϑ the distinct cases transform one to another continuously, then from here there follows the necessity that the argument of $\dfrac{1}{i}\ln\zeta$ has only one jump such that if the values of $\dfrac{1}{i}\ln\zeta$ in the space with no cuts are taken from the sheet M, then its value would be $-\pi$, and if these values are taken from the sheet L, then π.

Hence, first of all, the corollary follows for the case when cut III passes through the given point ξ.

Indeed, by the same arguments of continuity, we again must require that $\arg\dfrac{1}{i}\ln\zeta$ has on the interval $-\infty < \vartheta < +\infty$ an increment equal either to $-\pi$ or π. The analysis similar to the one conducted above allows us to get convinced that if we take the values of $\dfrac{1}{i}\ln\zeta$ at the point without cut from the sheet L, then also at the points on cut III we should take the sheet L, and conversely, if the values, with no cut present, were taken from the sheet M, then also on cut III we should take the values from the sheet M.

Obviously, the choice of the sheet number depends on which sheet the values of $\dfrac{1}{i}\ln\zeta$ are chosen on cut II, and has to be made so this function is changing continuously from case to case.

It is not difficult to verify directly that for any point on cut II we have only once a jump of the argument of $\dfrac{1}{i}\ln\zeta$. In such way, the method of the sign choice of this jump is completely determined.

Let us point out a simple mnemonic rule for the determination of this sign.

If we would slightly deform our integration contour so that our point in question would lie inside the disk $|\xi| \leq 1$, then during the motion of the root of the function $\frac{1}{i}\ln\zeta$ along cut II in the positive direction the argument of $\frac{1}{i}\ln\zeta$ would increase by π in the interval $-\infty < \vartheta < +\infty$, and during the motion of the root of this function in the negative direction, this argument would get in this interval the increment of $-\pi$.

Similarly, if the considered value of ξ would lie outside the disk $|\xi| \leq 1$ for those values of ϱ and t, where the root is moving near ξ in the positive direction, the increment of $\arg\frac{1}{i}\ln\zeta$ would be $-\pi$, and for those, where the root is moving near ξ in the negative direction, this increment would be π. Thus, if on the interval with no cuts we choose values of the function from the sheet M, then we have to place mentally the roots of $\frac{1}{i}\ln\zeta$, moving in the positive direction, inside the contour, and the roots moving in the negative direction in its exterior.

Conversely, if the values of $\frac{1}{i}\ln\zeta$ were selected in the domain without cuts from the sheet L, then positively increasing roots should be placed into the exterior of the disk, and negatively increasing into the interior.

If all conditions that we examined are satisfied, then the function $\frac{1}{\xi}\ln\frac{1}{i}\ln\zeta$ is the (defined for all values of ϱ, ϑ, and t) weak solution of the wave equation depending on the parameter ξ, and we can integrate this solution by using the theory developed in Chap. 1.

2. We now move on to the construction of our solution.

As indicated above, let us integrate the function $\frac{1}{\xi}\ln\frac{1}{i}\ln\zeta$ over the contour described earlier,

$$\frac{1}{2\pi i}\int\frac{1}{\xi}\ln\frac{1}{i}\ln\zeta\, d\xi.$$

To give a definite meaning to this integral, it only remains to agree about the choice of values of this function.

On the two rectilinear intervals of the contour $\arg\xi = \vartheta_0$ we can deal either with the absence of cut or with appearing cut II.

In the cases of the absence of cut we will take on both these intervals values from the sheet L_0, which, as is not difficult to see, cancel each other.

However, in the case of cut II, the values of $\frac{1}{i}\ln\zeta$ on the interval, going from the circle to the center, will be taken from the cut $M_0^{(+)}\,L_0^{(-)}$, and on the opposite interval, going from the center to the circle, we take these values from the cut $M_0^{(-)}\,L_0^{(+)}$.

On the semicircle $|\xi| = 1$, $\vartheta_0 < \arg\xi < \vartheta_0 + \pi$ we can have either the case when there is no cut at all, or the case of cut II or III.

In the first case, we take the values of the function from the sheet M_0, in the second case, i.e., on cut II, from the cut $M_0^{(-)}\,L_0^{(+)}$, and, finally, in the case of III from the cut $M_0^{(-)}\,M_1^{(+)}$.

Similarly, on the semicircle $|\xi| = 1$, $\vartheta_0 - \pi < \arg \xi < \vartheta_0$ we will take, in the case of the cut absence, the values from the sheet M_0, in the case of cut II from the cut $M_0^{(+)} L_0^{(-)}$, and, finally, on cut III we choose these values from the cut $M_0^{(+)} M_{-1}^{(-)}$.

At last, on the two rectilinear intervals $\arg \zeta = \vartheta_0 + \pi$ we can have either one cut I, or joint cut I and II, or joint cut I and III. On joint cut I and III we take the values on both intervals from the seams $L_{-1}^{(+)} L_1^{(-)}$. The integrals over these intervals are obviously canceling each other.

On joint cut I–II we take values on the interval, directed from the circle to the center, on the seam $M_0^{(-)} L_{-1}^{(+)}$, and on the interval, directed from the center to the circle, from the seam $M_0^{(+)} L_1^{(-)}$. Finally, on cut I for the interval going from the circle to the center we choose a value from the seam $L_0^{(-)} L_{-1}^{(+)}$, and for the interval going from the center to the circle – from the seam $L_0^{(+)} L_1^{(-)}$.

We now move on to the computation of our integral for different values of ϱ, t, and ϑ.

Our goal is to conduct this computation in the closed form in cases 1), 2), and 3). In case 4), this integral is not expressible in closed form.

Let us agree, first, on how we choose the values of $\ln \frac{1}{i} \ln \zeta$ on different parts of our contour. We begin with case 3).

In view of the mnemonic rule that we obtained earlier, on the circle $|\xi| = 1$ we have to always take the values of $\ln \frac{1}{i} \ln \zeta$ such that the roots of $\frac{1}{i} \ln \zeta$ would be outside this circle. However, in this case the part of the integral computed over this circle is simply the integral over the closed contour of the function single-valued inside this contour.

For convenience, we choose, for example, these values such that the value of the regular function $\ln \frac{1}{i} \ln \zeta$ at the point $\xi = 0$ on the sheet M_0, where this contour is located, would be, for $\vartheta = \vartheta_0$, equal to $i\frac{\pi}{2} + \ln \ln \frac{\varrho_0}{\varrho}$.

On the parts of radii $\arg \xi = \vartheta_0$ and $\arg \xi = \vartheta_0 + \pi$ we can choose the value of $\ln \frac{1}{i} \ln \zeta$ completely arbitrary, because the integrals over these parts cancel out at the end, as we have already pointed out above.

It is not difficult as well to compute directly this integral for discussed case 3).

Its value is obtained directly from the Cauchy formula and is equal to the value of the function $\ln \frac{1}{i} \ln \zeta$ at the origin, i.e.,

$$W = \frac{1}{2\pi i} \int\limits_C \frac{1}{\xi} \ln \frac{1}{i} \ln \zeta \, d\xi$$

$$= \ln\left[(\vartheta-\vartheta_0)+\frac{1}{i}\ln\frac{\varrho}{\varrho_0}\right] = \ln\frac{1}{i}\ln\frac{z}{z_0}, \tag{24}$$

where

$$z=\varrho e^{i\vartheta} \quad \text{and} \quad z_0=\varrho_0 e^{i\vartheta_0}. \tag{25}$$

We now move on to the study of case 2). First, we have to note that here we cannot use the Cauchy formula in its original form, since the integral written by us is not, generally speaking, the contour integral of a single-valued function yet. The roots of the function $\frac{1}{i}\ln\zeta$ can get inside the contour, and ln of this function is not coming back to the original value as we move around the contour.

If we recall the mnemonic rule given above, then we can easily select the domains where the roots have to be moved inside the contour.

Let us draw in the (ϱ,ϑ)-plane the circle of radius *at* centered at the point ϱ_0, ϑ_0, and construct two circles tangent to the given one with centers at the origin. Then, we draw through the tangency points of these circles their common tangents. In this case, our entire plane with the cut $\vartheta_0-\pi<\vartheta<\vartheta_0+\pi$ will be divided into domains marked on the drawing (see Fig. 5).

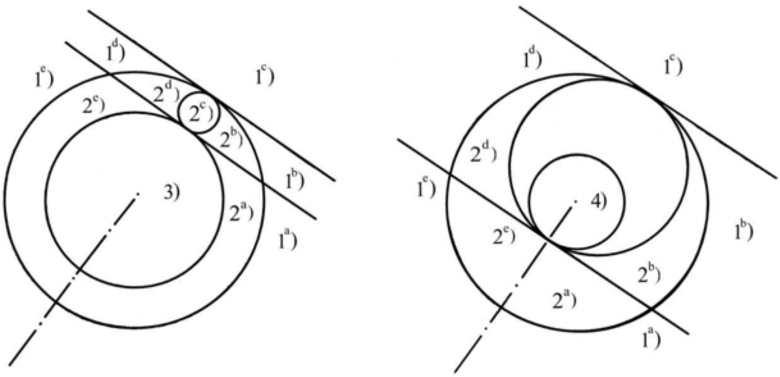

Fig. 5.

Tracing the motion of the roots of $\ln\frac{1}{i}\ln\zeta$ for $\vartheta<\vartheta_0-\pi$ or for $\vartheta>\vartheta_0+\pi$, we see that these roots are not related at all to the chosen contour. In the plane $\vartheta_0-\pi<\vartheta<\vartheta_0+\pi$ we brake case 2) into 5 subcases 2^a), 2^b), 2^c), 2^d), 2^e). Schematically, the motion of the roots in this case can be represented on the following drawing (see Fig. 6).

In interval 2^a) the increment ϑ of roots is not present. In interval 2^b) one of the roots enters inside the contour at the point $\xi=e^{i(\vartheta_0-\pi)}$ and moves inside the disk along the contour in the positive direction. In interval 2^c) the root

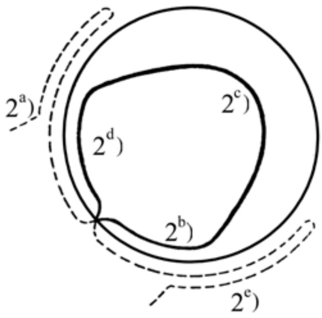

Fig. 6.

enters inside the disk $|\xi| \le 1$ and moves along a certain path to another point of the contour. Then, in interval 2^d) this root moves along the circle on the inner side, and, finally, for $\xi = e^{-i(\vartheta_0+\pi)}$ leaves this interval. On interval 2^e) there are no roots inside the domain. All computations for subcases 2^a) and 2^e) are carried out similarly to the previous one, and the result is as before

$$W = \frac{1}{2\pi i}\int\limits_C \frac{1}{\xi}\ln\frac{1}{i}\ln\zeta\, d\xi = \ln\frac{1}{i}\ln\frac{z}{z_0}. \tag{24}$$

Indeed, in this case, adding to the integrands in both integrals taken over the segments of the radius $\arg\xi = \vartheta_0 \pm \pi$, the multiple $2\pi i$, in this case, we can achieve that the integrals computed over intervals of these radii along the joint cut II and I would comprise one closed contour together with integrals computed over the circle $|\xi| = 1$.

Obviously, adding multiples of $2\pi i$ does not change the value of the integral in the large, since added terms simply cancel out like the integrals over parts of the radii.

It is interesting that these additional terms differ on parts 2^a) and 2^e), since the integrands in integrals over the inner contours, when ϑ changes from $-\infty$ to $+\infty$, get an increment of $-\pi$, while the integrands in integrals over the circle get an increment of π. In this case, it is obvious that the continuous continuation of the integrands from the circle to the radii is possible only when the additional terms in 2^a) and 2^e) also differ by 2π.

Let us compute now the integral W in cases 2^b), 2^c), and 2^d). This computation in all three cases is the same. We have

$$W = \frac{1}{2\pi i}\int\limits_C \ln\frac{1}{i}\ln\zeta\,\frac{d\xi}{\xi} = \frac{1}{2\pi i}\int\limits_C \ln\frac{1}{i}\frac{\ln\zeta}{\xi-\zeta^{(1)}}\,\frac{d\xi}{\xi} + \frac{1}{2\pi i}\int\limits_C \ln(\xi-\zeta^{(1)})\,\frac{d\xi}{\xi},$$

where $\zeta^{(1)}$ is a root of the function $\dfrac{1}{i}\ln\zeta$.

It is easy to see that the first integral, adding again some multiple of $2\pi i$ to the integrand on the radii, can be transformed to the integral computed over a closed contour, because the function $\dfrac{\ln\zeta}{\xi-\zeta^{(1)}}$ has no zeros inside the contour.

In this case, the Cauchy formula gives

$$\frac{1}{2\pi i}\int\limits_C \ln\frac{1}{i}\frac{\ln\zeta}{\xi-\zeta^{(1)}}\frac{d\xi}{\xi} = \ln\frac{1}{i}\ln\zeta|_{\xi=0} - \ln(-\zeta^{(1)})$$

$$= \ln\frac{1}{i}\ln\frac{z}{z_0} - \ln\zeta^{(1)} + \pi i.$$

To compute the second integral, we write it in the form

$$\frac{1}{2\pi i}\int\limits_C \ln\left(\frac{\xi-\zeta^{(1)}}{\xi}\right)\frac{d\xi}{\xi} + \frac{1}{2\pi i}\int\limits_C \ln\xi\,\frac{d\xi}{\xi}.$$

Here, the first term is the integral over a closed contour of the function regular at infinity and vanishing there as $\dfrac{1}{\xi^2}$. Therefore, it vanishes. Regarding the second term, it is equal to $\dfrac{1}{2\pi i}\ln^2\xi|_C$, i.e., to the difference of values of the functions $\dfrac{1}{2\pi i}\ln^2\xi$ going around the contour C.

Obviously, in this case, we have to start at the point $\xi = e^{i(\vartheta_0-\pi)}$ and proceed to the point $\xi = e^{i(\vartheta_0+\pi)}$. In this case, we have

$$\frac{1}{2\pi i}\int\limits_C \ln\xi\frac{d\xi}{\xi} = \frac{1}{4\pi i}\{[i(\vartheta_0+(2k+1)\pi)]^2 - [i(\vartheta_0+(2k-1)\pi)]^2\} = i(\vartheta_0+2k\pi).$$

Obviously, the constant k can be attributed to the value of $\ln\zeta^{(1)}$.

Finally, in this case,

$$W = \ln\frac{1}{i}\ln\frac{z}{z_0} - \ln\zeta^{(1)} + i\vartheta_0 + \pi i. \tag{26}$$

The value of $\ln\zeta^{(1)}$ can be determined by continuity moving from cases 2^a) and 2^e).

In exactly the same way, the integral in case 1) is computed.

Similarly to what has occurred in case 2^a) and 2^e), in cases 1^a) and 1^e) we should not include any roots inside the contour. In this case, it is obvious that

$$W = \ln\frac{1}{i}\ln\frac{z}{z_0}.$$

For cases 1^b) and 1^d), which are similarly to studied case 2^b), 2^c), and 2^d), we introduce one root inside the contour. In this case, computations give

$$\begin{aligned} &1^{b}) \quad W = \ln \tfrac{1}{i} \ln \tfrac{z}{z_0} - \ln \zeta^{(1)} + i\vartheta_0 + \pi i, \\ &1^{d}) \quad W = \ln \tfrac{1}{i} \ln \tfrac{z}{z_0} - \ln \zeta^{(2)} + i\vartheta_0 + \pi i. \end{aligned} \tag{27}$$

Here, $\zeta^{(1)}$ denotes the root located on the semicircle $\vartheta_0 - \pi < \arg \zeta^{(1)} < \vartheta_0$, and $\zeta^{(2)}$ is the root located on the semicircle $\vartheta_0 < \arg \zeta^{(2)} < \vartheta_0 + \pi$.

In perfect analogy, we can compute the integral in case 1^c), where we move both roots $\zeta^{(1)}$ and $\zeta^{(2)}$ inside the contour.

We have

$$W = \frac{1}{2\pi i} \int\limits_C \ln \frac{1}{i} \ln \zeta \, \frac{d\xi}{\xi} = \frac{1}{2\pi i} \int\limits_C \ln \frac{1}{i} \frac{\ln \zeta}{(\xi - \zeta^{(1)})(\xi - \zeta^{(2)})} \frac{d\xi}{\xi}$$

$$+ \frac{1}{2\pi i} \int\limits_C \ln \frac{(\xi - \zeta^{(1)})(\xi - \zeta^{(2)})}{\xi^2} \frac{d\xi}{\xi} + \frac{2}{2\pi i} \int\limits_C \ln \xi \, \frac{d\xi}{\xi},$$

and again, in view of the Cauchy theorem and because of the regularity of the integrand in the second integral,

$$W = \ln \frac{1}{i} \ln \frac{z}{z_0} - \ln \zeta^{(1)} - \ln \zeta^{(2)} + 2i\vartheta_0 - 2\pi i. \tag{28}$$

Let us now compose the function

$$\omega = \ln \frac{1}{i} \ln \frac{z}{z_0} - W - \pi i. \tag{29}$$

From the entire previous discussion, it is not difficult to see what this function is equal to in cases 1), 2), and 3).

This function vanishes on all sheets except the zero sheet.

On the zero sheet, it is zero in zone 3).

In zones 2) and 3) of the zero sheet, ω is simply $\ln \zeta^{(1)}$, moreover, inside the disk centered at ϱ_0, ϑ_0 with the radius *at* its values are defined by the usual rule in the interior of the unit disk, and in the exterior they are continued in the following way.

The contour C_1 is divided into two parts by the diameter passing through the coordinate origin. One of these parts I gives the values of $\arg \zeta^{(1)}$ located between $\vartheta_0 - \pi$ and ϑ_0, and part II gives the values of $\arg \zeta^{(1)}$ located between ϑ_0 and $\vartheta_0 + \pi$.

The function $\ln \zeta^{(1)}$ is divided into the sum of two functions, each of which preserves a constant value on one of our semicircles

$$\ln \zeta^{(1)} = \varphi_1(\zeta') + \varphi_2(\zeta'').$$

The function $\varphi_1(\zeta')$, preserving constant value on cut I, continues into the exterior of the circle C_1 via a family of half-tangents in the counterclockwise

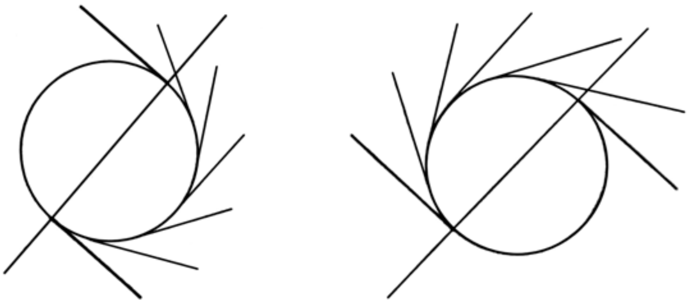

Fig. 7.

direction, and the function $\varphi_2(\zeta'')$, keeping constant value on cut II, continues via a system of half-tangents in the clockwise direction (see Fig. 7).

The real part of this function ω, as is easy to verify directly on the zero sheet of the Riemann surface, is exactly equal to the Volterra solution[8]

$$V = \ln\left(\frac{at}{r_1} - \sqrt{\frac{a^2t^2}{r_1^2} - 1}\right). \tag{30}$$

From the general reasoning that we developed in the first part, it follows that in case 4) as well we obtain the solution of the wave equation such that the function ω is the limiting solution in the entire domain.

This function is the diffraction of the elementary solution.

Before we begin to solve the Cauchy problem by using this function, let us study in more detail some of its properties.

Let us estimate values of this function as $\varrho \to 0$ and $\vartheta \to \pm\infty$, assuming ϱ_0, t, and ϑ_0 are fixed. Obviously, we have to study only case 4).

Evaluating ω, we notice that

$$\ln\frac{1}{i}\ln\frac{z}{z_0}$$

can be easily represented in the form

$$\frac{1}{2\pi i}\int\limits_C \ln\frac{1}{i}\ln\frac{z}{z_0}\,\frac{d\xi}{\xi}.$$

Whence we obtain for case 4)

$$\omega = \frac{1}{2\pi i}\int\limits_C \left\{\ln\frac{1}{i}\ln\frac{z}{z_0} - \ln\frac{1}{i}\ln\zeta\right\}\frac{d\xi}{\xi} - \pi i.$$

[8] Here, $r_1 = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. – *Ed.*

Since we are interested only in the real part of the function ω, then πi can be omitted. We have

$$\operatorname{Re}\omega = \operatorname{Re}\left\{-\frac{1}{2\pi i}\int\limits_C \ln\left[\frac{\vartheta - \frac{1}{i}\ln\xi + \arccos\frac{at_1}{\varrho}}{\vartheta - \vartheta_0 + \frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0}\right]\frac{d\xi}{\xi}\right\}. \tag{31}$$

Let us estimate this integral for large values of ϑ and small ϱ. For this purpose we denote

$$\frac{\vartheta - \frac{1}{i}\ln\xi + \arccos\frac{at_1}{\varrho}}{\vartheta - \vartheta_0 + \frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0} = 1 + \chi.$$

The quantity χ tends to zero as ϑ growths, in this case

$$\operatorname{Re}\omega = \operatorname{Re}\left\{-\frac{1}{2\pi i}\int\limits_C \ln(1+\chi)\frac{d\xi}{\xi}\right\}.$$

It is not important for the computation of the real part which branch of ln in the integrand is chosen.

Therefore, we choose the branch which vanishes for large values of ϑ.

We transform formula (31) to the form

$$\operatorname{Re}\omega = -\operatorname{Re}\left\{\frac{1}{2\pi i}\int\limits_C [\ln(1+\chi) - \chi]\frac{d\xi}{\xi} + \frac{1}{2\pi i}\int\limits_C \chi\frac{d\xi}{\xi}\right\}. \tag{32}$$

First, we calculate the second term.

Since χ is expressed rationally by $\dfrac{1}{i}\ln\zeta$, then the Riemann surface for χ is the same as for $\dfrac{1}{i}\ln\zeta$.

Obviously,

$$\chi = \frac{-\frac{1}{i}\ln\left[\xi\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)\right] - \frac{1}{i}\ln\varrho + \frac{1}{i}\ln\varrho_0 + \vartheta_0}{\vartheta - \vartheta_0 + \frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0}. \tag{33}$$

Whence, on the sheet M_0 we have[9] $\chi(0) = 0$.

Next, it is possible to note that the contour, which we use to integrate χ, can be transformed into a closed one on the Riemann surface.

Only the integrals, taken over both sides of a segment of the line $\arg\xi = \vartheta_0 - \pi$, along the part where we have only cut I, can obstruct doing this.

[9] See formulas (20)–(22). – *Ed.*

Denoting this segment by l, in the integral taken over C we have two terms of the form

$$\frac{1}{2\pi i}\int\limits_{l^+} \chi|_{L_0^{(-)}=L_{-1}^{(+)}}\frac{d\xi}{\xi} + \frac{1}{2\pi i}\int\limits_{l^-} \chi|_{L_0^{(+)}=L_1^{(-)}}\frac{d\xi}{\xi}.$$

Here, l^+ denotes the direction from the circle $|\xi| = 1$ to the critical point ξ_2. Developing in more detail these integrals, we have

$$\frac{1}{2\pi i}\int\limits_{l^+} \chi|_{L_0^{(-)}}\frac{d\xi}{\xi} = \frac{1}{2\pi i}\int\limits_{-e^{i\vartheta_0}}^{\xi_2} \frac{\frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0 + \pi + \frac{1}{i}\ln|\xi| - \arccos_0\frac{at_1}{\varrho}}{(\vartheta-\vartheta_0) + \frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0}\frac{d\xi}{\xi},$$

$$\frac{1}{2\pi i}\int\limits_{l^-} \chi|_{L_0^{(+)}}\frac{d\xi}{\xi} = \frac{1}{2\pi i}\int\limits_{\xi_2}^{-e^{i\vartheta_0}} \frac{\frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0 - \pi + \frac{1}{i}\ln|\xi| + \arccos_0\frac{at_1}{\varrho}}{(\vartheta-\vartheta_0) + \frac{1}{i}\ln\varrho - \frac{1}{i}\ln\varrho_0}\frac{d\xi}{\xi}.$$

Combining these integrals, after simple computations we obtain

$$\frac{1}{2\pi i}\int\limits_{l^+}\left\{\chi|_{L_0^{(-)}} - \chi|_{L_0^{(+)}}\right\}\frac{d\xi}{\xi} = \frac{1}{2\pi i}\int\limits_{l^+}\left\{\chi|_{M_0^{(-)}} - \chi|_{M_0^{(+)}}\right\}\frac{d\xi}{\xi}. \tag{34}$$

Taking into account this equality and replacing the contour C by the closed contour, located on the sheet M_0 and consisting of the circle $|\xi| = 1$ and pieces of the radius $\arg\xi = \vartheta_0 - \pi$, we obtain

$$\frac{1}{2\pi i}\int\limits_C \chi\frac{d\xi}{\xi} = \chi(0) = 0.$$

Thus, it remains only to estimate the integral

$$\frac{1}{2\pi i}\int\limits_C [\ln(1+\chi) - \chi]\frac{d\xi}{\xi}. \tag{35}$$

Because of our selection of the values of ln, for all χ with the module less than one, $|\chi| < q < 1$, we have a very simple estimate of the module of the integrand in (35)

$$\left|[\ln(1+\chi) - \chi]\frac{1}{\xi}\right| \le M\frac{|\chi|^2}{|\xi|},$$

where M is a certain absolute constant. Let us note that on the contour C, for fixed t, $|\xi| > |\xi_1| > K$, where K is a constant not depending on ϱ and ϑ. Because of this, the last integral goes to zero as χ tends to zero, i.e., with the growth of ϑ, and vanishes as $\dfrac{1}{\vartheta^2}$. However, for our purposes we need to make a somewhat more exact estimate.

We estimate the function $|\chi^2|$ separately on the circle $|\xi| = 1$, on the interval of the rectilinear part of the contour where only cut I is present, and on the interval where cuts I and II are present. The integral over the intervals, where cuts I and III are joint, cancels out, as we have repeatedly noted.

On the circle the values of χ are taken from the sheet M_0, therefore, $\arccos \dfrac{at_1}{\varrho}$ is represented in the form

$$-\frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right) = -\frac{1}{i}\ln(at_1 + \sqrt{a^2t_1^2 - \varrho^2}) + \frac{1}{i}\ln\varrho.$$

The function $\ln(at_1 + \sqrt{a^2t_1^2 - \varrho^2})$ satisfies the obvious inequalities

$$\ln at_1 < \ln(at_1 + \sqrt{a^2t_1^2 - \varrho^2}) < \ln at_1 + \ln 2.$$

Taking into account that at_1 on our circle, for the case in question, varies in the following limits $at - \varrho_0 < at_1 < at + \varrho_0$, we obtain

$$\left|-\frac{1}{i}\ln(at_1 + \sqrt{a^2t_1^2 - \varrho^2})\right| < M_1, \tag{36}$$

where M_1 is a constant depending only on ϱ_0.

Using inequality (36), we can already estimate the quantity $|\chi|$ more precisely.

To avoid the value $\varrho = 1$, we divide our study into two separate cases, first discussing only the values $\varrho < \dfrac{1}{2}$.

Dividing the numerator and the denominator of our expression of χ by $\dfrac{1}{i}\ln\varrho$, we obtain

$$\chi = \frac{\frac{i\vartheta_0}{\ln\varrho} - 1 + \frac{\ln\varrho_0}{\ln\varrho} - \frac{\ln\xi}{\ln\varrho} - \frac{\ln(at_1 + \sqrt{a^2t_1^2 - \varrho^2})}{\ln\varrho}}{\frac{i(\vartheta - \vartheta_0)}{\ln\varrho} + 1 - \frac{\ln\varrho_0}{\ln\varrho}}.$$

Whence we easily see that $|\chi^2|$ in this case satisfies the inequality

$$|\chi^2| \le \frac{M}{A + \frac{(\vartheta - \vartheta_0)^2}{(\ln\varrho)^2}}, \tag{37}$$

where A and M are constants for fixed ϱ_0, ϑ_0, and t.

If we consider now $\dfrac{1}{2} < \ln\varrho < N$, then immediately it is possible to obtain similarly

$$|\chi^2| \le \frac{M}{A + (\vartheta - \vartheta_0)^2}. \tag{38}$$

To estimate $|\chi|$ on the interval, where cuts I and III are made, it suffices to take into account that for $\left(\dfrac{at_1}{\varrho}\right) > 1$ the ratio

$$\frac{\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)}{\ln\varrho}$$

converges to a finite limit as $\varrho \to 0$ and, hence, has an exact upper boundary independent of ϱ.

Whence, for our interval we have

$$|\chi^2| \leq \frac{M}{A + \frac{(\vartheta-\vartheta_0)^2}{(\ln\varrho)^2}}$$

for $\varrho < \dfrac{1}{2}$. The estimate is even simpler,

$$|\chi^2| \leq \frac{M}{A + (\vartheta - \vartheta_0)^2}$$

for $\dfrac{1}{2} < \varrho < N$.

It remains for us to estimate the quantity $|\chi^2|$ on cuts I and II.

Taking into account that there $\arccos\dfrac{at_1}{\varrho}$ is a real number bounded by finite limits, we easily obtain the same estimate as in the previous case.

The estimate obtained for $|\chi|^2$ is carried out to the function $\mathrm{Re}\,\omega$, if we recall that the module of $\dfrac{1}{\xi}$ on the contour C is bounded by a constant independent of ϱ and ϑ. Let us also note that this estimate is uniform for bounded t and ϱ_0. This fact is easily verified directly.

Next, it is also necessary to estimate $\dfrac{\partial \mathrm{Re}\,\omega}{\partial t}$. We have

$$\frac{\partial \mathrm{Re}\,\omega}{\partial t} = \mathrm{Re}\,\frac{1}{2\pi i}\int\limits_C \frac{\frac{\partial}{\partial t}\left[\pm\frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)\right]}{\vartheta - \frac{1}{i}\ln\xi_0 \pm \frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)}\frac{d\xi}{\xi}. \tag{39}$$

Let us expand this integral in more detail. First, we denote different parts of the contour where

$$\pm\frac{1}{i}\ln\left(\frac{at_1}{\varrho} + \sqrt{\frac{a^2t_1^2}{\varrho^2} - 1}\right)$$

is computed from the different rules. Let l_1 be an arc of the circle $|\xi| = 1$, $\vartheta_0 - \pi < \arg\xi < \vartheta_0 + \pi$, let l_2 be the segment of the radius $\arg\xi = \vartheta_0 + \pi$

from the unit circle to the point ξ_2. We denote by l_3 the interval along the segment of the same radius from the point ξ_2 to the point ξ_4, and by l_4 the segment l_3 in the opposite direction, and by l_5 the interval l_2 taken in the opposite direction.

Then, taking into account the notes made above regarding the values of

$$\pm\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right),$$

we obtain

$$\begin{aligned}\frac{\partial \operatorname{Re}\omega}{\partial t} = \operatorname{Re}\frac{1}{2\pi i}\Biggl\{\int\limits_{l_1} & \frac{-\frac{\partial}{\partial t}\left[\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}{\vartheta-\frac{1}{i}\ln_0\xi-\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)-\pi}\frac{d\xi}{\xi}\\
&+\int\limits_{l_2}\frac{\frac{\partial}{\partial t}\left[\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}{\vartheta-\frac{1}{i}\ln|\xi|+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)-\pi}\frac{d\xi}{\xi}\\
&+\int\limits_{l_3}\frac{\frac{\partial}{\partial t}\operatorname{arccos}_*\frac{at_1}{\varrho}}{\vartheta-\frac{1}{i}\ln|\xi|+\operatorname{arccos}_*\frac{at_1}{\varrho}-\pi}\frac{d\xi}{\xi}\\
&+\int\limits_{l_4}\frac{-\frac{\partial}{\partial t}\operatorname{arccos}_*\frac{at_1}{\varrho}}{\vartheta-\frac{1}{i}\ln|\xi|-\operatorname{arccos}_*\frac{at_1}{\varrho}+\pi}\frac{d\xi}{\xi}\\
&+\int\limits_{l_5}\frac{\frac{\partial}{\partial t}\left[\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}{\vartheta-\frac{1}{i}\ln|\xi|+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)+\pi}\frac{d\xi}{\xi}\Biggr\}.\end{aligned}$$

Here, $\operatorname{arccos}_*\dfrac{at_1}{\varrho}$ denotes the function real in the interval

$$-1<\frac{at_1}{\varrho}<1$$

and equals

$$\frac{\pi}{2}-\int\limits_0^{\frac{at_1}{\varrho}}\frac{dy}{\sqrt{1-y^2}}.$$

It is easy to see that on l_1, l_2, and l_5,

$$\frac{\partial}{\partial t}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)=\frac{a}{\varrho}\frac{1}{\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}}=\frac{a}{\sqrt{a^2t_1^2-\varrho^2}}$$

$$=\frac{a}{\sqrt{\frac{\varrho_0^2}{4\xi^2}(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$=\frac{-2a\xi}{\varrho_0\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}.$$

Here, the root $\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}$ is chosen negative for positive values of ξ, and with the continuous change of ξ this root becomes positive for negative ξ. Moreover,

$$\frac{\partial}{\partial t}\operatorname{arccos}_*\frac{at_1}{\varrho}=\frac{a}{\varrho}\frac{-1}{\sqrt{1-\frac{a^2t_1^2}{\varrho^2}}}$$

$$=\frac{-a}{\sqrt{\frac{-\varrho_0^2}{4\xi^2}(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$=\frac{2a\xi}{\varrho_0\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}},$$

where

$$\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}$$

is chosen positive for negative ξ.

Taking into account these relations, we rewrite $\dfrac{\partial \operatorname{Re}\omega}{\partial t}$ in the form

$$\frac{\partial \operatorname{Re}\omega}{\partial t}=\operatorname{Re}\frac{1}{2\pi i}\left\{\frac{-i}{\varrho_0}\int\limits_{l_1}\frac{2a}{\left[\vartheta-\frac{1}{i}\ln|\xi|-\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}\right.$$

$$\times\frac{d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$+\frac{-i}{\varrho_0}\int\limits_{l_2}\frac{2a}{\left[\vartheta-\frac{1}{i}\ln|\xi|+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)-\pi\right]}$$

$$\times\frac{d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$+\frac{1}{\varrho_0}\int\limits_{l_3}\frac{2a\,d\xi}{\left[\vartheta-\frac{1}{i}\ln|\xi|+\operatorname{arccos}_*\frac{at_1}{\varrho}-\pi\right]\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$+\frac{1}{\varrho_0}\int\limits_{l_4}\frac{2a\,d\xi}{\left[\vartheta-\frac{1}{i}\ln|\xi|+\arccos_*\frac{at_1}{\varrho}+\pi\right]\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$-\frac{i}{\varrho_0}\int\limits_{l_5}\frac{2a}{\left[\vartheta-\frac{1}{i}\ln|\xi|+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)+\pi\right]}$$

$$\left.\times\frac{d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}\right\}. \tag{40}$$

To estimate it, let us now combine the integrals taken over the intervals l_2 and l_5. Changing the order of integration on l_5, we obtain

$$\int\limits_{l_2}F_2\,d\xi+\int\limits_{l_5}F_5\,d\xi=\int\limits_{l_2}\frac{2\pi}{\left[\left(\vartheta-\frac{1}{i}\ln|\xi|+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right)^2-\pi^2\right]}$$

$$\times\frac{2a\,d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}.$$

Then, we note the elementary equality

$$-\frac{i}{\varrho_0}\int\limits_{l_1}\frac{2a\,d\xi}{\vartheta\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$+\frac{1}{\varrho_0}\int\limits_{l_3}\frac{2a\,d\xi}{\vartheta\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$-\frac{1}{\varrho_0}\int\limits_{l_4}\frac{2a\,d\xi}{\vartheta\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}=0$$

following from the Cauchy theorem on contour integrals. Subtracting the left side of this equality term by term from $\dfrac{\partial\operatorname{Re}\omega}{\partial t}$, we present this last expression in the form

$$\frac{\partial\operatorname{Re}\omega}{\partial t}=\operatorname{Re}\frac{1}{2\pi i}\left\{\frac{-2ai}{\varrho_0}\int\limits_{l_1}\frac{\left[\frac{1}{i}\ln_0\xi+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}{\vartheta\left[\vartheta-\frac{1}{i}\ln_0\xi-\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]}\right.$$

$$\times\frac{d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$

$$-\frac{2ai}{\varrho_0}\int\limits_{l_2}\frac{2\pi}{\left[\vartheta-\frac{1}{i}\ln\xi+\frac{1}{i}\ln\left(\frac{at_1}{\varrho}+\sqrt{\frac{a^2t_1^2}{\varrho^2}-1}\right)\right]^2-\pi^2}$$
$$\times\frac{d\xi}{\sqrt{(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$
$$+\frac{2a}{\varrho_0}\int\limits_{l_3}\frac{\left(\frac{1}{i}\ln|\xi|-\arccos_*\frac{at_1}{\varrho}+\pi\right)d\xi}{\vartheta\left[\vartheta-\frac{1}{i}\ln|\xi|+\arccos_*\frac{at_1}{\varrho}-\pi\right]\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}$$
$$-\frac{2a}{\varrho_0}\int\limits_{l_4}\frac{\left(\frac{1}{i}\ln|\xi|+\arccos_*\frac{at_1}{\varrho}-\pi\right)}{\vartheta\left[\vartheta-\frac{1}{i}\ln|\xi|-\arccos_*\frac{at_1}{\varrho}+\pi\right]}$$
$$\left.\times\frac{d\xi}{\sqrt{-(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}}\right\}. \qquad (41)$$

Now, the estimate makes no longer any problems.

As is easy to verify, in the assumption that $\varrho_0 < at$ we have for $\dfrac{\partial \mathrm{Re}\,\omega}{\partial t}$ the same inequalities as the ones obtained for $\mathrm{Re}\,\omega$,

$$\left|\frac{\partial \mathrm{Re}\,\omega}{\partial t}\right| \le \frac{M}{A+\frac{(\vartheta-\vartheta_0)^2}{(\ln\varrho)^2}}, \quad |\mathrm{Re}\,\omega| \le \frac{M}{A+\frac{(\vartheta-\vartheta_0)^2}{(\ln\varrho)^2}} \quad \text{for} \quad \varrho<\frac{1}{2}, \qquad (42)$$

$$\left|\frac{\partial \mathrm{Re}\,\omega}{\partial t}\right| \le \frac{M}{A+(\vartheta-\vartheta_0)^2}, \quad |\mathrm{Re}\,\omega| \le \frac{M}{A+(\vartheta-\vartheta_0)^2} \text{ for } \frac{1}{2}<\varrho<N. \qquad (43)$$

The estimates on $\mathrm{Re}\,\omega$ are uniform for ϱ_0 and t, and the estimates for $\dfrac{\partial \mathrm{Re}\,\omega}{\partial t}$ are uniform for ϱ_0 and t, if $at > \varrho_0+\delta$, where δ is a fixed number.

We could show how the constants A and M change with the change of ϱ_0 and t, but because of absolute simplicity of these estimates we leave them aside.

To finish the study of this question, let us also consider $\dfrac{\partial \mathrm{Re}\,\omega}{\partial \vartheta}$. It is obvious that this function approaches zero as

$$\frac{M}{A+\frac{(\vartheta-\vartheta_0)^2}{(\ln\varrho)^2}}$$

with ϑ increasing for small values of ϱ, and as

$$\frac{M}{A+(\vartheta-\vartheta_0)^2}$$

for values of $\varrho > \dfrac{1}{2}$.

3. Now we can move on to the search for weak solutions of the Cauchy problem on the Riemann surface.

Since this solution is in perfect analogy with the one already carried out for the regular Cauchy problem on the simple plane, then we only outline here the basic points of the derivation.

In correspondence to a certain point ϱ_0, ϑ_0, t_0 or, which is the same, x_0, y_0, t_0 we construct in our Riemann (x, y, t)-space, with the branching axis $x_0 = 0$, $y_0 = 0$, the cone of characteristics with its surface bounded in limits of that part of the space, where $\vartheta_0 - \pi < \vartheta < \vartheta_0 + \pi$. Next, in the entire remaining part we supplement this surface by the multi-sheeted surface of the cone of characteristics with an apex at the point $x_1 = 0$, $y_1 = 0$, $t_1 = t_0 - \dfrac{\varrho_0}{a}$. Its equation is

$$x^2 + y^2 = a^2(t - t_1)^2.$$

Thus, we obtain a surface which we call the surface of a wave for the given point x_0, y_0, t_0.

Let us again consider the domain D_ε and the function ψ_ε, which we determined in Sect. 5 of Chap. 1. We define the function v_ε in our Riemann space, inside the surface of a wave, by using the previous conditions

$$v_\varepsilon\big|_{(x-x_0)^2+(y-y_0)^2=a^2(t-t_0)^2} = 0 \tag{44}$$

and

$$\Box v_\varepsilon = \psi_\varepsilon. \tag{45}$$

It can be easily verified that this function can be defined by using the formula

$$v_\varepsilon(x, y, t; x_0, y_0, t_0) = -\frac{1}{2\pi}\frac{\partial}{\partial t}\iiint\limits_{D_\varepsilon} \psi_\varepsilon(x_1 - x_0, y_1 - y_0, t_1 - t_0)$$

$$\times \omega(x, y, t; x_1, y_1, t_1)\, dx_1\, dy_1\, dt_1. \tag{46}$$

Indeed, if we calculate it, for example, at points with coordinates satisfying the condition $t > t_0 - \dfrac{\varrho_0}{a}$, then this function entirely coincides for $\vartheta_0 - \dfrac{\pi}{2} < \vartheta < \vartheta_0 + \dfrac{\pi}{2}$ with the function given in formula (79) of the first part, and vanishes at remaining ϑ. For all other points, it is directly obvious that it satisfies both conditions stated.

Now we solve the problem of integration of the wave equation on the Riemann surface with initial data

$$u|_{t=0} = u_0(\varrho, \vartheta), \tag{47}$$

$$\iint\limits_{\Omega} v\frac{\partial u}{\partial t}\bigg|_{t=0} dS = L(\Omega, v). \tag{48}$$

Let us construct the domain bounded by the surface of a wave for the point x_0, y_0, t_0, by the plane $t = 0$ and two half-planes $\vartheta = \alpha$ and $\vartheta = \beta$.

By the Lebesgue theorem, it is always possible to choose α and β such that they would be the surfaces of complete summability.

Applying to this domain the Green formula obtained in Chap. 1 and taking into account that on the surface of wave v_ε and $\dfrac{\partial v_\varepsilon}{\partial \nu}$ vanish, we obtain

$$\iint\limits_{D_\varepsilon} \psi_\varepsilon u \, dr = \iint\limits_{S_1} \left(v \overline{\frac{\partial u}{\partial \nu}} - u \overline{\frac{\partial v}{\partial \nu}} \right) dS$$

$$+ \iint\limits_{S_2} \left(v \overline{\frac{\partial u}{\partial \nu}} - u \overline{\frac{\partial v}{\partial \nu}} \right) dS + \iint\limits_{S_3} \left(v \overline{\frac{\partial u}{\partial t}} - u \overline{\frac{\partial v}{\partial t}} \right) dS. \tag{49}$$

(The application of this formula, strictly speaking, requires also an additional study of the behavior of v_ε near $\varrho = 0$. However, it is not difficult to clarify the validity of all these operations under sufficiently wide assumptions.)

Here, S_1 and S_2 denote the parts of surfaces $\vartheta = \alpha$ and $\vartheta = \beta$, which are the boundaries of the domain G, and S_3 denotes the part of the plane $t = 0$, which is such boundary.

We pass now to the limit as $\alpha \to -\infty$ and $\beta \to \infty$.

If the integrals over S_1 and S_2 cancel out in this case, then we obtain the solution of the problem repeating word by word all arguments of Chap. 1. The uniqueness of the solution in this case follows from the algorithm itself.

It is not difficult to indicate the cases when the solution probably exists. It happens, for example, when the function $u|_{t=0}$ is continuous and bounded on all sheets, and the function $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ exists and is summable.

This easily follows from the estimates we gave for ω, $\dfrac{\partial \omega}{\partial t}$, and $\dfrac{\partial \omega}{\partial \vartheta}$.

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8. The Problem of Propagation of a Plastic State*

S. L. Sobolev

Summary. The author considers the problem on propagation of a plastic state in an infinite plane with a circular hole, subject to the action of symmetrical forces causing assigned displacements on the boundary.

Author's hypothesis are reduced to the following.

1. Matter of the plane can be only in two states: elastic or plastic; moreover, the plastic zone is expressed in Lagrangian coordinates (ϱ_0, ϑ_0) by the inequalities

$$r_0 < \varrho_0 < R_0(t),$$

and the elastic zone is expressed by the inequality

$$\varrho_0 > R_0(t).$$

The author restricts himself to the consideration of only the motions when $R_0(t)$ is an increasing function.

2. The elastic state of the matter is described by the usual equations of linear elasticity within small quantities of the highest orders.

3. The plastic state is described by the known Saint-Venant equations.

4. The displacement vector, stress tensor, and velocity vector of a particle remain continuous while crossing the boundary between the elastic and plastic zones.

The author indicates the method of computation of all quantities characterizing the motion, i.e., the radius $R_0(t)$, components of the displacement at any point of time in both zones, components of the stress tensor also in both zones and flow lines in the plastic zone.

The research shows that if the angle of rotation given on the internal contour increases too rapidly relative to the radius of expansion of the inner hole, then the characteristics of the plastic zone become tangent to the inner contour, and the problem losses its meaning. From a physical standpoint, this is related to the fact that the break of the matter occurs along the inner boundary.

The rate of growth of the maximal angle of rotation for large values of the radius of expansion of the inner hole is approximately equal to $3 \ln r$.

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This paper was written by S. L. Sobolev in English, its summary was written in Russian. – *Ed.*

1. We consider in the present paper the following problem. An infinite plane with a circular hole is exposed to the action of forces applied to its boundary. We suppose that the values of displacements of the material particles are given at this boundary. Let these values depend only on time and be symmetrical with regard to the center.

According to Lagrange, let us denote the polar coordinates of a material point in undisturbed state by ϱ_0, ϑ_0 and the coordinates of this point at the time t by ϱ, ϑ.

We suppose that the function $\varrho(\varrho_0, t)$ is independent of ϑ_0 and the function $\vartheta(\varrho_0, \vartheta_0, t)$ can be represented in the form

$$\vartheta = \vartheta_0 + \psi(\varrho_0, t),$$

where the function ψ depends only on ϱ_0 and t.

The stresses $\widehat{\varrho\varrho}$, $\widehat{\varrho\vartheta}$, and $\widehat{\vartheta\vartheta}$ (symbols adopted from A. E. H. Love's treatise[1]), being supposed symmetrical, will also depend only on ϱ_0 and t.

At infinity the stresses are supposed to vanish. Accordingly we suppose that our plane consists of two parts: the plastic part

$$r_0 < \varrho_0 < R_0(t)$$

and the elastic one

$$R_0(t) < \varrho_0 < \infty.$$

Let us consider the matter of our plane according to the following hypotheses.

I. The elastic state can be defined by the usual formulas of linear elasticity. The stresses $\widehat{\varrho\varrho}$, $\widehat{\varrho\vartheta}$, and $\widehat{\vartheta\vartheta}$ satisfy the well-known equations of mechanics of continuous media which can be written as

$$\begin{aligned} &\frac{\partial\, \widehat{\varrho\varrho}}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial\, \widehat{\varrho\vartheta}}{\partial \vartheta} + \frac{\widehat{\varrho\varrho} - \widehat{\vartheta\vartheta}}{\varrho} = 0, \\ &\frac{\partial\, \widehat{\varrho\vartheta}}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial\, \widehat{\vartheta\vartheta}}{\partial \vartheta} + \frac{2\, \widehat{\varrho\vartheta}}{\varrho} = 0, \end{aligned} \tag{1}$$

and the displacements are connected with the stresses by the formulas

$$\begin{aligned} \widehat{\varrho\varrho} &= \lambda\left[\frac{1}{\varrho}\frac{\partial}{\partial\varrho}(\varrho u_\varrho) + \frac{1}{\varrho}\frac{\partial u_\vartheta}{\partial\vartheta}\right] + 2\mu\frac{\partial u_\varrho}{\partial\varrho}, \\ \widehat{\vartheta\vartheta} &= \lambda\left[\frac{1}{\varrho}\frac{\partial}{\partial\varrho}(\varrho u_\varrho) + \frac{1}{\varrho}\frac{\partial u_\vartheta}{\partial\vartheta}\right] + 2\mu\left(\frac{1}{\varrho}\frac{\partial u_\vartheta}{\partial\vartheta} + \frac{u_\varrho}{\varrho}\right), \\ \widehat{\varrho\vartheta} &= \mu\left[\frac{\partial u_\vartheta}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial u_\varrho}{\partial\vartheta} - \frac{u_\vartheta}{\varrho}\right], \end{aligned} \tag{2}$$

[1] Love, A. E. H.: A Treatise on the Mathematical Theory of Elasticity. 4th ed. University Press, Cambridge (1927) – *Ed.*

usually called the Hooke law.

We shall study below the finite displacements and velocities in the plastic part of our plane, and we have to demand that these quantities are equal on both sides of the circumference $\varrho_0 = R_0$.

As the expressions for these quantities are exact in the plastic part, they must be exact in the elastic one as well. Therefore, we are induced to consider in our investigation the so-called "nonlinear elasticity".

In order to be exact, we put down all our conditions without neglecting the small terms.

Thus, we suppose that the solutions of equations (1) and (2) give only the principal parts of displacements and stresses. The full expressions differ from these solutions by certain small quantities.

II. The plastic state can be exactly defined by means of the Saint-Venant equations: equations of mechanics of continuous media (1), the condition of plasticity

$$(\overset{\frown}{\varrho\varrho} - \overset{\frown}{\vartheta\vartheta})^2 + 4\,\overset{\frown}{\varrho\vartheta}^{\,2} = 4k^2 \tag{3}$$

and the equation for velocities which can be written in polar coordinates in the form

$$\left(\frac{1}{\varrho}\frac{\partial u'_\vartheta}{\partial\vartheta} + \frac{u'_\varrho}{\varrho} - \frac{\partial u'_\varrho}{\partial\varrho}\right) 2\,\overset{\frown}{\varrho\vartheta} = \left(\frac{\partial u'_\vartheta}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial u'_\varrho}{\partial\vartheta} - \frac{u'_\vartheta}{\varrho}\right)(\overset{\frown}{\vartheta\vartheta} - \overset{\frown}{\varrho\varrho}). \tag{4}$$

We suppose again that our matter is incompressible, i.e.,

$$\frac{\partial u'_\varrho}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial u'_\vartheta}{\partial\vartheta} + \frac{u'_\varrho}{\varrho} = 0. \tag{5}$$

We apply the last hypotheses to the elastic part of our medium too, though in this part we have no new results, the symmetrical solution of linear elasticity equations for an infinite simply connected domain being the same in the case of compressibility or incompressibility.

III. At the boundary between the elastic and the plastic state all the functions ϱ, ϑ, $\overset{\frown}{\varrho\varrho}$, $\overset{\frown}{\vartheta\vartheta}$, $\overset{\frown}{\varrho\vartheta}$, u'_ϱ, and u'_ϑ are continuous:

$$\begin{gathered}\varrho^{(e)} = \varrho^{(p)}, \quad \vartheta^{(e)} = \vartheta^{(p)}, \quad \overset{\frown}{\varrho\varrho}^{(e)} = \overset{\frown}{\varrho\varrho}^{(p)}, \quad \overset{\frown}{\vartheta\vartheta}^{(e)} = \overset{\frown}{\vartheta\vartheta}^{(p)}, \\ \overset{\frown}{\varrho\vartheta}^{(e)} = \overset{\frown}{\varrho\vartheta}^{(p)}, \quad u'^{(e)}_\varrho = u'^{(p)}_\varrho, \quad u'^{(e)}_\vartheta = u'^{(p)}_\vartheta.\end{gathered} \tag{6}$$

The above given symbols are evident. The continuity of the functions ϱ, ϑ, $\overset{\frown}{\varrho\vartheta}$, and $\overset{\frown}{\vartheta\vartheta}$ is obvious.

The continuity of $\overset{\frown}{\varrho\varrho}$ is the mathematical formulation of the hypotheses that the plastic state occupies the smallest zone imaginable.

Indeed, if the elastic zone cannot be extended up to the disk $\varrho_0 < R_0$, it follows that at the outer side of this disk equation (3) is satisfied.

From the continuity on the left side of equation (3) and from that of $\widehat{\varrho\vartheta}$ and $\widehat{\vartheta\vartheta}$ it follows that either $\widehat{\varrho\varrho}$ is continuous or

$$\widehat{\varrho\varrho}^{(e)} + \widehat{\varrho\varrho}^{(p)} = 2\,\widehat{\vartheta\vartheta}\,.$$

We choose the former of these two hypotheses.

The continuity of u'_ϱ and u'_ϑ is a new admission which results from the natural requirement of the slowness of deformation and of the smallness of accelerations.

Indeed, remembering that $\varrho(\varrho_0, \vartheta_0)$ and $\vartheta(\varrho_0, \vartheta_0)$ are continuous, i.e.,

$$\begin{aligned}\varrho^{(e)}(R_0(t), t) &= \varrho^{(p)}(R_0(t), t),\\ \psi^{(e)}(R_0(t), t) &= \psi^{(p)}(R_0(t), t),\end{aligned}$$

and differentiating both sides of these equalities with respect to t, we obtain

$$\frac{\partial\varrho^{(e)}}{\partial\varrho_0}\frac{\partial R_0}{\partial t} + \frac{\partial\varrho^{(e)}}{\partial t} = \frac{\partial\varrho^{(p)}}{\partial\varrho_0}\frac{\partial R_0}{\partial t} + \frac{\partial\varrho^{(p)}}{\partial t},$$

$$\frac{\partial\vartheta^{(e)}}{\partial\varrho_0}\frac{\partial R_0}{\partial t} + \frac{\partial\vartheta^{(e)}}{\partial t} = \frac{\partial\vartheta^{(p)}}{\partial\varrho_0}\frac{\partial R_0}{\partial t} + \frac{\partial\vartheta^{(p)}}{\partial t}.$$

Supposing that $\dfrac{\partial R_0}{\partial t} = 0$, we see that

$$\frac{\partial\varrho^{(e)}}{\partial t} = \frac{\partial\varrho^{(p)}}{\partial t} \quad\text{and}\quad \frac{\partial\vartheta^{(e)}}{\partial t} = \frac{\partial\vartheta^{(p)}}{\partial t}.$$

Our supposition is proved.

When on the contrary $\dfrac{\partial R_0}{\partial t} \neq 0$, then the radius of the plastic zone changes, and the particles of our medium will change from one state to another.

If the velocities in the plastic and the elastic zones differ, the velocities of these particles must change abruptly, which is impossible.

It must be noted that, according to our suppositions, the function $\varrho(\varrho_0, t)$ is completely determined.

Indeed, from the condition of compressibility it follows that

$$\iint \varrho\, d\varrho\, d\vartheta = \iint \varrho_0\, d\varrho_0\, d\vartheta_0.$$

Therefore,

$$\frac{D(\varrho, \vartheta)}{D(\varrho_0, \vartheta_0)} = \frac{\varrho_0}{\varrho}$$

or

$$\varrho\frac{\partial\varrho}{\partial\varrho_0} = \varrho_0. \tag{7}$$

Integrating with respect to ϱ_0, we get

$$\varrho^2 = \varrho_0^2 + \frac{k}{\mu} C(t). \tag{8}$$

Here $C(t)$ is a function which can be determined from the boundary condition.

Differentiating this formula with respect to t, we obtain

$$2\varrho \frac{\partial \varrho}{\partial t} = \frac{k}{\mu} C'(t)$$

or

$$u'_\varrho = \frac{kC'(t)}{2\mu\varrho}. \tag{9}$$

2. Now let us consider the solution of the Saint-Venant equations for a plastic zone.

This can be done in different ways. We follow that of Carathéodory–Schmidt. Let us remember the main features of this method.

It consists of the construction of two orthogonal systems of lines, so-called characteristics.

These characteristics are supposed to be determined by the equations

$$\begin{aligned} u(\varrho, \vartheta) &= \text{const}, \\ v(\varrho, \vartheta) &= \text{const}, \end{aligned} \tag{10}$$

the functions u and v satisfying the equations

$$\begin{aligned} (\widehat{\varrho\varrho} - \widehat{\vartheta\vartheta}) \left(\frac{\partial u}{\partial \varrho}\right)^2 + \frac{4\,\widehat{\varrho\vartheta}}{\varrho} \frac{\partial u}{\partial \varrho} \frac{\partial u}{\partial \vartheta} + \frac{\widehat{\vartheta\vartheta} - \widehat{\varrho\varrho}}{\varrho^2} \left(\frac{\partial u}{\partial \vartheta}\right)^2 &= 0, \\ (\widehat{\varrho\varrho} - \widehat{\vartheta\vartheta}) \left(\frac{\partial v}{\partial \varrho}\right)^2 + \frac{4\,\widehat{\varrho\vartheta}}{\varrho} \frac{\partial v}{\partial \varrho} \frac{\partial v}{\partial \vartheta} + \frac{\widehat{\vartheta\vartheta} - \widehat{\varrho\varrho}}{\varrho^2} \left(\frac{\partial v}{\partial \vartheta}\right)^2 &= 0. \end{aligned} \tag{11}$$

Some interesting theorems about these lines were worked out by Hencky and later on by Carathéodory and Schmidt. Let us recall some of them.

1. Supposing that the characteristics are known, we can express the solution of the problem of plastic stresses by means of the formulas

$$\frac{\widehat{\varrho\varrho} + \widehat{\vartheta\vartheta}}{2k} = \frac{\sigma}{2k} = \pm(f_1(u) + f_2(v)), \tag{12}$$

$$\beta = f_1(u) - f_2(v), \tag{13}$$

$$\widehat{\varrho\varrho} = \pm k \sin 2(\beta - \vartheta) + \sigma, \quad \widehat{\vartheta\vartheta} = \mp k \sin 2(\beta - \vartheta) + \sigma, \quad \widehat{\varrho\vartheta} = \pm k \cos 2(\beta - \vartheta). \tag{14}$$

Here β is the angle between the positive direction of the line $u = \text{const}$ and the direction of the axis $\vartheta = 0$.

We must choose the upper or the lower signs of the first terms on the right side of formulas (14) in all these formulas in the same way. The sign in the right side of formula (12) is connected with the signs in formulas (14) and depends on the orientation of the positive direction of axes u and v.

We shall choose this sign substituting directly expressions (14) into the equation of continuous media (1).

2. Formulas (12), (13), (14) are equivalent to the Saint-Venant equation, i.e., if we determine the orthogonal system of the lines $u = \text{const}$, $v = \text{const}$ in such a way that the angle β will be given by means of formula (13) with certain f_1 and f_2, equations (12) and (14) give us the solution of the problem of a plastic state.

From equality (13) it follows that

$$\frac{\partial^2 \beta}{\partial u \partial v} = 0. \tag{15}$$

This last equation is the only condition which has to be satisfied, if we want to construct a solution of our type.

Now let us consider our principal problem.

The two orthogonal systems of lines invariable with regard to the rotation about the origin can generally be represented in the form

$$\begin{aligned} u &= \vartheta - \vartheta_1(\varrho) = \text{const}, \\ v &= \vartheta - \vartheta_2(\varrho) = \text{const}, \end{aligned} \tag{16}$$

the functions ϑ_1 and ϑ_2 satisfying the relation

$$1 + \varrho^2 \vartheta_1'(\varrho) \vartheta_2'(\varrho) = 0. \tag{17}$$

Now let us write condition (13).

It is known that the angle β can be represented in the form

$$\beta = \vartheta + \arctan(\varrho \vartheta_1'). \tag{18}$$

Remembering that

$$\frac{\partial}{\partial u} = \frac{\partial \vartheta}{\partial u} \frac{\partial}{\partial \vartheta} + \frac{\partial \varrho}{\partial u} \frac{\partial}{\partial \varrho}, \qquad \frac{\partial}{\partial v} = \frac{\partial \vartheta}{\partial v} \frac{\partial}{\partial \vartheta} + \frac{\partial \varrho}{\partial v} \frac{\partial}{\partial \varrho} \tag{19}$$

and

$$\frac{\partial \vartheta}{\partial u} = -\frac{1}{\Delta} \frac{\partial v}{\partial \varrho}, \quad \frac{\partial \vartheta}{\partial v} = \frac{1}{\Delta} \frac{\partial u}{\partial \varrho}, \quad \frac{\partial \varrho}{\partial u} = \frac{1}{\Delta} \frac{\partial v}{\partial \vartheta}, \quad \frac{\partial \varrho}{\partial v} = -\frac{1}{\Delta} \frac{\partial u}{\partial \vartheta},$$

$$\Delta = \frac{D(u, v)}{D(\varrho, \vartheta)} = \vartheta_2'(\varrho) - \vartheta_1'(\varrho), \tag{20}$$

we obtain

$$\frac{\partial \beta}{\partial u} = \frac{\vartheta_2'}{\vartheta_2' - \vartheta_1'} + \frac{1}{\vartheta_2' - \vartheta_1'} \frac{\vartheta_1' + \varrho \vartheta_1''}{1 + \varrho^2 \vartheta_1'^2}$$

$$= \frac{(1+\varrho^2\vartheta_1'^2) - (\varrho\vartheta_1'' + \vartheta_1')\varrho^2\vartheta_1'}{(1+\varrho^2\vartheta_1'^2)^2},$$

$$\frac{\partial^2\beta}{\partial u \partial v} = \frac{d}{d\varrho}\left[\frac{(1+\varrho^2\vartheta_1'^2) - (\varrho\vartheta_1'' + \vartheta_1')\varrho^2\vartheta_1'}{(1+\varrho^2\vartheta_1'^2)^2}\right]\frac{-1}{\vartheta_2 - \vartheta_1} = 0. \tag{21}$$

Integrating with respect to ϱ, we have

$$\frac{(1+\varrho^2\vartheta_1'^2) - (\varrho\vartheta_1'' + \vartheta_1')\varrho^2\vartheta_1'}{(1+\varrho^2\vartheta_1'^2)^2} = G$$

or

$$\frac{1}{2\varrho}\frac{d}{d\varrho}\left[\frac{\varrho^2}{1+\varrho^2\vartheta_1'^2}\right] = G.$$

Integrating once more, we obtain

$$\frac{\varrho^2}{1+\varrho^2\vartheta_1'^2} = G\varrho^2 + C_1, \tag{22}$$

and finally

$$\vartheta_1' = \pm\frac{1}{\varrho}\sqrt{\frac{(1-G)\varrho^2 - C_1}{G\varrho^2 + C_1}}.$$

If we denote

$$\frac{C_1}{1-G} = a^2, \qquad \frac{C_1}{G} = b^2,$$

we obtain

$$\vartheta_1' = \frac{b}{a\varrho}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}}, \qquad \vartheta_2' = -\frac{a}{b\varrho}\sqrt{\frac{\varrho^2 + b^2}{\varrho^2 - a^2}}. \tag{23}$$

Integrating with respect to ϱ, we get

$$\begin{aligned}
\vartheta_1(\varrho) &= -\arctan\frac{b}{a}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}} + \frac{1}{2}\frac{b}{a}\ln\frac{\sqrt{\varrho^2 + b^2} + \sqrt{\varrho^2 - a^2}}{\sqrt{\varrho^2 + b^2} - \sqrt{\varrho^2 - a^2}}, \\
\vartheta_2(\varrho) &= -\arctan\frac{b}{a}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}} - \frac{1}{2}\frac{a}{b}\ln\frac{\sqrt{\varrho^2 + b^2} + \sqrt{\varrho^2 - a^2}}{\sqrt{\varrho^2 + b^2} - \sqrt{\varrho^2 - a^2}}.
\end{aligned} \tag{24}$$

The constants of integration are not necessary and may be neglected without any restriction of the generality.

Thus, we have

$$\begin{aligned}
u &= \vartheta + \arctan\frac{b}{a}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}} - \frac{b}{a}\ln(\sqrt{\varrho^2 + b^2} + \sqrt{\varrho^2 - a^2}), \\
v &= \vartheta + \arctan\frac{b}{a}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}} + \frac{a}{b}\ln(\sqrt{\varrho^2 + b^2} + \sqrt{\varrho^2 - a^2}),
\end{aligned} \tag{25}$$

and

$$\beta = \vartheta + \arctan \varrho\vartheta_1' = \vartheta + \arctan \frac{b}{a}\sqrt{\frac{\varrho^2 - a^2}{\varrho^2 + b^2}} = \frac{a^2 u + b^2 v}{a^2 + b^2}. \tag{26}$$

By means of formulas (12) and (13) we get

$$\frac{\sigma}{2k} = \pm\left(\frac{a^2 u}{a^2 + b^2} - \frac{b^2 v}{a^2 + b^2}\right) + \text{const.} \tag{27}$$

Remembering that with our hypothesis σ will be independent of ϑ, we conclude that $a = b$.

Finally, when we put $a^2 = \lambda$, we obtain

$$\begin{aligned} u &= \vartheta + \arctan\sqrt{\frac{\varrho^2 - \lambda}{\varrho^2 + \lambda}} - \ln(\sqrt{\varrho^2 + \lambda} + \sqrt{\varrho^2 - \lambda}), \\ v &= \vartheta + \arctan\sqrt{\frac{\varrho^2 - \lambda}{\varrho^2 + \lambda}} + \ln(\sqrt{\varrho^2 + \lambda} + \sqrt{\varrho^2 - \lambda}). \end{aligned} \tag{28}$$

In these formulas, λ can be either positive or negative. Complex values of λ give us complex values of ϑ and shall be neglected.

Formulas (12) and (14) with regard to (13) give us

$$\frac{\sigma}{2k} = \varepsilon \ln(\sqrt{\varrho^2 + \lambda} + \sqrt{\varrho^2 - \lambda}) + C, \tag{29}$$

where $\varepsilon = \pm 1$,

$$\overset{\frown}{\varrho\varrho} = \sigma + \eta k \frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho^2}, \quad \overset{\frown}{\vartheta\vartheta} = \sigma - \eta k \frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho^2}, \quad \overset{\frown}{\varrho\vartheta} = -\eta k \frac{\lambda}{\varrho^2}, \tag{30}$$

where $\eta = \pm 1$.

To determine the connection between η and ε, we substitute expressions (30) into the equations of mechanics of continuous media (1). Then we get $\eta = -\varepsilon$. Finally, we can write the general solution of equations of plasticity with polar symmetry in the form

$$\begin{gathered} \overset{\frown}{\varrho\varrho} = \varepsilon k \left[2\ln(\sqrt{\varrho^2 + \lambda} + \sqrt{\varrho^2 - \lambda}) - \frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho^2}\right] + C_1, \\ \overset{\frown}{\vartheta\vartheta} = \varepsilon k \left[2\ln(\sqrt{\varrho^2 + \lambda} + \sqrt{\varrho^2 - \lambda}) + \frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho^2}\right] + C_1, \\ \overset{\frown}{\varrho\vartheta} = \varepsilon k \frac{\lambda}{\varrho^2}, \end{gathered} \tag{31}$$

where $\varepsilon = \pm 1$. Now let us calculate the values of velocities.

From equations (4) and (9) it follows that

$$\frac{\partial u'_\vartheta}{\partial \varrho} - \frac{u'_\vartheta}{\varrho} = \frac{\lambda C'(t)}{\varrho^2\sqrt{\varrho^4 - \lambda^2}}. \tag{32}$$

From (32) we deduce

$$u'_\vartheta = \frac{k}{\mu}\frac{C'(t)}{\lambda}\frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho} + \varrho C'_2(t), \tag{33}$$

$C'_2(t)$ is the constant of integration, which can depend on t.

Remembering that $u'_\vartheta = \varrho'_\vartheta$, we get

$$\vartheta' = \frac{k}{\mu}\frac{C'(t)}{2\lambda}\frac{\sqrt{\varrho^4 - \lambda^2}}{\varrho} + C'_2(t). \tag{34}$$

It must be noted that our solution given by means of formulas (28), (31), (9), and (34) is determined only in the domain outside the circle

$$\varrho^2 = |\lambda| \tag{35}$$

or

$$\varrho_0^2 = |\lambda| - \frac{k}{\mu}C,$$

because for smaller values of ϱ the square root $\sqrt{\varrho^4 - \lambda^2}$ becomes imaginary. The characteristics are tangent to circumference (35). The maximum value of shearing stresses at this circumference will be obtained in that direction of a small element of line which coincides with the direction of the contour itself.

The characteristics are given in Fig. 1.

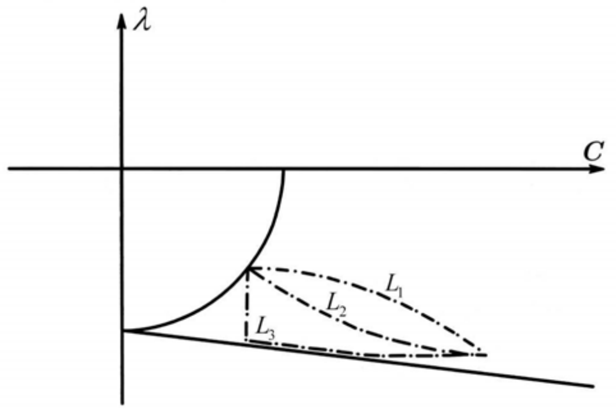

Fig. 1.

The solution of our equations for the elastic part of the medium can be obtained by means of well-known methods.

The solutions of the linear elastic equations are

$$\begin{gathered} u_\varrho = \frac{kC(t)}{2\mu\varrho}, \qquad u_\vartheta = -\frac{kB(t)}{2\mu\varrho}, \\ \widehat{\varrho\varrho} = \frac{kC(t)}{\varrho^2}, \quad \widehat{\vartheta\vartheta} = -\frac{kC(t)}{\varrho^2}, \quad \widehat{\varrho\vartheta} = -\frac{kB(t)}{\varrho^2}. \end{gathered} \tag{36}$$

From the condition that the maximum value of shearing stresses will be equal to k it follows that

$$C^2(t) + B^2(t) = R^4(t). \tag{37}$$

But we shall also consider equations (36) and (37) as approximately exact and introduce certain corrections which are small quantities.

Notice that the only supposition we have made for simplifying the elasticity equation is that the values of the partial derivatives

$$\frac{\partial u_\varrho}{\partial \varrho}, \quad \frac{\partial u_\vartheta}{\partial \varrho}, \quad \frac{1}{\varrho}\frac{\partial u_\varrho}{\partial \vartheta}, \quad \frac{1}{\varrho}\frac{\partial u_\vartheta}{\partial \vartheta} \tag{38}$$

are small quantities.

In this hypothesis we have found that the ratio $\dfrac{k}{\mu}$ will be a small quantity.

Indeed, k is the value of the maximum tangential stress for small strain, and μ is the value of the tangential stress which would take place, if the strain were of finite value; in cartesian coordinates

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 1.$$

Then we see that the order of strain components (36) will be $O\left(\dfrac{k}{\mu}\right)$.

Accordingly, we suppose that the exact expressions for our unknown quantities will be

$$u_\varrho = \frac{k}{\mu}\frac{C(t)}{2\varrho} + u_\varrho^*, \qquad u_\vartheta = -\frac{k}{\mu}\frac{B(t)}{2\varrho} + u_\vartheta^*, \tag{39}$$

$$u_\varrho^* = R\,O\left(\frac{k^2}{\mu^2}\right), \qquad u_\vartheta^* = R\,O\left(\frac{k^2}{\mu^2}\right),$$

$$\widehat{\varrho\varrho} = \frac{kC(t)}{\varrho^2} + \widehat{\varrho\varrho}^{\,*}, \quad \widehat{\vartheta\vartheta} = -\frac{kC(t)}{\varrho^2} + \widehat{\vartheta\vartheta}^{\,*}, \quad \widehat{\varrho\vartheta} = -\frac{kB(t)}{\varrho^2} + \widehat{\varrho\vartheta}^{\,*}, \tag{40}$$

$$\widehat{\varrho\varrho}^{\,*} = k\,O\left(\frac{k}{\mu}\right), \quad \widehat{\vartheta\vartheta}^{\,*} = k\,O\left(\frac{k}{\mu}\right), \quad \widehat{\varrho\vartheta}^{\,*} = k\,O\left(\frac{k}{\mu}\right),$$

the corrections u_ϱ^*, u_ϑ^*, $\widehat{\varrho\varrho}^*$, $\widehat{\vartheta\vartheta}^*$, $\widehat{\varrho\vartheta}^*$ being supposed of higher order of smallness.

Formulas (39) and (40) permit us to proceed to the solution of our problem. Let us write all the boundary conditions.

First, equating the values of stresses at the boundary, we get

$$\frac{kC(t)}{R^2}+\widehat{\varrho\varrho}^*=\varepsilon k\left[2\ln(\sqrt{R^2+\lambda}+\sqrt{R^2-\lambda})-\frac{\sqrt{R^4+\lambda^2}}{R^2}\right]+C_1(t),$$

$$-\frac{kB(t)}{R^2}=\varepsilon k\frac{\lambda}{R^2}-\widehat{\varrho\vartheta}^*, \tag{41}$$

$$\frac{2\varepsilon k\sqrt{R^4-\lambda^2}}{R^2}=-\frac{2kC(t)}{R^2}+\widehat{\varrho\varrho}^*-\widehat{\vartheta\vartheta}^*.$$

Thus, we see that

$$\varepsilon=-\operatorname{sign}C(t). \tag{42}$$

The function $C(t)$ is known, because at the boundary $\varrho=r_0$ we have

$$\varrho(r_0,t)=\sqrt{r_0^2+\frac{k}{\mu}C(t)}, \tag{43}$$

and the function $\varrho(r_0,t)$ is supposed to be given.

Suppose $C(t)>0$, i.e., $\varepsilon=-1$.

The first equation in (41) gives the value of $C_1(t)$, if $B(t)$, $C(t)$, $\lambda(t)$ and $R(t)$ are supposed to be known. Approximate relation (37) and the other equations in (41) give expressions for two of these functions by means of the last one. Then, in order to solve our problem, we shall determine only one new relation.

Making equal at the end the tangential velocities on the contour $\varrho=R$ and using the given values of these velocities for $\varrho=r_0$, we obtain

$$\frac{1}{R(t)}\left[-\frac{k}{\mu}\frac{B'(t)}{2R(t)}+\frac{d}{dt}u_\vartheta^*|_{\varrho=R(t)}\right]+\zeta^*=\frac{k}{\mu}\frac{C(t)}{2\lambda(t)}\frac{\sqrt{R^2-\lambda^2}}{R^2}+C_2'(t),$$

$$\vartheta'(r_0,t)=\frac{k}{\mu}\frac{C'(t)}{2\lambda(t)}\frac{\sqrt{\left[r_0^2+\frac{k}{\mu}C(t)\right]^2-[\lambda(t)]^2}}{r_0^2+\frac{k}{\mu}C(t)}+C_2'(t). \tag{44}$$

Here, ζ^* is a small introduced quantity, because u_ϑ does not coincide with the small change of $\vartheta(\varrho_0,\vartheta_0,t)$, divided by ϱ.

Excluding $C_2'(t)$ and using (41), we get

$$-\frac{k}{\mu}\frac{\lambda'(t)}{2R^2(t)}-\frac{k}{\mu}\frac{C'(t)C(t)}{2\lambda(t)R^2(t)}+\frac{k}{\mu}\frac{C'(t)}{2\lambda(t)}$$

$$\times \frac{\sqrt{\left[r_0^2 + \frac{k}{\mu}C(t)\right]^2 - [\lambda(t)]^2}}{r_0^2 + \frac{k}{\mu}C(t)} - \vartheta'(r_0, t) = \chi^*, \tag{45}$$

where χ^* is a small quantity

$$\chi^* = k\frac{R'(t)}{R(t)} O\left(\frac{k^2}{\mu^2}\right).$$

Substituting $R^2 = \sqrt{\lambda^2 + C^2}$ and neglecting the small terms, we get

$$\vartheta'(r_0, t) = -\frac{k}{\mu}\frac{\lambda'}{2\sqrt{\lambda^2 + C^2}} - \frac{k}{\mu}\frac{C'C}{2\lambda\sqrt{\lambda^2 + C^2}} + \frac{k}{\mu}\frac{C'}{2\lambda}\frac{\sqrt{\left[r_0^2 + \frac{k}{\mu}C\right]^2 - \lambda^2}}{r_0^2 + \frac{k}{\mu}C}. \tag{46}$$

Equation (46) serves to determine the function $\lambda(t)$, and when integrating it, we get the full solution of our problem.

The integration of this equation in a finite form is in general impossible and can be realized only by approximate methods.

Yet let us begin with some investigations concerning the existence of the solution with $\vartheta(r_0, t)$ arbitrary.

As the law of the inverse change of a material particle from the plastic into the elastic state is unknown, we limit ourselves to the consideration of such types of motion where

$$R_0(t) = \sqrt{R^2(t) - \frac{k}{\mu}C(t)}$$

is a function which cannot decrease.

For instance, this condition will be satisfied, if we suppose that $C(t)$ and $\vartheta(r_0, t)$ are nondecreasing functions.

Let us consider λ and C in the (C, λ)-plane varying along a curve L. The parametric expression of this curve will be $C = C(t)$, $\lambda = \lambda(t)$. Let us consider $\vartheta(r_0, t)$ as unknown.

We suppose that the motion began at the moment $t = 0$ and that at this moment condition (3) is satisfied on the circle $\varrho_0 = r_0$, i.e., this circle coincides with the boundary between the elastic state and the plastic one $R_0(0) = \varrho_0$.

The right side of equation (46) is obviously positive, when λ is a negative nonincreasing function. Let us limit ourselves to this case.

From the condition $R_0^2(0) = r_0^2$, i.e.,

$$R^2(0) - \frac{k}{\mu}C(0) = r_0^2, \tag{47}$$

it follows that all the possible curves L will begin at the contour of the quarter of ellipse (47) very little different from the circle

$$\lambda^2(0) + C^2(0) = \varrho_0^4. \tag{48}$$

The second condition we obtain, if we remember that the square root

$$\sqrt{\left[r_0^2 + \frac{k}{\mu}C\right]^2 - \lambda^2}$$

is real.

Then we have

$$r_0^2 + \frac{k}{\mu}C > -\lambda, \tag{49}$$

and our curves L can be situated only as we observe in Fig. 2.

Let us compare the value of $\vartheta(\varrho_0, t)$ at the same point, when we follow different paths L_1 and L_2, starting from the same initial state (see Fig. 2).

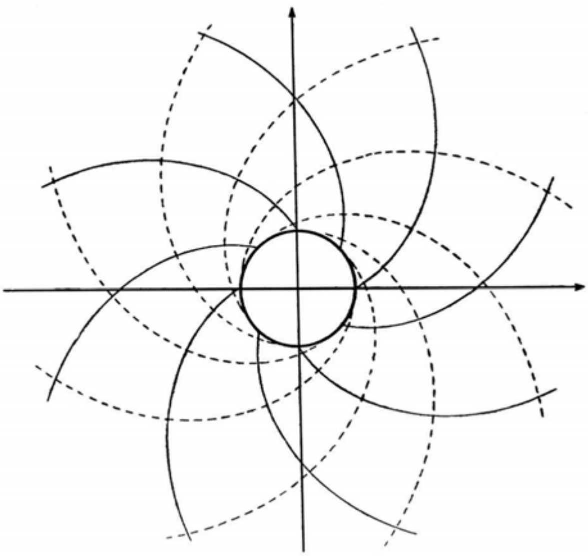

Fig. 2.

Equation (46) can be written as

$$d\vartheta = -\frac{k}{\mu}\left\{\frac{d\lambda}{2\sqrt{\lambda^2 + C^2}} + \left(\frac{C}{2\lambda\sqrt{\lambda^2 + C^2}} - \frac{\sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 - \lambda^2}}{2\left(r_0^2 + \frac{k}{\mu}C\right)\lambda}\right)dC\right\}. \tag{50}$$

The difference between the two values ϑ_{L_1} and ϑ_{L_2} at the same point, when the paths L_2 and L_1 are situated as in Fig. 2, is obviously equal to the contour integral

$$\vartheta_{L_2} - \vartheta_{L_1} = -\frac{k}{\mu} \int\limits_L \left[\frac{d\lambda}{2\sqrt{\lambda^2 + C^2}} + \left(\frac{C}{2\lambda\sqrt{\lambda^2 + C^2}} - \frac{\sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 - \lambda^2}}{2\left(r_0^2 + \frac{k}{\mu}C\right)\lambda} \right) dC \right],$$

taken along a closed contour L.

Let this integral be transformed into an integral on the area S, limited by L. We get

$$\vartheta_{L_2} - \vartheta_{L_1} = -\frac{k}{\mu} \iint\limits_S \left\{ \frac{\partial}{\partial C} \left(\frac{1}{2\sqrt{\lambda^2 + C^2}} \right) \right.$$

$$\left. - \frac{\partial}{\partial \lambda} \left[\frac{C}{2\lambda\sqrt{\lambda^2 + C^2}} - \frac{\sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 - \lambda^2}}{2\left(r_0^2 + \frac{k}{\mu}C\right)\lambda} \right] \right\} dC\, d\lambda$$

$$= \frac{k}{\mu} \iint\limits_S \left\{ -\frac{C}{2\lambda^2\sqrt{\lambda^2 + C^2}} + \frac{r_0^2 + \frac{k}{\mu}C}{2\lambda^2\sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 - \lambda^2}} \right\} dC\, d\lambda. \tag{51}$$

The expression under the sign of the integral is obviously positive.

It follows that the integral over L_2 is larger than the integral over L_1, and the largest possible value of ϑ can be secured by means of integration over the contour L_3 which consists of a line parallel to the λ-axis, also a part of the line

$$r_0^2 + \frac{k}{\mu}C = -\lambda,$$

and finally the line parallel to the C-axis. The contour is given in Fig. 2.

For the given value of C the largest possible value of ϑ can be obtained on the line

$$\lambda = -r_0^2 - \frac{k}{\mu}C.$$

Let us calculate this value following a path from the point $C_0 = 0$, $\lambda_0 = r_0^2$ to a point $C_1 \lambda_1 = -r_0^2 - \frac{k}{\mu}C_1$.

We get

$$\vartheta = \vartheta_0 + \frac{k}{\mu} \left\{ \frac{1}{2} + \int\limits_0^{C_1} \left[\frac{\frac{k}{\mu}}{2\sqrt{C^2 + \left(r_0^2 + \frac{k}{\mu}C\right)^2}} \right. \right.$$

$$+ \frac{C}{2\left(r_0^2 + \frac{k}{\mu}C\right)\sqrt{C^2 + \left(r_0^2 + \frac{k}{\mu}C\right)^2}}\Bigg] dC \Bigg\}$$

$$= \frac{k}{2\mu} + \frac{1}{2}\left\{\sqrt{\frac{k^2}{\mu^2}+1}\,\ln\left[\sqrt{\frac{k^2}{\mu^2}+1}\,C + \frac{\frac{k}{\mu}r_0^2}{\sqrt{\frac{k^2}{\mu^2}+1}} + \sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 + C^2}\right]\right.$$

$$\left. + \ln\left[-C + \sqrt{\left(r_0^2 + \frac{k}{\mu}C\right)^2 + C^2}\right] - \ln\left(r_0^2 + \frac{k}{\mu}C\right)\right\}\Bigg|_0^{C_1}$$

Whence it follows that when C_1 is increasing, the largest possible value of ϑ is asymptotically equal to $\frac{3}{2}\ln C_1$.

If our boundary conditions are such that ϑ increases more rapidly than it is permitted for the corresponding increase of C, differential equation (46) becomes unsolvable. In this case, at a definite moment, the characteristics become tangent to our circle $\varrho = r_0$. From a physical standpoint, the matter of the plane will be torn, because it is impossible to force an internal particle to turn with such a velocity.

We neglect here the elementary calculation of the case, when C is a negative decreasing function, i.e., our hole is contracting to the centre. The result will be analogous.

At the end we regard it as a pleasant duty to express our sincere thanks to our friend S. G. Mikhlin, who has much contributed to this work.

9. On a New Problem of Mathematical Physics*

S. L. Sobolev

Summary. We consider a system of partial differential equations that is not a Kovalevskaya system. The Cauchy problem and the mixed problem in a smooth domain are studied. We prove the existence of a solution in a Hilbert space H and the continuous dependence on the initial conditions. The Cauchy problem in an unbounded space is solved explicitly.

1 Statement of the Problem

The following system of partial differential equations appears in some problems of mathematical physics and mechanics:

$$\begin{aligned}
\frac{\partial v_x}{\partial t} &= v_y - \frac{\partial p}{\partial x} + F_x, \\
\frac{\partial v_y}{\partial t} &= -v_x - \frac{\partial p}{\partial y} + F_y, \\
\frac{\partial v_z}{\partial t} &= -\frac{\partial p}{\partial z} + F_z, \\
\frac{\partial v_x}{\partial x} &+ \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = g,
\end{aligned} \tag{1}$$

where v_x, v_y, v_z are components of some vector $\overrightarrow{v}$ and p is a scalar function in a domain Ω with the boundary S. The coordinates of points of the domain Ω are denoted by x, y, z.

Depending on the physical problem, some conditions can be given on the boundary of the domain, for example,

$$p|_S = 0 \tag{2.1}$$

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or

$$[v_x \cos nx + v_y \cos ny + v_z \cos nz]_S = 0. \tag{2.2}$$

To find the solutions of these problems, it is necessary to set the initial values of the vector $\overrightarrow{v}$

$$\begin{aligned} v_x|_{t=0} &= v_x^{(0)}(x, y, z), \\ v_y|_{t=0} &= v_y^{(0)}(x, y, z), \\ v_z|_{t=0} &= v_z^{(0)}(x, y, z). \end{aligned} \tag{3}$$

In some problems, the boundary conditions can be more complicated.

In addition to this main problem, we also study system (1) with initial conditions (3) in an unbounded space. In such case, boundary conditions (2) disappear and should be replaced with some condition at infinity. It is convenient to write system (1) and conditions (2) and (3) in vector form. Denote by i, j, and k the unit vectors parallel to the coordinate axes and write system (1) as

$$\overrightarrow{N}(\overrightarrow{v}, p) \equiv \frac{\partial \overrightarrow{v}}{\partial t} - [\overrightarrow{v} \times \overrightarrow{\mathrm{k}}] + \operatorname{grad} p = \overrightarrow{F},$$
$$\operatorname{div} \overrightarrow{v} = g.$$

Boundary condition (2.2) takes the form

$$v_n|_S = 0,$$

and initial condition (3) is written in the form of the equality

$$\overrightarrow{v}|_{t=0} = \overrightarrow{v}^{(0)}(x, y, z).$$

We consider both the case where $\overrightarrow{v}$ is given in a bounded domain Ω, and the case where $\overrightarrow{v}$ is given in the entire space. However, in the first case we restrict ourselves to the simplest qualitative research of solutions of system (1) with conditions (2.1) or (2.2), without consideration, for example, of the behavior of these solutions for large values of t which would require the study of detailed spectral properties of the corresponding operators. The main problem is to investigate solutions of system (1) with initial conditions (3) in an unbounded space with the corresponding conditions at infinity. We obtain a formula for the solution of this problem, which allows us to make a number of qualitative conclusions about the behavior of solutions of system (1).

2 The Main Equations in a Function Space

We consider a Hilbert space of complex vectors $\overrightarrow{v}$ such that $|\overrightarrow{v}|^2$ is integrable over a domain Ω. We denote this space by H.

The inner product in H is defined by the formula

$$(\overrightarrow{v}^{(1)}, \overrightarrow{v}^{(2)}) = \iiint\limits_{\Omega} (v_x^{(1)}\overline{v}_x^{(2)} + v_y^{(1)}\overline{v}_y^{(2)} + v_z^{(1)}\overline{v}_z^{(2)})d\Omega. \tag{4}$$

We consider two cases: the case where Ω coincides with the whole space and the case of a bounded domain Ω homeomorphic to a ball. The condition on the topology of the domain is not essential and is introduced only for the sake of simplicity. In the space H there is a linear manifold $\widetilde{G}_1$ of vectors

$$\overrightarrow{v}_1 = \operatorname{grad} \varphi, \tag{5}$$

where φ is a function having continuous derivatives of any order inside of the domain.

Vectors of form (5) have continuous derivatives of any order and satisfy the equation

$$\operatorname{rot} \overrightarrow{v}_1 = 0. \tag{6}$$

Vectors satisfying (6) are usually said to be *potential*. It is known that for any infinitely differentiable vector, condition (6) is necessary and sufficient for it can be represented in form (5).

Another linear manifold $\widetilde{J}_1 \subset H$ consists of vectors of the form

$$\overrightarrow{v}_2 = \operatorname{rot} \overrightarrow{\Psi}, \tag{7}$$

where the vector $\overrightarrow{\Psi}$ has continuous derivatives of any order. Vectors of form (7) satisfy the equation

$$\operatorname{div} \overrightarrow{v}_2 = 0. \tag{8}$$

Vectors satisfying (8) are usually said to be *solenoidal*. It is well-known that for any infinitely differentiable vector, condition (8) is necessary and sufficient for it can be represented in form (7).

Let $\widetilde{H}_0$ be a linear manifold of smooth vectors $\overrightarrow{v}$, where each vector vanishes outside some (depending on this vector) finite domain $C_{\overrightarrow{v}}$ lying, together with its boundary, inside Ω and has derivatives of any order. Such vectors are called cut-off vectors.

We denote by $\widetilde{J}_0$ the manifold of smooth solenoidal cut-off vectors $\widetilde{J}_0 \subset \widetilde{H}_0$ and by $\widetilde{G}_0$ the manifold of smooth potential cut-off vectors $\widetilde{G}_0 \subset \widetilde{H}_0$.

Lemma 1. *If Ω is the entire space, any element $\overrightarrow{v}$ of H orthogonal to all elements of $\widetilde{G}_0$ and $\widetilde{J}_0$ can be equal only to zero.*

Proof. We note that the orthogonality of $\overrightarrow{v}$ to all elements of $\widetilde{G}_0$ and $\widetilde{J}_0$ implies that the vector $\overrightarrow{v}$ is orthogonal to the image of any vector $\overrightarrow{\omega}$ in $\widetilde{H}_0$ under the action of the Laplace operator, i.e., to the image of any smooth vector vanishing outside some finite interior subdomain $C_{\overrightarrow{\omega}}$. Indeed,

$$\Delta\overrightarrow{\omega} = \operatorname{grad}\operatorname{div} \overrightarrow{\omega} - \operatorname{rot}\operatorname{rot} \overrightarrow{\omega}. \tag{9}$$

However,

$$\begin{aligned} \overrightarrow{\omega}_1 &= \operatorname{grad}\operatorname{div}\overrightarrow{\omega} \in \widetilde{G}_0, \\ \overrightarrow{\omega}_2 &= \operatorname{rot}\operatorname{rot}\overrightarrow{\omega} \in \widetilde{J}_0. \end{aligned} \tag{10}$$

Hence,

$$(\overrightarrow{v}, \varDelta\overrightarrow{\omega}) = (\overrightarrow{v}, \overrightarrow{\omega}_1) - (\overrightarrow{v}, \overrightarrow{\omega}_2) = 0. \tag{11}$$

Thus, each component of $\overrightarrow{v}$ is orthogonal to all functions of the form $\varDelta\psi$, where ψ is a smooth function vanishing outside some domain C_ψ. In other words,

$$\iiint\limits_{-\infty}^{\infty} v_i \varDelta\psi \, d\varOmega = 0, \quad i = 1, 2, 3. \tag{12}$$

Hence v_i is a harmonic function [1] and can be represented in the form

$$v_i = \sum_{n=0}^{\infty} r^n Y_n^{(i)}(\theta, \varphi), \tag{13}$$

where $Y_n^{(i)}(\theta, \varphi)$ are some spherical Laplace functions.

Additionally, the function v_i should be square-integrable over the whole space

$$\iiint\limits_{-\infty}^{\infty} |v_i|^2 \, d\varOmega < +\infty. \tag{14}$$

Let

$$|b_n^{(i)}|^2 = \int\limits_0^{2\pi}\int\limits_0^{\pi} |Y_n^{(i)}(\theta, \varphi)|^2 \sin\theta \, d\theta \, d\varphi,$$

then

$$\int\limits_0^{A}\int\limits_0^{2\pi}\int\limits_0^{\pi} |v_i|^2 \, d\varOmega = \sum_0^{\infty} |b_n^{(i)}|^2 \frac{A^{2n+1}}{2n+1}. \tag{15}$$

If at least one of $|b_n^{(i)}|^2$ is nonzero, then the sum on the right side of (15) unboundedly increases if A increases. Since this contradicts (14), all $|b_n^{(i)}|^2$ are equal to zero. Hence v_1, v_2, v_3 are equal to zero, as required. □

Lemma 2. *The manifold $\widetilde{G}_0$ is orthogonal to $\widetilde{J}_1$.*

Proof. Let $\overrightarrow{v}_1 \in \widetilde{G}_0$, $\overrightarrow{v}_2 \in \widetilde{J}_1$. Then, by (7),

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2}) \, d\varOmega = \iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\operatorname{rot}\overrightarrow{\psi}}) \, d\varOmega$$

$$= -\iiint\limits_{-\infty}^{\infty} \operatorname{div}\,[\overrightarrow{v}_1 \times \overline{\overrightarrow{\psi}}] \, d\varOmega + \iiint\limits_{-\infty}^{\infty} (\overline{\overrightarrow{\psi}}, \operatorname{rot}\overrightarrow{v}_1) \, d\varOmega. \tag{16}$$

The first integral on the right side of (16) is reduced to the integral over a finite domain Ω_{v_1}, whereas the second integral vanishes in view of (6). Hence,

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2})\, d\Omega = -\iiint\limits_{\Omega_{v_1}} \mathrm{div}\,[\overrightarrow{v}_1 \times \overline{\overrightarrow{\Psi}}]\, d\Omega = \iint\limits_{S_{v_1}} ([\overrightarrow{v}_1 \times \overline{\overrightarrow{\Psi}}], \overrightarrow{n})\, dS,^{1}$$

where S_{v_1} is the surface bounding the volume Ω_{v_1}. The last integral is equal to zero since $\overrightarrow{v}_1$ vanishes on S_{v_1}. Hence,

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2})\, d\Omega = 0. \tag{17}$$

The lemma is proved. □

Lemma 3. *The manifold $\widetilde{G}_1$ is orthogonal to $\widetilde{J}_0$.*

Proof. Let $\overrightarrow{v}_1 \in \widetilde{G}_1$, $\overrightarrow{v}_2 \in \widetilde{J}_0$. Then,

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2})\, d\Omega = \iiint\limits_{-\infty}^{\infty} (\mathrm{grad}\,\varphi, \overline{\overrightarrow{v}_2})\, d\Omega$$

$$= \iiint\limits_{-\infty}^{\infty} \mathrm{div}\,(\varphi \overline{\overrightarrow{v}_2})\, d\Omega - \iiint\limits_{-\infty}^{\infty} \varphi \mathrm{div}\, \overline{\overrightarrow{v}_2}\, d\Omega. \tag{18}$$

By (8), the second integral on the right side of (18) is equal to zero, and the first integral is reduced to the integral over finite domain Ω_{v_1}. We have

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2})\, d\Omega = \iiint\limits_{\Omega_{v_2}} \mathrm{div}\,(\varphi \overline{\overrightarrow{v}_2})\, d\Omega = -\iint\limits_{S_{v_2}} \varphi \overline{v_{2_n}}\, dS. \tag{19}$$

But v_{2_n} vanishes on S_{v_2}. Consequently,

$$\iiint\limits_{-\infty}^{\infty} (\overrightarrow{v}_1, \overline{\overrightarrow{v}_2})\, d\Omega = 0.$$

The lemma is proved. □

From Lemmas 2 and 3 we obtain the following assertion.

Corollary. *The manifolds $\widetilde{J}_0$ and $\widetilde{G}_0$ are orthogonal.*

---

[1] $\overrightarrow{n}$ is the inward normal to the surface S_{v_1}. – *Ed.*

We denote by G_0, J_0, G_1, and J_1 the closures $\widetilde{G}_0$, $\widetilde{J}_0$, $\widetilde{G}_1$, and $\widetilde{J}_1$, respectively.

The following assertion holds.

Theorem. *If Ω is the entire space, then the Hilbert space H can be represented in the form*

$$H = J \oplus G,$$

where $J = J_0 = J_1$ and $G = G_0 = G_1$.

Proof. Indeed, J_0 and G_0 are orthogonal as the closures of two orthogonal manifolds. Moreover, H does not contain any element orthogonal to G_0 and J_0 simultaneously. Consequently, $H = J_0 \oplus G_0$. However, $J_1 \supseteq J_0$ and J_1 is orthogonal to G_0. Hence J_1 coincides with J_0. In the same way, $G_1 \supseteq G_0$ and G_1 is orthogonal to J_0. Consequently, G_1 coincides with G_0. The theorem is proved. □

We proceed with the consideration of the finite domains.

Lemma 4. *If $\overrightarrow{v}$ in H is orthogonal to both manifolds $\widetilde{J}_0$ and $\widetilde{G}_0$, then $\overrightarrow{v}$ is a harmonic vector. In other words, both the curl and the divergence of this vector vanish.*

Proof. Arguing in the same way as in the proof of Lemma 1, we conclude that all the components of the vector $\overrightarrow{v}$ are harmonic functions of variables x, y, z and have continuous derivatives of any order.

Further, let $\overrightarrow{v}_1 = \operatorname{grad}\varphi_1 \in \widetilde{G}_0$, and let φ_1 satisfy the condition

$$\varphi_1 \equiv 0 \quad \text{outside} \quad V_1 \subset \Omega. \tag{20}$$

By the assumptions of the lemma, we obtain

$$\begin{aligned} \iiint\limits_{\Omega} (\overrightarrow{v}, \overline{\overrightarrow{v}_1})\, d\Omega = 0 = \iiint\limits_{V_1} (\overrightarrow{v}, \overline{\operatorname{grad}\varphi_1})\, d\Omega = \iiint\limits_{V_1} \operatorname{div}(\overline{\varphi_1}\overrightarrow{v})\, d\Omega \\ - \iiint\limits_{V_1} \overline{\varphi_1}\operatorname{div}\overrightarrow{v}\, d\Omega = -\iint\limits_{S_1} \overline{\varphi_1} v_n\, dS - \iiint\limits_{V_1} \overline{\varphi_1}\operatorname{div}\overrightarrow{v}\, d\Omega. \end{aligned} \tag{21}$$

The first term on the right side of (21) is equal to zero since $\varphi_1 = 0$ on the surface S_1 and, consequently, for any φ_1 satisfying (20) we have

$$\iiint\limits_{\Omega} \overline{\varphi_1}\operatorname{div}\overrightarrow{v}\, d\Omega = 0.$$

This is possible only if

$$\operatorname{div}\overrightarrow{v} = 0. \tag{22}$$

Let $\overrightarrow{v}_2 \in \widetilde{J}_0$ and $\overrightarrow{v}_2 = \operatorname{rot}\overrightarrow{\Psi}_2$, where

$$\overrightarrow{\Psi}_2 \equiv 0 \quad \text{outside} \quad V_2 \subset \Omega. \tag{23}$$

Then,

$$\iiint\limits_{\Omega} (\overrightarrow{v}, \overline{\overrightarrow{v}_2})\, d\Omega = 0 = \iiint\limits_{V_2} (\overrightarrow{v}, \overline{\operatorname{rot} \overrightarrow{\Psi}_2})\, d\Omega$$

$$= \iiint\limits_{V_2} \operatorname{div} [\overline{\overrightarrow{\Psi}} \times \overrightarrow{v}]\, d\Omega + \iiint\limits_{V_2} (\overline{\overrightarrow{\Psi}_2}, \operatorname{rot} \overrightarrow{v})\, d\Omega$$

$$= -\iint\limits_{S_2} ([\overline{\overrightarrow{\Psi}}_2 \times \overrightarrow{v}], \overrightarrow{n})\, dS + \iiint\limits_{V_2} (\overline{\overrightarrow{\Psi}}_2, \operatorname{rot} \overrightarrow{v})\, d\Omega. \tag{24}$$

The first term on the right side of (24) is equal to zero since $\overrightarrow{\Psi}_2 \equiv 0$ on S_2. Hence, for any $\overrightarrow{\Psi}_2$ satisfying (23), we have

$$\iiint\limits_{\Omega} (\overline{\overrightarrow{\Psi}}_2, \operatorname{rot} \overrightarrow{v})\, d\Omega = 0.$$

This is possible only if

$$\operatorname{rot} \overrightarrow{v} = 0. \tag{25}$$

Thus, any vector $\overrightarrow{v}$ orthogonal to $\widetilde{J}_0$ and $\widetilde{G}_0$ simultaneously is a harmonic vector.

The lemma is proved. □

This lemma has two important consequences.

Lemma 5. *If a vector $\overrightarrow{v}$ is orthogonal to $\widetilde{G}_0$ and $\widetilde{J}_1$ simultaneously, then it is equal to zero identically.*

Proof. Indeed, since $\overrightarrow{v}$ is orthogonal to $\widetilde{G}_0$ and $\widetilde{J}_0$, it is harmonic and, consequently, $\operatorname{div} \overrightarrow{v} = 0$. Hence it admits the representation $\overrightarrow{v} = \operatorname{rot} \overrightarrow{\Psi}$ and, consequently, $\overrightarrow{v} \in \widetilde{J}_1$. Since $\overrightarrow{v}$ is orthogonal to $\widetilde{J}_1$, we have $\overrightarrow{v} \equiv 0$. □

Lemma 6. *If $\overrightarrow{v}$ is orthogonal to $\widetilde{G}_1$ and $\widetilde{J}_0$ simultaneously, then it is equal to zero identically.*

Proof. Indeed, $\overrightarrow{v}$ is orthogonal to $\widetilde{G}_1$ and $\widetilde{J}_0$, it is harmonic and, consequently, admits the representation $\overrightarrow{v} = \operatorname{grad} \varphi$. Thus, $\overrightarrow{v} \in \widetilde{G}_1$. Since $\overrightarrow{v}$ is orthogonal to $\widetilde{G}_1$, we have $\overrightarrow{v} \equiv 0$. □

We now prove the main assertion.

Theorem. *The space H admits the representation*

$$H = G_0 \oplus I \oplus J_0,$$

where $I = G_1 \cdot J_1$ is the intersection of G_1 and J_1, i.e., the set of vectors that are common to these two spaces.

Proof. Indeed, Lemmas 5 and 6 imply

$$H = G_0 \oplus J_1 \quad \text{and} \quad H = G_1 \oplus J_0.$$

If the vector $\overrightarrow{v}$ is orthogonal to G_0, J_0 and I, then it is identically equal to zero. Indeed, since this vector is orthogonal to G_0, it belongs to J_1. Since it is orthogonal to J_0, it belongs to G_1. Consequently, this vector belongs to I and, by the orthogonality to I, is equal to zero identically. □

We note that I consists of harmonic vectors, which follows from Lemma 4.

Returning to our system of equations, we reduce it to a more convenient form. We construct a vector $\overrightarrow{v}^*$ satisfying the condition $\operatorname{div} \overrightarrow{v}^* = g$. In the case of (2.1) this vector is arbitrary, in the case of (2.2) it satisfies the additional condition

$$v_n^*|_S = 0.$$

This can be done, for example, if we set

$$\overrightarrow{v}^* = \operatorname{grad} v, \quad \varDelta v = g, \quad \frac{\partial v}{\partial n}|_S = 0.$$

Making the change of unknown functions by the formula $\overrightarrow{v} = \overrightarrow{v}^* + \overrightarrow{v}_1$, we obtain for $\overrightarrow{v}_1$ the same system of equations but with the condition $\operatorname{div} \overrightarrow{v}_1 = 0$.

Thus, we can restrict ourselves to the case $g = 0$.

We study system (1) in a Hilbert space H. As the unknown we take an element $\overrightarrow{v}$ of the Hilbert space. By the equation

$$\operatorname{div} \overrightarrow{v} = 0, \tag{1.1}$$

the vector $\overrightarrow{v}$ is solenoidal.

Our next goal is to find solutions $\overrightarrow{v}$ satisfying condition (2.2)

$$v_n|_S = 0.$$

In the case of the smooth boundary S, for smooth functions $\overrightarrow{v}$ we have

$$\iiint\limits_{\varOmega} (\overrightarrow{v}, \operatorname{grad} \varphi)\, d\varOmega = 0, \tag{26}$$

where φ is an arbitrary infinitely differentiable function.

We consider weak solutions of the problem. For this purpose, we replace condition (2.2) by the requirement that $\overrightarrow{v}$ is an arbitrary element of J_0. Let $\overrightarrow{v}$ be a sufficiently smooth vector and have the limit value v_n on the sufficiently smooth surface S. From the fact that $\overrightarrow{v}$ belongs to J_0 we have condition (2.2) and equation (1.1). Indeed, the left side of (26) vanishes for any φ, which can occur only if

$$\operatorname{div} \overrightarrow{v} = 0, \quad v_n|_S = 0.$$

To find $\dfrac{\partial \vec{v}}{\partial t}$ from (1), we should subtract the vector $\operatorname{grad} p$ from the vector $[\vec{v} \times \vec{\mathrm{k}}] + \vec{F}$ in such a way that the resulting vector belongs to J_0.

The vector $\operatorname{grad} p$ is an element of G_1. We define $\operatorname{grad} p$ in the weak sense as an arbitrary element $\vec{v}_1$ of G_1.

From (1) it follows that the vector $\vec{v}_1$ satisfying our conditions is defined in a unique way by the formula

$$\vec{v}_1 = P_0^*\{[\vec{v} \times \vec{\mathrm{k}}] + \vec{F}\}, \tag{27}$$

where P_0^* is the projection from H into G_1. Furthermore,

$$\frac{\partial \vec{v}}{\partial t} = P_0\{[\vec{v} \times \vec{\mathrm{k}}] + \vec{F}\}, \tag{28}$$

where P_0 is the projection from H into J_0.

Thus, system (1) and condition (2.2) can be written as a vector equation (28).

We now consider the problem on integration of system (1) under condition (2.1).

To generalize the statement of this problem, we can regard $\vec{v}$ as an arbitrary element of J_1 since no boundary conditions are imposed on this vector.

To compute $\dfrac{\partial \vec{v}}{\partial t}$, we should subtract from $[\vec{v} \times \vec{\mathrm{k}}] + \vec{F}$ a potential vector $\operatorname{grad} p$ such that p is equal to zero on the boundary and after subtracting we obtain a solenoidal vector.

It is easy to see that for sufficiently smooth p and smooth boundary S the vector $\operatorname{grad} p$ is orthogonal to any element $\vec{v}_2 \in \widetilde{J}_1$. Indeed,

$$\iiint\limits_{\Omega} (\vec{v}_2, \operatorname{grad} p)\, d\Omega = \iiint\limits_{\Omega} \operatorname{div}(p\vec{v}_2)\, d\Omega$$

$$- \iiint\limits_{\Omega} p \operatorname{div} \vec{v}_2\, d\Omega = -\iint\limits_{S} p v_{2_n}\, dS - \iiint\limits_{\Omega} p \operatorname{div} \vec{v}_2\, d\Omega. \tag{29}$$

Both terms on the right side are equal to zero and, consequently,

$$\iiint\limits_{\Omega} (\vec{v}_2, \operatorname{grad} p)\, d\Omega = 0,$$

if $\vec{v}_2 \in \widetilde{J}_1$. Therefore, to generalize the problem, it is natural to replace $\operatorname{grad} p$ with an arbitrary vector $\vec{v}_1 \in G_0$. In this statement, the computation of $\dfrac{\partial \vec{v}}{\partial t}$ is possible for any $\vec{v} \in H$ and $\vec{F} \in H$ and leads to a unique result. As it follows from (1), it suffices to take

$$\vec{v}_1 = P_1^*\{[\vec{v} \times \vec{\mathrm{k}}] + \vec{F}\},$$

where P_1^* is the projection from H into G_0; moreover, we find

$$\frac{\partial \overrightarrow{v}}{\partial t} = P_1\{[\overrightarrow{v} \times \overrightarrow{\mathrm{k}}] + \overrightarrow{F}\}, \tag{30}$$

where P_1 is the projection from H into J_1.

Thus, system (1) and condition (2.1) can be written as a equation (30).

Note that if the above expression

$$P_1^*\{[\overrightarrow{v} \times \overrightarrow{\mathrm{k}}] + \overrightarrow{F}\}$$

is actually the gradient of a smooth function p, then we can assume that this function is equal to zero on the boundary. We write the orthogonality condition for $\operatorname{grad} p$ to any element of J_1. We transform this expression according to (29). For v_{2_n} we can take any function with zero mean value. The right side of (29) can be identically equal to zero only if $p = \mathrm{const}$.

As is known [1], if $\operatorname{grad} p \in G_1$ then the function p always exists.

Finally, we consider the last case, where the space Ω is unbounded. To obtain $\dfrac{\partial \overrightarrow{v}}{\partial t}$ from $[\overrightarrow{v} \times \overrightarrow{\mathrm{k}}] + \overrightarrow{F}$, we should subtract $\operatorname{grad} p \in G$.

To generalize the result we can, as above, write the problem in the form

$$\frac{\partial \overrightarrow{v}}{\partial t} = P\{[\overrightarrow{v} \times \overrightarrow{\mathrm{k}}] + \overrightarrow{F}\}, \tag{31}$$

where P is the projection from H into J.

3 Representation of Solution as Power Series

Using the representation of the solution in the Hilbert space, we can easily construct a solution as a power series. In this section, we consider only equation (31) since equations (28) and (30) can be studied in a similar way. We start with the homogeneous equation. We denote by $A\overrightarrow{v}$ the operator $P[\overrightarrow{v} \times \overrightarrow{\mathrm{k}}]$. Obviously,

$$\|A\overrightarrow{v}\| \le \|\overrightarrow{v} \times \overrightarrow{\mathrm{k}}\| \le \|\overrightarrow{v}\|.$$

Consequently, the norm of the operator A does not exceed 1. The equation

$$\frac{d\overrightarrow{v}}{dt} = A\overrightarrow{v} \tag{32}$$

has a solution

$$\overrightarrow{v} = e^{tA}\overrightarrow{v}_0 \equiv \overrightarrow{v}_0 + \frac{t}{1}A\overrightarrow{v}_0 + \frac{t^2}{2!}A^2\overrightarrow{v}_0 + \cdots . \tag{33}$$

Indeed, series (33) converges uniformly with respect to t since the norm of its nth term satisfies the inequality

$$\|\frac{t^n}{n!}A^n\vec{v}_0\| \leq \frac{t^n}{n!}\|\vec{v}_0\|.$$

Obviously,

$$\frac{d\vec{v}}{dt} = A\vec{v}_0 + \frac{t}{1}A^2\vec{v}_0 + \frac{t^2}{2!}A^3\vec{v}_0 + \cdots,$$

and the series of the derivatives converges uniformly. Hence,

$$\frac{d\vec{v}}{dt} = A\vec{v}.$$

Moreover,

$$\vec{v}|_{t=0} = \vec{v}_0. \tag{34}$$

Thus, the problem is solved. We establish that it is well posed. For this purpose, we need to show the continuous dependence of the solution on the initial data. Let

$$\|\vec{v}_0 - \vec{v}_0^1\| < \varepsilon.$$

We consider the vectors

$$\vec{v} = e^{tA}\vec{v}_0$$

and

$$\vec{v}^1 = e^{tA}\vec{v}_0^1.$$

Then

$$\|\vec{v} - \vec{v}^1\| = \|e^{tA}(\vec{v}_0 - \vec{v}_0^1)\| < e^t\varepsilon. \tag{35}$$

Consequently, the solution in the space H continuously depends on $\vec{v}_0$ which is also given in the space H. Hence our problem is well posed.

By analogy, we can find a solution of a nonhomogeneous equation. We write this equation in the form

$$\frac{\partial\vec{v}}{\partial t} = A\vec{v} + P\vec{F}$$

and obtain the solution of this problem according to the general formula for an ordinary linear equation with constant coefficients, namely,

$$\vec{v} = e^{tA}\vec{v}_0 + \int_0^t e^{(t-t_1)A}P\vec{F}(t_1)\,dt_1. \tag{36}$$

Indeed, the integral on the right side of (36) has meaning because the norm of the integrand does not exceed

$$\|\vec{F}(t_1)\|e^{|t-t_1|}.$$

Differentiating both sides of (36) with respect to t, we find

$$\frac{d\overrightarrow{v}}{dt} = Ae^{tA}\overrightarrow{v}_0 + A\int\limits_0^t e^{(t-t_1)A}P\overrightarrow{F}(t_1)\,dt_1 + P\overrightarrow{F}(t).$$

Hence formula (36) expresses a solution of the problem. It is obvious that the constructed solution satisfies the initial conditions. The proof of the well-posedness of this problem is obvious.

4 Potential Function for Solution

Multiplying the second equation in (1) by $\pm i$ and adding with the first equation, we can write system (1) in the form

$$\begin{aligned}
&\frac{\partial}{\partial t}(v_x + iv_y) + i(v_x + iv_y) = -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)p + (F_x + iF_y),\\
&\frac{\partial}{\partial t}(v_x - iv_y) - i(v_x - iv_y) = -\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)p + (F_x - iF_y),\\
&\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + F_z,\\
&\frac{1}{2}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(v_x + iv_y) + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(v_x - iv_y)\right] + \frac{\partial v_z}{\partial z} = g.
\end{aligned}\tag{37}$$

Using (37), we can find a solution of system (1) in a simple form. Introducing for brevity the notation

$$\begin{aligned}
&\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \frac{\partial}{\partial \zeta}, &&\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \frac{\partial}{\partial \overline{\zeta}},\\
&v_x + iv_y = w, &&v_x - iv_y = \overline{w},\\
&F_x + iF_y = U, &&F_x - iF_y = \overline{U},
\end{aligned}\tag{38}$$

we find

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + i\right)w = -\frac{\partial p}{\partial \overline{\zeta}} + U,\\
&\left(\frac{\partial}{\partial t} - i\right)\overline{w} = -\frac{\partial p}{\partial \zeta} + \overline{U},\\
&\frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + F_z,\\
&\frac{1}{2}\left(\frac{\partial w}{\partial \zeta} + \frac{\partial \overline{w}}{\partial \overline{\zeta}}\right) + \frac{\partial v_z}{\partial z} = g.
\end{aligned}\tag{39}$$

The solution of system (39) can be represented in the form

$$w = w^{I} + w^{II} + w^{III},$$

$$\overline{w} = \overline{w}^{I} + \overline{w}^{II} + \overline{w}^{III},$$

$$v_z = v_z^{I} + v_z^{II} + v_z^{III},$$

$$p = p^{II} + p^{III},$$

where w^I, $\overline{w}^I$, v_z^I is a particular solution of the system

$$\left(\frac{\partial}{\partial t} + i\right) w^I = U, \quad \left(\frac{\partial}{\partial t} - i\right) \overline{w}^I = \overline{U}, \quad \frac{\partial v_z^I}{\partial z} = F_z, \tag{40}$$

w^{II}, $\overline{w}^{II}$, v_z^{II}, p^{II} is a particular solution of the system

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + i\right) w^{II} = -\frac{\partial p^{II}}{\partial \overline{\zeta}},\\
&\left(\frac{\partial}{\partial t} - i\right) \overline{w}^{II} = -\frac{\partial p^{II}}{\partial \zeta},\\
&\frac{\partial v_z^{II}}{\partial t} = -\frac{\partial p^{II}}{\partial z},\\
&\frac{1}{2}\left(\frac{\partial w^{II}}{\partial \zeta} + \frac{\partial \overline{w}^{II}}{\partial \overline{\zeta}}\right) + \frac{\partial v_z^{II}}{\partial z} = g - \frac{1}{2}\left(\frac{\partial w^{I}}{\partial \zeta} + \frac{\partial \overline{w}^{I}}{\partial \overline{\zeta}}\right) - \frac{\partial v_z^{I}}{\partial z},
\end{aligned} \tag{41}$$

and, finally, w^{III}, $\overline{w}^{III}$, v_z^{III}, p^{III} is a solution of the corresponding homogeneous system

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + i\right) w^{III} = -\frac{\partial p^{III}}{\partial \overline{\zeta}},\\
&\left(\frac{\partial}{\partial t} - i\right) \overline{w}^{III} = -\frac{\partial p^{III}}{\partial \zeta},\\
&\frac{\partial v_z^{III}}{\partial t} = -\frac{\partial p^{III}}{\partial z},\\
&\frac{1}{2}\left(\frac{\partial w^{III}}{\partial \zeta} + \frac{\partial \overline{w}^{III}}{\partial \overline{\zeta}}\right) + \frac{\partial v_z^{III}}{\partial z} = 0.
\end{aligned} \tag{42}$$

Obviously, it is easy to construct a solution of system (40) because this system is a system of ordinary differential equations.

The first three equations in (41) connect each unknown function w^{II}, $\overline{w}^{II}$, v_z^{II} with the unknown p^{II}; moreover, all the operators

$$\left(\frac{\partial}{\partial t} + i\right), \quad \left(\frac{\partial}{\partial t} - i\right), \quad \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial \overline{\zeta}}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial z} \tag{43}$$

commute in these equations. Therefore, we can look for a particular solution of equation (41) in the form

$$p^{II} = M_p \Phi^{II}, \quad w^{II} = M_w \Phi^{II}, \quad \overline{w}^{II} = \overline{M}_w \Phi^{II}, \quad v_z^{II} = M_v \Phi^{II},$$

where Φ^{II} is a potential. The operator M_p is the least common multiple (product) of the operators on the left sides of the first three equations in (41), and each of the operators M_w, $\overline{M}_w$, M_v is the product of the operator on the right side of the corresponding equation and the completion of the operator on the left side of the same equation to the operator M_p. Therefore,

$$\begin{aligned}
p^{II} &= -\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + i\right)\left(\frac{\partial}{\partial t} - i\right)\Phi^{II}, \\
w^{II} &= \left(\frac{\partial}{\partial t} - i\right)\frac{\partial}{\partial \overline{\zeta}}\frac{\partial}{\partial t}\Phi^{II}, \\
\overline{w}^{II} &= \left(\frac{\partial}{\partial t} + i\right)\frac{\partial}{\partial \zeta}\frac{\partial}{\partial t}\Phi^{II}, \\
v_z^{II} &= \left(\frac{\partial}{\partial t} + i\right)\left(\frac{\partial}{\partial t} - i\right)\frac{\partial}{\partial z}\Phi^{II}.
\end{aligned} \tag{44}$$

Formulas (44) can be written in the form

$$\begin{aligned}
v_x^{II} &= \frac{\partial^3 \Phi^{II}}{\partial x \partial t^2} + \frac{\partial^2 \Phi^{II}}{\partial y \partial t}, \\
v_y^{II} &= \frac{\partial^3 \Phi^{II}}{\partial y \partial t^2} - \frac{\partial^2 \Phi^{II}}{\partial x \partial t}, \\
v_z^{II} &= \frac{\partial^3 \Phi^{II}}{\partial z \partial t^2} + \frac{\partial \Phi^{II}}{\partial z}, \\
p^{II} &= -\frac{\partial^3 \Phi^{II}}{\partial t^3} - \frac{\partial \Phi^{II}}{\partial t}.
\end{aligned} \tag{45}$$

Substituting expressions (44) into the last equation in (41), we obtain the following equation for the potential Φ^{II}:

$$L\Phi^{II} = \left\{\frac{1}{2}\frac{\partial^2}{\partial \zeta \partial \overline{\zeta}}\frac{\partial}{\partial t}\left[\left(\frac{\partial}{\partial t} - i\right) + \left(\frac{\partial}{\partial t} + i\right)\right] + \frac{\partial^2}{\partial z^2}\left(\frac{\partial}{\partial t} + i\right)\left(\frac{\partial}{\partial t} - i\right)\right\}\Phi^{II}$$

$$= \left(\frac{\partial^2}{\partial t^2}\Delta + \frac{\partial^2}{\partial z^2}\right)\Phi^{II} = g - \frac{1}{2}\left\{\frac{\partial}{\partial \zeta}w^I + \frac{\partial}{\partial \overline{\zeta}}w^I\right\} - \frac{\partial v_z^I}{\partial z}. \tag{46}$$

Later we will show how to find a particular solution of equation (46).

The general solution of equation (42) can be represented in the same form as in the case of equations (41)

$$p^{III} = -\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + i\right)\left(\frac{\partial}{\partial t} - i\right)\Phi^{III}, \quad w^{III} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - i\right)\frac{\partial}{\partial \overline{\zeta}}\Phi^{III},$$

$$\overline{w}^{III} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + i\right)\frac{\partial}{\partial \zeta}\Phi^{II}, \qquad v_z^{III} = \left(\frac{\partial}{\partial t} + i\right)\left(\frac{\partial}{\partial t} - i\right)\frac{\partial}{\partial z}\Phi^{III}$$

or

$$\begin{aligned} p^{III} &= -\left(\frac{\partial^3}{\partial t^3} + \frac{\partial}{\partial t}\right)\Phi^{III}, \\ v_x^{III} &= \left(\frac{\partial^3}{\partial x \partial t^2} + \frac{\partial^2}{\partial y \partial t}\right)\Phi^{III}, \\ v_y^{III} &= \left(\frac{\partial^3}{\partial y \partial t^2} - \frac{\partial^2}{\partial x \partial t}\right)\Phi^{III}, \\ v_z^{III} &= \left(\frac{\partial^3}{\partial z \partial t^2} + \frac{\partial}{\partial z}\right)\Phi^{III}, \end{aligned} \tag{47}$$

where the function Φ^{III} is a solution of the homogeneous equation[2]

$$L\Phi^{III} \equiv \left(\frac{\partial^2}{\partial t^2}\Delta + \frac{\partial^2}{\partial z^2}\right)\Phi^{III} = 0. \tag{48}$$

We show that such representation is always possible.

We preliminarily establish that the vectors $\overrightarrow{v}$ and p satisfy the equations

$$Lp = 0, \tag{49}$$

$$L\overrightarrow{v} = 0. \tag{50}$$

Instead of equation (50), it is sufficient to consider the equations

$$Lw = 0, \tag{51}$$

$$Lv_z = 0. \tag{52}$$

To prove (51) and (52), we apply some of the operators M_p, M_w, $\overline{M}_w$, M_z to the equation

$$\frac{1}{2}\left(\frac{\partial w^{III}}{\partial \zeta} + \frac{\partial \overline{w}^{III}}{\partial \overline{\zeta}}\right) + \frac{\partial v_z^{III}}{\partial z} = 0.$$

For example,

$$M_w = \frac{\partial}{\partial \overline{\zeta}}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - i\right).$$

We obtain

$$\frac{1}{2}\frac{\partial^2}{\partial \zeta \partial \overline{\zeta}}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - i\right)w^{III} + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - i\right)\frac{\partial^2}{\partial \overline{\zeta}^2}\overline{w}^{III}$$

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[2] Equation (48) is called the Sobolev equation. – *Ed.*

$$+\frac{\partial^2}{\partial z\partial\overline{\zeta}}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}-i\right)v_z^{III}=0$$

or, using the equations

$$\left(\frac{\partial}{\partial t}-i\right)\overline{w}^{III}=-\frac{\partial p^{III}}{\partial\zeta},\quad \left(\frac{\partial}{\partial t}+i\right)w^{III}=-\frac{\partial p^{III}}{\partial\overline{\zeta}},$$

$$\frac{\partial}{\partial t}v_z^{III}=-\frac{\partial p^{III}}{\partial z},$$

we have

$$\left\{\frac{1}{2}\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}-i\right)+\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+i\right)\right]\right.$$
$$\left.+\frac{\partial^2}{\partial z^2}\left(\frac{\partial}{\partial t}+i\right)\left(\frac{\partial}{\partial t}-i\right)\right\}w^{III}=0,$$

i.e.,

$$Lw^{III}=0.$$

Remaining equations (52) and (49) are proved in a similar way.

Assuming that the functions v_x, v_y, v_z, p are given and satisfy system (42), we consider system (47) as a system of equations with respect to an unknown function Φ^{III}.

Let us show that equations (48) and (47) are compatible and determine a function Φ up to a harmonic function $\chi(x,y)$ of two variables (we omit the notation III for brevity).

Indeed, the general solution of the first equation in (47) has the form

$$\frac{\partial\Phi}{\partial t}=C_2(x,y,z)\cos t+C_3(x,y,z)\sin t-\int\limits_0^t\sin(t-t_1)p(x,y,z,t_1)\,dt_1. \quad (53)$$

For $L\dfrac{\partial\Phi}{\partial t}$ we have

$$L\frac{\partial\Phi}{\partial t}=\left(\frac{\partial^2C_2}{\partial z^2}-\Delta C_2\right)\cos t+\left(\frac{\partial^2C_3}{\partial z^2}-\Delta C_3\right)\sin t$$
$$-\int\limits_0^t\sin(t-t_1)\frac{\partial^2p(x,y,z,t_1)}{\partial z^2}\,dt_1-\Delta\frac{\partial^2}{\partial t^2}\int\limits_0^t\sin\tau p(x,y,z,t-\tau)\,d\tau$$
$$=-\left(\frac{\partial^2C_2}{\partial x^2}+\frac{\partial^2C_2}{\partial y^2}\right)\cos t-\left(\frac{\partial^2C_3}{\partial x^2}+\frac{\partial^2C_3}{\partial y^2}\right)\sin t$$
$$-\int\limits_0^t\sin(t-t_1)\frac{\partial^2p(x,y,z,t_1)}{\partial z^2}\,dt_1-\int\limits_0^t\sin\tau\Delta\frac{\partial^2}{\partial t^2}p(x,y,z,t-\tau)\,d\tau$$

$$-\sin t \frac{\partial}{\partial t} \Delta p|_{t=0} - \cos t \Delta p|_{t=0}$$

$$= -\sin t \left[\frac{\partial}{\partial t} \Delta p|_{t=0} + \frac{\partial^2 C_3}{\partial x^2} + \frac{\partial^2 C_3}{\partial y^2} \right] - \cos t \left[\Delta p|_{t=0} + \frac{\partial^2 C_2}{\partial x^2} + \frac{\partial^2 C_2}{\partial y^2} \right].$$

Choosing C_2 and C_3 in a suitable way, we can achieve that $L\dfrac{\partial \Phi}{\partial t}$ is equal to zero.

Obviously, the solutions of the following equations exist

$$\begin{aligned} \frac{\partial^2 C_3}{\partial x^2} + \frac{\partial^2 C_3}{\partial y^2} &= -\frac{\partial}{\partial t} \Delta p|_{t=0}, \\ \frac{\partial^2 C_2}{\partial x^2} + \frac{\partial^2 C_2}{\partial y^2} &= -\Delta p|_{t=0}. \end{aligned} \tag{54}$$

Choosing C_2 and C_3, we obtain the value $\dfrac{\partial \Phi}{\partial t}$ up to two arbitrary functions $\chi_1(x, y, z)$ and $\chi_2(x, y, z)$ that are harmonic with respect to x and y. We have

$$\frac{\partial \Phi}{\partial t} = \Omega + \chi_1(x, y, z) \cos t - \chi_2(x, y, z) \sin t,$$

where Ω denotes all the terms on the right side of (53) except for terms containing χ_1 and χ_2.

For Φ we obtain the equality

$$\Phi = C_1(x, y, z) + \chi_1(x, y, z) \sin t + \chi_2(x, y, z) \cos t + \int_0^t \Omega \, dt_1$$

$$= C_1(x, y, z) + \chi_1(x, y, z) \sin t + \chi_2(x, y, z) \cos t + \int_0^t \Omega(x, y, z, t - \tau) \, d\tau.$$

Computing $L\Phi$, we find

$$L\Phi = \frac{\partial^2 C_1}{\partial z^2} + \int_0^t L\Omega(x, y, z, t - \tau) \, d\tau + \left.\frac{\partial \Delta \Omega}{\partial t}\right|_{t=0} = \frac{\partial^2 C_1}{\partial z^2} + \left.\frac{\partial \Delta \Omega}{\partial t}\right|_{t=0}.$$

Choosing C_1 from the equation

$$\frac{\partial^2 C_1}{\partial z^2} = -\left.\frac{\partial \Delta \Omega}{\partial t}\right|_{t=0},$$

we obtain for Φ the final expression

$$\Phi = \Phi_0 + D_0(x, y) + z D_1(x, y) + \chi_1(x, y, z) \sin t + \chi_2(x, y, z) \cos t,$$

where D_0 and D_1 are arbitrary functions and $\chi_1(x,y,z)$ and $\chi_2(x,y,z)$ are arbitrary harmonic functions of x and y. Furthermore, Φ_0 is a solution of the equation $L\Phi = 0$.

Let us show that under a suitable choice of these functions, all the remaining equations of system (47) are also satisfied.

Indeed, consider the differences

$$\begin{aligned} \psi_z^{(0)} &= \frac{\partial^3 \Phi_0}{\partial z \partial t^2} + \frac{\partial \Phi_0}{\partial z} - v_z, \\ \psi_x^{(0)} &= \frac{\partial^3 \Phi_0}{\partial x \partial t^2} + \frac{\partial^2 \Phi_0}{\partial y \partial t} - v_x, \\ \psi_y^{(0)} &= \frac{\partial^3 \Phi_0}{\partial y \partial t^2} - \frac{\partial^2 \Phi_0}{\partial x \partial t} - v_y. \end{aligned} \tag{55}$$

We have

$$\frac{\partial \psi_z^{(0)}}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\partial^3 \Phi_0}{\partial t^3} + \frac{\partial \Phi_0}{\partial t}\right) - \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} - \frac{\partial v_z}{\partial t} = 0.$$

Consequently, $\psi_z^{(0)}(x,y,z)$ is independent of t. On the other hand, by (50) we have $L\psi_z^{(0)} = 0$. Consequently,

$$\frac{\partial^2 \psi_z^{(0)}}{\partial z^2} = 0 \quad \text{and} \quad \psi_z^{(0)} = A_0(x,y) + zA_1(x,y). \tag{56}$$

Consider the expressions

$$\frac{\partial \psi_x^{(0)}}{\partial t} - \psi_y^{(0)} \quad \text{and} \quad \frac{\partial \psi_y^{(0)}}{\partial t} + \psi_x^{(0)}.$$

By equations (1), (47) and (55), we find

$$\frac{\partial \psi_x^{(0)}}{\partial t} - \psi_y^{(0)} = \frac{\partial^4 \Phi_0}{\partial x \partial t^3} + \frac{\partial^2 \Phi_0}{\partial x \partial t} - \left(\frac{\partial v_x}{\partial t} - v_y\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial v_x}{\partial t} - v_y\right) = 0,$$

$$\frac{\partial \psi_y^{(0)}}{\partial t} + \psi_x^{(0)} = \frac{\partial^4 \Phi_0}{\partial y \partial t^3} + \frac{\partial^2 \Phi_0}{\partial y \partial t} - \left(\frac{\partial v_y}{\partial t} + v_x\right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial v_y}{\partial t} + v_x\right) = 0.$$

Consequently,

$$\frac{\partial^2 \psi_x^{(0)}}{\partial t^2} + \psi_x^{(0)} = 0, \qquad \frac{\partial^2 \psi_y^{(0)}}{\partial t^2} + \psi_y^{(0)} = 0. \tag{57}$$

Hence,

$$\begin{aligned} \psi_x^{(0)} &= B_1(x,y,z)\cos t + B_2(x,y,z)\sin t, \\ \psi_y^{(0)} &= -B_1(x,y,z)\sin t + B_2(x,y,z)\cos t. \end{aligned} \tag{58}$$

On the other hand, using (48) and homogeneous system (1), we have

$$\frac{\partial \psi_x^{(0)}}{\partial x} + \frac{\partial \psi_y^{(0)}}{\partial y} + \frac{\partial \psi_z^{(0)}}{\partial z} = 0 \tag{59}$$

and

$$L\psi_x^{(0)} = 0, \tag{60}$$

$$L\psi_y^{(0)} = 0. \tag{61}$$

From (60) and (61) it follows that

$$\frac{\partial^2 B_1}{\partial x^2} + \frac{\partial^2 B_1}{\partial y^2} = \frac{\partial^2 B_2}{\partial x^2} + \frac{\partial^2 B_2}{\partial y^2} = 0. \tag{62}$$

Equation (59), together with (56) and (58), yields

$$A_1(x,y) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y}\right)\cos t + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\sin t = 0. \tag{63}$$

From this it follows that

$$A_1(x,y) = 0, \quad \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} = 0, \quad \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = 0. \tag{64}$$

We see that

$$\psi_z^{(0)} = A_0(x,y), \tag{65}$$

$$B_1 = \frac{\partial u}{\partial x}, \quad B_2 = \frac{\partial u}{\partial y}, \tag{66}$$

where $u(x,y,z)$ is a harmonic function of the variables x and y. We consider the expressions for ψ_x, ψ_y, and ψ_z,

$$\begin{aligned}
\psi_x &= \frac{\partial^3 \Phi}{\partial x \partial t^2} + \frac{\partial^2 \Phi}{\partial y \partial t} - v_x \\
&= \psi_x^{(0)} + \sin t\left(-\frac{\partial \chi_1}{\partial x} - \frac{\partial \chi_2}{\partial y}\right) + \cos t\left(\frac{\partial \chi_1}{\partial y} - \frac{\partial \chi_2}{\partial x}\right), \\
\psi_y &= \frac{\partial^3 \Phi}{\partial y \partial t^2} - \frac{\partial^2 \Phi}{\partial x \partial t} - v_y \\
&= \psi_y^{(0)} + \sin t\left(-\frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial x}\right) + \cos t\left(-\frac{\partial \chi_2}{\partial y} - \frac{\partial \chi_1}{\partial x}\right), \\
\psi_z &= \frac{\partial^3 \Phi}{\partial z \partial t^2} + \frac{\partial \Phi}{\partial z} - v_z = \psi_z^{(0)} + D_1(x,y).
\end{aligned} \tag{67}$$

Choosing D_1, χ_1, χ_2 to satisfy the equations

$$A_0(x,y) + D_1(x,y) = 0, \quad \frac{\partial \chi_1}{\partial x} + \frac{\partial \chi_2}{\partial y} = B_2, \quad \frac{\partial \chi_2}{\partial x} - \frac{\partial \chi_1}{\partial y} = B_1, \tag{68}$$

for example, setting $\chi_1 = 0$, $\chi_2 = u$, from (58), (65)–(67) we obtain $\psi_x = \psi_y = \psi_z = 0$. Consequently, (47) is proved.

5 Green-Type Integral Formulas

We consider two systems of functions v_x, v_y, v_z, p and w_x, w_y, w_z, q.

Introduce the function

$$Z \equiv \left(\frac{\partial v_x}{\partial t} - v_y + \frac{\partial p}{\partial x}\right) w_x + \left(\frac{\partial v_y}{\partial t} + v_x + \frac{\partial p}{\partial y}\right) w_y + \left(\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z}\right) w_z$$

$$+ \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) q + \left(\frac{\partial w_x}{\partial t} - w_y + \frac{\partial q}{\partial x}\right) v_x + \left(\frac{\partial w_y}{\partial t} + w_x + \frac{\partial q}{\partial y}\right) v_y$$

$$+ \left(\frac{\partial w_z}{\partial t} + \frac{\partial q}{\partial z}\right) v_z + \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right) p.$$

It is easy to see that Z can be written as

$$Z = \frac{\partial}{\partial t}(v_x w_x + v_y w_y + v_z w_z) + \frac{\partial}{\partial x}(p w_x + q v_x)$$

$$+ \frac{\partial}{\partial y}(p w_y + q v_y) + \frac{\partial}{\partial z}(p w_z + q v_z). \tag{69}$$

Integrating the equality over a four-dimensional cylinder (Ω_3, $0 \le t \le t_0$) with axis parallel to the t-axis and using the Ostrogradskii formula, we obtain

$$\iiiint\limits_{\Omega_4} \left\{ \left(\frac{\partial v_x}{\partial t} - v_y + \frac{\partial p}{\partial x}\right) w_x \right.$$

$$+ \left(\frac{\partial v_y}{\partial t} + v_x + \frac{\partial p}{\partial y}\right) w_y + \left(\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z}\right) w_z + \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) q$$

$$+ \left(\frac{\partial w_x}{\partial t} - w_y + \frac{\partial q}{\partial x}\right) v_x + \left(\frac{\partial w_y}{\partial t} + w_x + \frac{\partial q}{\partial y}\right) v_y + \left(\frac{\partial w_z}{\partial t} + \frac{\partial q}{\partial z}\right) v_z$$

$$\left. + \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right) p \right\} d\Omega\, dt = \iiint\limits_{\Omega_3} (v_x w_x + v_y w_y + v_z w_z)\Big|_0^{t_0} d\Omega$$

$$- \int\limits_0^{t_0} \iint\limits_{S_3} \{(p w_x + q v_x) \cos nx$$

$$+ (p w_y + q v_y) \cos ny + (p w_z + q v_z) \cos nz\}\, dS_3\, dt, \tag{70}$$

where n denotes the inward normal to the surface S_3 bounding Ω_3 in a three-dimensional space.

Consider the case when the functions w_x, w_y, w_z, q satisfy the homogeneous system (1). In this case, we obtain

$$\int_0^{t_0}\iiint_\Omega\left\{w_x\left(\frac{\partial v_x}{\partial t}-v_y+\frac{\partial p}{\partial x}\right)+w_y\left(\frac{\partial v_y}{\partial t}+v_x+\frac{\partial p}{\partial y}\right)+w_z\left(\frac{\partial v_z}{\partial t}+\frac{\partial p}{\partial z}\right)\right.$$

$$\left.+q\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)d\Omega\, dt=\iiint_\Omega(v_xw_x+v_yw_y+v_zw_z)\right|_0^{t_0}d\Omega$$

$$-\int_0^{t_0}\iint_S(pw_n+qv_n)\,dS\,dt, \tag{71}$$

where v_n and w_n are the normal components of $\overrightarrow{v}$ and $\overrightarrow{w}$.

We will use the Green formula in form (71).

If v_x, v_y, v_z, p satisfy homogeneous system (1), then (71) becomes

$$\iiint_\Omega(v_xw_x+v_yw_y+v_zw_z)\Big|_0^{t_0}d\Omega=\int_0^{t_0}\left\{\iint_S(pw_n+qv_n)\,dS\right\}dt. \tag{72}$$

Formulas (71) and (72) are obtained for a bounded domain Ω. We prove that if the domain Ω contains infinity, but the functions v_x, v_y, v_z, $\operatorname{grad} p$ and w_x, w_y, w_z, $\operatorname{grad} q$ are square-integrable over this domain, then these formulas remain valid. It suffices to establish these formulas for the exterior of a sufficiently large ball, because any unbounded domain can be represented as the union of a bounded domain and its exterior. The formulas for the union of domains can be obtained by adding the formulas for each term.

Remark. Any solenoidal vector $\overrightarrow{v}$ having continuous derivatives and defined outside some ball Ω can be continuously extended to the whole space such that the extended vector is also solenoidal and has continuous derivatives.

Indeed, any solenoidal vector $\overrightarrow{v}$ can be represented outside the ball Ω by the formula

$$\overrightarrow{v}=\operatorname{rot}\overrightarrow{A}.$$

Extending $\overrightarrow{A}$ to the whole space such that the second-order derivatives of the extended function remain continuous, we obtain our assertion.

Let the vector $\operatorname{grad} p$ be square-integrable outside the domain Ω. Extending the function p continuously to the entire space and using the arguments from Sect. 2, we have

$$\iiint_{-\infty}^{+\infty}(\overrightarrow{v},\operatorname{grad} p)\,d\Omega=0.$$

Hence,

$$\iiint_{\infty-\Omega}(\overrightarrow{v},\operatorname{grad} p)\,d\Omega=-\iiint_\Omega(\overrightarrow{v},\operatorname{grad} p)\,d\Omega.$$

However,

$$\iiint\limits_{\Omega} (\vec{v}, \operatorname{grad} p)\, d\Omega = -\iint\limits_{S} v_{n^*} p\, dS, \tag{72.1}$$

where n^* is the inward normal to Ω. Replacing $v_{n^*} = -v_n$, where n is the normal to $\infty - \Omega$, we obtain

$$\iiint\limits_{\infty-\Omega} (\vec{v}, \operatorname{grad} p)\, d\Omega = -\iint\limits_{S} v_n p\, dS.$$

From this formula we have (71) and (72) for the exterior of the ball Ω and, consequently, for any domain if we recall the proof of this formula and use relation (72.1).

6 Particular Solutions of Main Equation (48)

In this section, we indicate some particular solutions of equation (48). Using the solutions, we can construct the general solution of the problem.

We set

$$\Phi = \varrho^m r^{-m-s} \Psi\left(\frac{\varrho\tau}{r}\right), \tag{73}$$

where

$$\begin{aligned}
&\varrho^2 = (x - x_0)^2 + (y - y_0)^2, \\
&r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, \\
&\tau = t - t_0.
\end{aligned}$$

We compute the function $L\Phi$. For simplicity, we first set

$$x_0 = y_0 = z_0 = t_0 = 0.$$

Using cylindrical coordinates, we obtain

$$\Phi = \varrho^m r^{-m-s} \Psi\left(\frac{\varrho t}{r}\right) = \varrho^m r^{-m-s} \Psi(\xi),$$

where $\xi = \dfrac{\varrho t}{r}$. Then,

$$\begin{aligned}
\frac{\partial^2 \Phi}{\partial z^2} &= \varrho^m r^{-m-s-4} \{[(m+s)(m+s+1)z^2 - (m+s)\varrho^2]\Psi(\xi) \\
&\quad + [(2m+2s+2)z^2 - \varrho^2]\xi\Psi'(\xi) + z^2\xi^2\Psi''(\xi)\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial z^2}\frac{\partial^2 \Phi}{\partial t^2} &= \varrho^m r^{-m-s-6} \{[(m+s+2)(m+s+4)\varrho^2 z^2 - (m+s+2)\varrho^2 r^2]\Psi''(\xi) \\
&\quad + [(2m+2s+7)\varrho^2 z^2 - \varrho^2 r^2]\xi\Psi'''(\xi) + z^2\varrho^2\xi^2\Psi^{(IV)}(\xi)\},
\end{aligned}$$

$$\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\varrho\frac{\partial}{\partial\varrho}\frac{\partial^2\Phi}{\partial t^2}=\varrho^m r^{-m-s-6}\{[(m+2)^2(\varrho^2+z^2)r^2-2(m+3)(m+s+2)\varrho^2r^2$$
$$+(m+s+2)(m+s+4)\varrho^4]\Psi''(\xi)$$
$$+[(2m+5)z^2r^2-(2m+2s+7)\varrho^2z^2]\xi\Psi'''(\xi)+z^4\xi^2\Psi^{(IV)}(\xi)\}.$$

Thus,

$$L\Phi\equiv\frac{\partial^2}{\partial t^2}\Delta\Phi+\frac{\partial^2\Phi}{\partial z^2}=\varrho^m r^{-m-s-4}\{[(m+s)(m+s+1)\Psi(\xi)$$
$$+(2m+2s+2)\xi\Psi'(\xi)+(\xi^2+(m+2)^2)\Psi''(\xi)+(2m+5)\xi\Psi'''(\xi)$$
$$+\xi^2\Psi^{(IV)}(\xi)]z^2-[(m+s)\Psi(\xi)+\xi\Psi'(\xi)$$
$$+(m+s+2-s^2)\Psi''(\xi)+\xi\Psi'''(\xi)]\varrho^2\}.$$

As we see, $L\Phi$ can be equal to zero only if the following two equalities hold:

$$\begin{aligned}&\Lambda_1\equiv\xi\Psi'''+[m-(s+1)(s-2)]\Psi''+\xi\Psi'+(m+s)\Psi=0,\\&\Lambda_2\equiv\xi^2\Psi^{(IV)}+(2m+5)\xi\Psi'''+[\xi^2+(m+2)^2]\Psi''\\&\quad+(2m+2s+2)\xi\Psi'+(m+s)(m+s+1)\Psi=0.\end{aligned}\tag{74}$$

A direct computation shows that

$$\Lambda_2-\xi\frac{d\Lambda_1}{d\xi}-(m+s+1)\Lambda_1=(s-1)^2[\xi\Psi'''+(m+s+2)\Psi''].$$

It is easy to see that the second equation in (74) follows from the first equation for $s=1$.

Thus, for $s=1$ we obtain for the unknown function the ordinary differential equation

$$\Lambda_1\equiv\xi\Psi'''+(m+2)\Psi''+\xi\Psi'+(m+1)\Psi=0.\tag{75}$$

The solutions of equation (75) form some class of solutions of the equation $L\Phi=0$.

Equation (75) can be solved in a finite form for any m by using the Lommel functions or Bessel functions. We are interested in some particular solutions of this equation.

For $m=0$ we have

$$\Lambda_1\equiv\xi\Psi'''+2\Psi''+\xi\Psi'+\Psi=0.$$

This equation can be written in the form

$$\Lambda_1\equiv\xi\left[\Psi'''+\frac{1}{\xi}\Psi''+\left(1-\frac{1}{\xi^2}\right)\Psi'\right]+\left(\Psi''+\frac{1}{\xi}\Psi'+\Psi\right)=\xi\frac{dN}{d\xi}+N,$$

where $N = \Psi'' + \frac{1}{\xi}\Psi' + \Psi$. Consequently, solutions of this equation are solutions of the Bessel equation

$$\Psi'' + \frac{1}{\xi}\Psi' + \Psi = 0$$

or solutions of the equation

$$\Psi'' + \frac{1}{\xi}\Psi' + \Psi = \frac{1}{\xi},$$

that are Lommel functions.

Thus, for example, the following function is a solution of (48):

$$\Phi_0 = \frac{1}{r} J_0\left(\frac{\varrho t}{r}\right). \tag{76}$$

For $m = -1$ we obtain a solution in the form

$$\Phi_1 = \frac{1}{\varrho}\int\limits_0^{\xi} J_0(\xi_1)\, d\xi_1. \tag{77}$$

Indeed, in this case, equation (75) takes the form

$$\xi\Psi''' + \Psi'' + \xi\Psi' = 0, \tag{78}$$

and the function $\Psi'(\xi) = J_0(\xi)$ is a solution of the last equation.

For convenience, we introduce the notation

$$\Xi(\xi) = \int\limits_0^{\xi} J_0(\xi_1)\, d\xi_1. \tag{79}$$

As is known, the function $\Xi(\xi)$ is expressed in terms of the Lommel functions, but it is of no interest for us.

Consider the case $m = -2$. In this case, we obtain a solution of the problem of the form

$$\Phi_2 = \frac{r}{\varrho^2}\xi[\Xi(\xi) + J_0'(\xi)], \tag{80}$$

which can be easily verified by substitution.

7 Another Class of Particular Solutions

We set

$$\Phi = x\varrho^m r^{-m-s}\Psi(\xi). \tag{81}$$

We compute the function $L\Phi$:

$$\frac{\partial^2 \Phi}{\partial t^2} = x \frac{\partial^2}{\partial t^2}[\varrho^m r^{-m-s} \Psi(\xi)],$$

$$\Delta \frac{\partial^2 \Phi}{\partial t^2} = x\Delta \frac{\partial^2}{\partial t^2}[\varrho^m r^{-m-s}\Psi(\xi)] + 2\frac{x}{\varrho}\frac{\partial}{\partial \varrho}[\varrho^{m+2} r^{-m-s-2}\Psi''(\xi)],$$

$$L\Phi = xL(\varrho^m r^{-m-s}\Psi(\xi)) + x\{[(2m+4)\varrho^m r^{-m-s-2}$$
$$-(2m+2s+4)\varrho^{m+2} r^{-m-s-4}]\Psi''(\xi) + 2z^2 t \varrho^m r^{-m-s-5}\Psi'''(\xi)\}$$
$$= x\varrho^m r^{-m-s-4}\left\{ z^2 \left[\left(\xi \frac{d\Lambda_1}{d\xi} + (m+s+1)\Lambda_1 \right)\right.\right.$$
$$\left. +(s-1)^2[\xi\Psi''' + (m+s+2)\Psi''] + (2m+4)\Psi'' + 2\xi\Psi''' \right]$$
$$\left. -\varrho^2[\Lambda_1 - 2s\Psi''] \right\} = x^2 \varrho^m r^{-m-s-4}\left\{ z^2 \left[\xi \frac{d}{d\xi}(\Lambda_1 + 2s\Psi'') \right.\right.$$
$$+(m+s+1)(\Lambda_1 + 2s\Psi'') + ((s-1)^2 + 2 - 2s)\xi\Psi'''$$
$$\left.\left. +((s-1)^2(m+s+2) - 2s(m+s+1) + (2m+4))\Psi'' \right] - \varrho^2[\Lambda_1 + 2s\Psi''] \right\}$$
$$= x^2 \varrho^m r^{-m-s-4}\left\{ z^2 \left[\xi \frac{d}{d\xi}(\Lambda_1 + 2s\Psi'') + (m+s+1)(\Lambda_1 + 2s\Psi'') \right.\right.$$
$$+(s-1)(s-3)\xi\Psi''' + ((m+2)[(s-1)^2 - 2s + 2]$$
$$\left.\left. +s(s-1)^2 - 2s(s-1))\Psi'' \right] - \varrho^2[\Lambda_1 + 2s\Psi''] \right\}$$
$$= x^2 \varrho^m r^{-m-s-4}\left\{ z^2 \left[\xi \frac{d}{d\xi}(\Lambda_1 + 2s\Psi'') + (m+s+1)(\Lambda_1 + 2s\Psi'') \right.\right.$$
$$\left.\left. +(s-1)(s-3)[\xi\Psi''' + (m+s+2)\Psi''] \right] - \varrho^2[\Lambda_1 + 2s\Psi''] \right\}.$$

We see that for $s = 1$ and $s = 3$ two equations for Ψ follow one from another. Consequently, for

$$\Lambda_1 + 2s\Psi'' = 0$$

function (81) satisfies the equation $L\Phi = 0$.

We could obtain the second solution by solving the equations

$$\begin{aligned} &\Lambda_1 + 2s\Psi'' = 0, \\ &\xi\Psi''' + (m+s+2)\Psi'' = 0 \end{aligned} \tag{82}$$

for any s.

The equations for Ψ take the form

$$\xi\Psi''' + (m+3s+2-s^2)\Psi'' + \xi\Psi' + (m+s)\Psi = 0. \tag{83}$$

We do not study equations (82) in detail, but rather consider equation (83) for $s = 1$ and $s = 3$.

For $s = 1$ from (83) we find

$$\xi\Psi''' + (m+4)\Psi'' + \xi\Psi' + (m+1)\Psi = 0. \tag{84}$$

For $s = 3$ from (83) we obtain

$$\xi\Psi''' + (m+2)\Psi'' + \xi\Psi' + (m+3)\Psi = 0. \tag{85}$$

The integration of these equations is again reduced to the Lommel functions and the Bessel functions.

We indicate one important solution of (85) for $m = -2$,

$$\xi\Psi''' + \xi\Psi' + \Psi = 0. \tag{86}$$

The following function is a solution of this equation,

$$\Psi = \xi J_0'(\xi),$$

which can be easily verified by direct differentiation.

Replacing x with y, we obtain additional solutions of equation (48).

Thus,

$$\Phi = \frac{x}{\varrho^2 r}\frac{\varrho t}{r}J_0'\left(\frac{\varrho t}{r}\right) = \frac{xt}{\varrho r^2}J_0'\left(\frac{\varrho t}{r}\right) \tag{87}$$

and

$$\Phi = \frac{yt}{\varrho r^2}J_0'\left(\frac{\varrho t}{r}\right). \tag{88}$$

Using the constructed particular solutions of (48), we can start to construct the general solution of our problem.

Differentiating the solution of (80) with respect to z, we obtain one more important solution

$$\Phi = \frac{z}{\varrho^2 r}[\xi\Xi(\xi) + \xi J_0'(\xi)]$$

$$+\frac{r}{\varrho^2}\frac{d}{d\xi}[\xi\Xi(\xi) + \xi J_0'(\xi)]\frac{d\left(\frac{\varrho t}{r}\right)}{dz} = \frac{zt}{\varrho r^2}J_0'\left(\frac{\varrho t}{r}\right). \tag{89}$$

For $m = -3$, $s = 3$ we obtain the equation

$$\xi\Psi''' - \Psi'' + \xi\Psi' = 0, \tag{90}$$

whose solution is, in particular, the function[3]

$$\Psi = \xi J_0(\xi) - \Xi(\xi).$$

[3] The function $\Xi(\xi)$ is defined by (79). – *Ed.*

Indeed,

$$\Psi' = \xi J_0'(\xi),$$

$$\Psi'' = -\xi J_0(\xi),$$

$$\Psi''' = -\xi J_0'(\xi) - J_0(\xi).$$

This implies our assertion.

Using the obtained solutions, we can construct the system of functions

$$\Phi_I, \quad \Phi_{II}, \quad \Phi_{III}, \quad Q, \tag{91}$$

where

$$\Phi_I = \Phi^{11} - \Phi^{12},$$

$$\Phi_{II} = \Phi^{21} + \Phi^{22},$$

$$\Phi^{11} = \frac{x-x_0}{\varrho^2 r} \frac{\varrho(t-t_0)}{r} J_0'(\varrho\frac{t-t_0}{r}),$$

$$\Phi^{12} = \frac{y-y_0}{\varrho^3} \frac{\varrho(t-t_0)}{r} J_0(\varrho\frac{t-t_0}{r}) - \frac{y-y_0}{\varrho^3} \Xi(\varrho\frac{t-t_0}{r}),$$

$$\Phi^{21} = \frac{y-y_0}{\varrho^2 r} \frac{\varrho(t-t_0)}{r} J_0'(\varrho\frac{t-t_0}{r}),$$

$$\Phi^{22} = \frac{x-x_0}{\varrho^3} \frac{\varrho(t-t_0)}{r} J_0(\varrho\frac{t-t_0}{r}) - \frac{x-x_0}{\varrho^3} \Xi(\varrho\frac{t-t_0}{r}),$$

$$\Phi_{III} = \frac{z-z_0}{\varrho^2 r} \frac{\varrho(t-t_0)}{r} J_0'(\varrho\frac{t-t_0}{r}),$$

$$Q = -\frac{1}{\varrho} \Xi(\varrho\frac{t-t_0}{r}).$$

It is obvious that each of these functions satisfies the equations

$$L\Phi = 0 \quad \text{and} \quad L_0\Phi = 0, \tag{92}$$

where L_0 denotes the operator obtained from the operator L by replacing the variables x, y, z, t with the variables x_0, y_0, z_0, t_0.

Differentiating, we see that functions (91) satisfy the equations

$$\begin{aligned} \frac{\partial \Phi_I}{\partial t_0} - \Phi_{II} &= -\frac{\partial Q}{\partial x_0}, \\ \frac{\partial \Phi_{II}}{\partial t_0} + \Phi_I &= -\frac{\partial Q}{\partial y_0}, \\ \frac{\partial \Phi_{III}}{\partial t_0} &= -\frac{\partial Q}{\partial z_0}, \end{aligned} \tag{93}$$

$$\frac{\partial \Phi_I}{\partial x_0} + \frac{\partial \Phi_{II}}{\partial y_0} + \frac{\partial \Phi_{III}}{\partial z_0} = 0. \tag{94}$$

8 Three Particular Solutions of System (1)

Using the potentials Φ_I, Φ_{II}, Φ_{III}, Q, we can construct three particular solutions of homogeneous system (1) by using formulas like (47). We obtain

$$\begin{aligned} w_x^I &= w_x^{111} + w_x^{112} - w_x^{121} - w_x^{122}, \\ w_y^I &= w_y^{111} + w_y^{112} - w_y^{121} - w_y^{122}, \\ w_z^I &= w_z^{111} + w_z^{113} - w_z^{121} - w_z^{123}, \\ q^I &= q^{111} + q^{113} - q^{121} - q^{123}, \end{aligned} \tag{95}$$

where

$$\begin{array}{llll} w_x^{111} = \dfrac{\partial^3 \Phi^{11}}{\partial x \partial t^2}, & w_x^{112} = \dfrac{\partial^2 \Phi^{11}}{\partial y \partial t}, & w_x^{121} = \dfrac{\partial^3 \Phi^{12}}{\partial x \partial t^2}, & w_x^{122} = \dfrac{\partial^2 \Phi^{12}}{\partial y \partial t}, \\ w_y^{111} = \dfrac{\partial^3 \Phi^{11}}{\partial y \partial t^2}, & w_y^{112} = -\dfrac{\partial^2 \Phi^{11}}{\partial x \partial t}, & w_y^{121} = \dfrac{\partial^3 \Phi^{12}}{\partial y \partial t^2}, & w_y^{122} = -\dfrac{\partial^2 \Phi^{12}}{\partial x \partial t}, \\ w_z^{111} = \dfrac{\partial^3 \Phi^{11}}{\partial z \partial t^2}, & w_z^{113} = \dfrac{\partial \Phi^{11}}{\partial z}, & w_z^{121} = \dfrac{\partial^3 \Phi^{12}}{\partial z \partial t^2}, & w_z^{123} = \dfrac{\partial \Phi^{12}}{\partial z}, \\ q^{111} = -\dfrac{\partial^3 \Phi^{11}}{\partial t^3}, & q^{113} = -\dfrac{\partial \Phi^{11}}{\partial t}, & q^{121} = -\dfrac{\partial^3 \Phi^{12}}{\partial t^3}, & q^{123} = -\dfrac{\partial \Phi^{12}}{\partial t}. \end{array}$$

Similarly,

$$\begin{aligned} w_x^{II} &= w_x^{211} + w_x^{212} + w_x^{221} + w_x^{222}, \\ w_y^{II} &= w_y^{211} + w_y^{212} + w_y^{221} + w_y^{222}, \\ w_z^{II} &= w_z^{211} + w_z^{213} + w_z^{221} + w_z^{223}, \\ q^{II} &= q^{211} + q^{213} + q^{221} + q^{223}, \end{aligned} \tag{96}$$

where

$$\begin{array}{llll} w_x^{211} = \dfrac{\partial^3 \Phi^{21}}{\partial x \partial t^2}, & w_x^{212} = \dfrac{\partial^2 \Phi^{21}}{\partial y \partial t}, & w_x^{221} = \dfrac{\partial^3 \Phi^{22}}{\partial x \partial t^2}, & w_x^{222} = \dfrac{\partial^2 \Phi^{22}}{\partial y \partial t}, \\ w_y^{211} = \dfrac{\partial^3 \Phi^{21}}{\partial y \partial t^2}, & w_y^{212} = -\dfrac{\partial^2 \Phi^{21}}{\partial x \partial t}, & w_y^{221} = \dfrac{\partial^3 \Phi^{22}}{\partial y \partial t^2}, & w_y^{222} = -\dfrac{\partial^2 \Phi^{22}}{\partial x \partial t}, \\ w_z^{211} = \dfrac{\partial^3 \Phi^{21}}{\partial z \partial t^2}, & w_z^{213} = \dfrac{\partial \Phi^{21}}{\partial z}, & w_z^{221} = \dfrac{\partial^3 \Phi^{22}}{\partial z \partial t^2}, & w_z^{223} = \dfrac{\partial \Phi^{22}}{\partial z}, \\ q^{211} = -\dfrac{\partial^3 \Phi^{21}}{\partial t^3}, & q^{213} = -\dfrac{\partial \Phi^{21}}{\partial t}, & q^{221} = -\dfrac{\partial^3 \Phi^{22}}{\partial t^3}, & q^{223} = -\dfrac{\partial \Phi^{22}}{\partial t}. \end{array}$$

Finally,

$$\begin{aligned} w_x^{III} &= w_x^{31} + w_x^{32}, \\ w_y^{III} &= w_y^{31} + w_y^{32}, \\ w_z^{III} &= w_z^{31} + w_z^{33}, \\ q^{III} &= q^{31} + q^{33}, \end{aligned} \tag{97}$$

where

$$w_x^{31} = \frac{\partial^3 \Phi_{III}}{\partial x \partial t^2}, \quad w_x^{32} = \frac{\partial^2 \Phi_{III}}{\partial y \partial t}, \quad w_y^{31} = \frac{\partial^3 \Phi_{III}}{\partial y \partial t^2}, \quad w_y^{32} = -\frac{\partial^2 \Phi_{III}}{\partial x \partial t},$$

$$w_z^{31} = \frac{\partial^3 \Phi_{III}}{\partial z \partial t^2}, \quad w_z^{33} = \frac{\partial \Phi_{III}}{\partial z}, \quad q^{31} = -\frac{\partial^3 \Phi_{III}}{\partial t^3}, \quad q^{33} = -\frac{\partial \Phi_{III}}{\partial t}.$$

We will use solutions (95), (96), and (97).

9 Computations of Some Auxiliary Definite Integrals

For our purpose, we need to compute some definite integrals. For completeness, we recall their computation.

We consider

$$\chi_s(t) = \int\limits_0^t \frac{\xi^{2s-1} J_0(\xi)\, d\xi}{\sqrt{t^2 - \xi^2}}, \tag{98}$$

$$\omega_s(t) = \int\limits_0^t \frac{\xi^{2s} J_0'(\xi)\, d\xi}{\sqrt{t^2 - \xi^2}}. \tag{99}$$

We show how some of these integrals are expressed in terms of other integrals.

First of all, we make the change of variables $\xi = t\zeta$,

$$\chi_s(t) = t^{2s-1} \int\limits_0^1 \frac{\zeta^{2s-1} J_0(\zeta t)\, d\zeta}{\sqrt{1 - \zeta^2}}, \tag{100}$$

$$\omega_s(t) = t^{2s} \int\limits_0^1 \frac{\zeta^{2s} J_0'(\zeta t)\, d\zeta}{\sqrt{1 - \zeta^2}}. \tag{101}$$

Differentiating the first equation with respect to t, we obtain

$$\frac{d}{dt}\left(\frac{\chi_s(t)}{t^{2s-1}}\right) = \frac{\omega_s(t)}{t^{2s}}. \tag{102}$$

Moreover,

$$\frac{\omega_s(t)}{t^{2s-1}} = \int_0^1 \frac{\zeta^{2s-1}[\zeta t J_0'(\zeta t)]\, d\zeta}{\sqrt{1-\zeta^2}}. \tag{103}$$

Using the equality

$$\frac{d}{d\xi}[\xi J_0'(\xi)] = \xi J_0''(\xi) + J_0'(\xi) = -\xi J_0(\xi)$$

and differentiating both sides of (103) with respect to t, we obtain

$$\frac{d}{dt}\left(\frac{\omega_s(t)}{t^{2s-1}}\right) = -\int_0^1 \frac{\zeta^{2s-1}\zeta \cdot \zeta t J_0(\zeta t)\, d\zeta}{\sqrt{1-\zeta^2}}$$

$$= -t\int_0^1 \frac{\zeta^{2s+1} J_0(\zeta t)\, d\zeta}{\sqrt{1-\zeta^2}} = -\frac{1}{t^{2s}}\int_0^1 \frac{(\zeta t)^{2s+1} J_0(\zeta t)\, d\zeta}{\sqrt{1-\zeta^2}} = -\frac{\chi_{s+1}(t)}{t^{2s}},$$

$$\frac{d}{dt}\left(\frac{\omega_s(t)}{t^{2s-1}}\right) = -\frac{\chi_{s+1}(t)}{t^{2s}}. \tag{104}$$

Thus, the computation of all χ_s and ω_s is reduced to the computation of some of them. Let us compute the integral

$$\chi_1(t) = \int_0^t \frac{\xi J_0(\xi)\, d\xi}{\sqrt{t^2-\xi^2}}. \tag{105}$$

Note that $\chi_1(t)$ can be represented as an integral in the complex plane ξ,

$$\chi_1(t) = \frac{1}{2}\int_C \frac{\xi J_0(\xi)\, d\xi}{\sqrt{t^2-\xi^2}},$$

where C is an open contour such that its endpoints are located at the point $\xi = 0$ on two sheets of the Riemann surface of the function $\sqrt{t^2-\xi^2}$ passing around the point $\xi = t$ (see Fig. 1).

Let us compute the integral

$$\frac{d^2\chi_1(t)}{dt^2} = \frac{1}{2}\int_C \xi J_0(\xi) \frac{\partial^2 \frac{1}{\sqrt{t^2-\xi^2}}}{\partial t^2}\, d\xi.$$

The function $\dfrac{1}{\sqrt{t^2-\xi^2}}$ satisfies the equation

$$\frac{\partial^2 \frac{1}{\sqrt{t^2-\xi^2}}}{\partial t^2} = \frac{\partial^2 \frac{1}{\sqrt{t^2-\xi^2}}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \frac{1}{\sqrt{t^2-\xi^2}}}{\partial \xi}$$

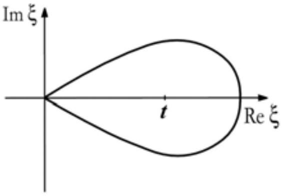

Fig. 1.

or

$$\xi\frac{\partial^2\frac{1}{\sqrt{t^2-\xi^2}}}{\partial t^2}=\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}\frac{1}{\sqrt{t^2-\xi^2}}.$$

Consequently,

$$\frac{d^2\chi_1(t)}{dt^2}=\frac{1}{2}\int\limits_C J_0(\xi)\frac{d}{d\xi}\xi\frac{d}{d\xi}\frac{1}{\sqrt{t^2-\xi^2}}d\xi$$

$$=\frac{1}{2}J_0(\xi)\xi\frac{d}{d\xi}\left(\frac{1}{\sqrt{t^2-\xi^2}}\right)\Bigg|_C-\frac{1}{2}\int\limits_C\xi\frac{d}{d\xi}\left(\frac{1}{\sqrt{t^2-\xi^2}}\right)J_0'(\xi)\,d\xi$$

$$=-\frac{1}{2}\xi J_0'(\xi)\ \frac{1}{\sqrt{t^2-\xi^2}}\Bigg|_C+\frac{1}{2}\int\limits_C\frac{1}{\sqrt{t^2-\xi^2}}\frac{d}{d\xi}\left(\xi J_0'(\xi)\right)\,d\xi$$

$$=-\frac{1}{2}\int\limits_C\frac{\xi J_0(\xi)d\xi}{\sqrt{t^2-\xi^2}}=-\chi_1(t).$$

Hence,

$$\frac{d^2\chi_1(t)}{dt^2}+\chi_1(t)=0.$$

Thus,

$$\chi_1(t)=a\cos t+b\sin t. \tag{106}$$

We note that $\chi_1(t)$ is equal to zero at $t=0$. Hence,

$$\chi_1(t)=b\sin t.$$

To determine the constant b, we note that

$$\frac{\chi_1(t)}{t}=\int\limits_0^1\frac{\zeta J_0(\zeta t)d\zeta}{\sqrt{1-\zeta^2}}.$$

Consequently,

$$\lim_{t\to 0}\frac{\chi_1(t)}{t} = \int\limits_0^1 \frac{\zeta d\zeta}{\sqrt{1-\zeta^2}} = -\sqrt{1-\zeta^2}\Big|_0^1 = 1. \tag{107}$$

Thus,

$$\chi_1(t) = \sin t. \tag{108}$$

Using (102) and (104), we write a number of equalities:

$$\int\limits_0^t \frac{\xi J_0(\xi)d\xi}{\sqrt{t^2-\xi^2}} = \chi_1(t) = \sin t,$$

$$\int\limits_0^t \frac{\xi^2 J_0'(\xi)d\xi}{\sqrt{t^2-\xi^2}} = \omega_1(t) = t^2\frac{d}{dt}\left(\frac{\sin t}{t}\right) = -\sin t + t\cos t,$$

$$\begin{aligned}\int\limits_0^t \frac{\xi^3 J_0(\xi)d\xi}{\sqrt{t^2-\xi^2}} &= \chi_2(t) = -t^2\frac{d}{dt}\left(\frac{\omega_1(t)}{t}\right) = -t^2\frac{d}{dt}\left[-\frac{\sin t}{t} + \cos t\right] \\ &= t^2\left[\sin t\left(1-\frac{1}{t^2}\right) + \frac{\cos t}{t}\right] = \sin t(t^2-1) + t\cos t,\end{aligned} \tag{109}$$

$$\begin{aligned}\int\limits_0^t \frac{\xi^4 J_0'(\xi)d\xi}{\sqrt{t^2-\xi^2}} &= \omega_2(t) = t^4\frac{d}{dt}\left(\frac{\chi_2(t)}{t^3}\right) \\ &= (3-2t^2)\sin t + (t^3-3t)\cos t,\end{aligned}$$

$$\int\limits_0^t \frac{\xi^5 J_0(\xi)d\xi}{\sqrt{t^2-\xi^2}} = \chi_3(t) = (t^4-5t^2+9)\sin t + (2t^3-9t)\cos t.$$

Further,

$$\int\limits_0^t \frac{J_0'(\xi)d\xi}{\sqrt{t^2-\xi^2}} = \omega_0(t) = \frac{\cos t - 1}{t}.$$

Obviously,

$$\frac{d(t\omega_0(t))}{dt} = -\chi_1(t) = -\sin t, \quad t\omega_0(t) = \cos t - 1.$$

It is convenient to write the integrals following from the above formulas

$$
\begin{aligned}
&\int_0^1 \frac{J_0'(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = \frac{\cos t - 1}{t},\\
&\int_0^1 \frac{\zeta J_0(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = \frac{\sin t}{t},\\
&\int_0^1 \frac{\zeta^2 J_0'(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = -\frac{\sin t}{t^2} + \frac{\cos t}{t},\\
&\int_0^1 \frac{\zeta^3 J_0(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = \left(\frac{1}{t} - \frac{1}{t^3}\right)\sin t + \frac{1}{t^2}\cos t,\\
&\int_0^1 \frac{\zeta^4 J_0'(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = \left(-\frac{2}{t^2} + \frac{3}{t^4}\right)\sin t + \left(\frac{1}{t} - \frac{3}{t^3}\right)\cos t,\\
&\int_0^1 \frac{\zeta^5 J_0(\zeta t)d\zeta}{\sqrt{1-\zeta^2}} = \left(\frac{1}{t} - \frac{5}{t^3} + \frac{9}{t^5}\right)\sin t + \left(\frac{2}{t^2} - \frac{9}{t^4}\right)\cos t.
\end{aligned}
\tag{110}
$$

We will use these formulas from now on.

10 Computation of v_x

We move on to solving our problem. We consider system (1), where $\overrightarrow{F}$ is an arbitrary vector of exterior forces. Assume that it is square-integrable over the entire space. We represent the vector $\overrightarrow{F}$ as the sum of two terms $\overrightarrow{F} = \overrightarrow{F}_1 + \overrightarrow{F}_2$, where $\overrightarrow{F}_1$ is a potential vector and $\overrightarrow{F}_2$ is a solenoidal vector; moreover, each of them is square-integrable. Let $\overrightarrow{F}_1 = \operatorname{grad}\Psi$ and $p' = p - \Psi$. Then system (1) can be written in the form

$$
\frac{d\overrightarrow{v}}{dt} = (\overrightarrow{v} \times \mathrm{k}) - \operatorname{grad} p' - \overrightarrow{F}_2.
$$

Hence, without loss of generality, we can assume that $\overrightarrow{F}$ in (1) is a solenoidal vector.

We cut off the cylinder

$$
|z - z_0| \le h, \quad \varrho \le \eta \tag{111}
$$

and apply (71) to the volume $\Omega_{h,\eta}$ obtained by this method. Let $\overrightarrow{v}$ and p be unknown functions satisfying (1), and let $\overrightarrow{w}$ and q be $\overrightarrow{w}^I$ and q^I. We obtain

$$\iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^I)\Big|_{t=t_0} d\Omega - \iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^I)\Big|_{t=0} d\Omega - \int\limits_0^{t_0} \left[\iint\limits_{S_{h,\eta}} (pw_n^I + q^I v_n)\, dS\right] dt$$

$$= \int\limits_0^{t_0} \left[\iiint\limits_{\Omega_{h,\eta}} [(\overrightarrow{w}^I, \overrightarrow{F}) + q^I g]\, d\Omega\right] dt. \tag{112}$$

We pass to the limit as $\eta \to 0$ and evaluate

$$\lim_{\eta\to 0} \iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^I)\Big|_{t=t_0} d\Omega$$

and

$$\lim_{\eta\to 0} \int\limits_0^{t_0} \left[\iint\limits_{S_{h,\eta}} (pw_n^I + q^I v_n)\, dS\right] dt.$$

First of all, we note that the components of the vector $\overrightarrow{w}^I$ at $t = t_0$ take the values

$$w_x^I = \frac{\partial^2 \frac{1}{r}}{\partial x^2}, \quad w_y^I = \frac{\partial^2 \frac{1}{r}}{\partial x \partial y}, \quad w_z^I = \frac{\partial^2 \frac{1}{r}}{\partial x \partial z}. \tag{113}$$

Indeed, at $t = t_0$ we have

$$\frac{\partial^2 \Phi^{11}}{\partial t^2} = -\frac{x - x_0}{r^3}, \quad \frac{\partial^2 \Phi^{12}}{\partial t^2} = 0, \quad \frac{\partial \Phi^{11}}{\partial t} = 0, \quad \frac{\partial \Phi^{12}}{\partial t} = 0.$$

Using (95), we obtain our assertion. By (113), we obtain

$$\iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^I)\Big|_{t=t_0} d\Omega = \iiint\limits_{\Omega_{h,\eta}} \left(\overrightarrow{v}, \operatorname{grad} \frac{\partial \frac{1}{r}}{\partial x}\right) d\Omega = -\iint\limits_{S_{h,\eta}} v_n \frac{\partial \frac{1}{r}}{\partial x}\, dS,^{4} \tag{114}$$

where n is the inward normal to the surface $S_{h,\eta}$ bounding $\Omega_{h,\eta}$. The last integral can be written in the form

$$\iint\limits_{S_{h,\eta}} v_n \frac{\partial \frac{1}{r}}{\partial x}\, dS = -\int\limits_{-h}^{+h} \int\limits_0^{2\pi} \left(v_x \frac{x - x_0}{\varrho} + v_y \frac{y - y_0}{\varrho}\right) \frac{x - x_0}{r^3} \eta \Big|_{\varrho=\eta} dz\, d\varphi$$

$$- \int\limits_0^{2\pi} \int\limits_0^{\eta} v_z \Big|_{z=z_0+h} \frac{x - x_0}{r^3} \varrho\, d\varrho\, d\varphi + \int\limits_0^{2\pi} \int\limits_0^{\eta} v_z \Big|_{z=z_0-h} \frac{x - x_0}{r^3} \varrho\, d\varrho\, d\varphi.$$

---

[4] Further, S. L. Sobolev assumes that the function g is zero in (1). – *Ed.*

The limits of the last two summands in this formula are zero since moduli of each of these terms do not exceed the quantity

$$\max|v_z| \int\limits_0^{2\pi} \int\limits_0^{\eta} \frac{x - x_0}{r^3} \varrho \, d\varrho \, d\varphi = \max|v_z| \omega',$$

where ω' is the solid angle under which one, being at the point x_0, y_0, z_0, can observe the bottom of a sufficiently narrow cylinder (ω' and both integrals are as small as desired).

Thus,

$$\lim_{\eta \to 0} \iint\limits_{S_{h,\eta}} v_n \frac{\partial \frac{1}{r}}{\partial x} \, dS$$

$$= - \lim_{\eta \to 0} \int\limits_{-h}^{+h} \int\limits_0^{2\pi} \left(v_x \frac{(x - x_0)^2}{r^3} + v_y \frac{(y - y_0)(x - x_0)}{r^3} \right) dz \, d\varphi.$$

We can verify that

$$\lim_{\eta \to 0} \int\limits_{-h}^{+h} \int\limits_0^{2\pi} v_y \frac{(y - y_0)(x - x_0)}{r^3} \, dz \, d\varphi = 0.$$

Indeed,

$$\int\limits_{-h}^{+h} \int\limits_0^{2\pi} v_y|_{x_0, y_0, z_0} \frac{(y - y_0)(x - x_0)}{r^3} \, dz \, d\varphi = 0,$$

since the function $\dfrac{(y - y_0)(x - x_0)}{r^3}$ is odd. At the same time, we have

$$\lim_{\eta \to 0} \int\limits_{-h}^{+h} \int\limits_0^{2\pi} (v_y - v_y|_{x_0, y_0, z_0}) \frac{(y - y_0)(x - x_0)}{r^3} \, dz \, d\varphi = 0,$$

since the integrand

$$\frac{\eta^2 \sin\varphi \cos\varphi}{[\eta^2 + z^2]^{\frac{3}{2}}}$$

tends to zero everywhere except for a neighborhood of $z = z_0$. The integral over this neighborhood does not exceed the quantity

$$\max|v_y - v_y|_{x_0, y_0, z_0}| 2\pi \int\limits_{-\infty}^{+\infty} \frac{dz}{\sqrt{\eta^2 + z^2}} = 2\pi \max|v_y - v_y^{(0)}|$$

and is as small as desired.

Likewise, we prove that

$$\lim_{\eta\to 0}\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi} v_x\frac{(x-x_0)^2}{r^3}\,dz\,d\varphi = v_x(x_0,y_0,z_0,t_0)\lim_{\eta\to 0}\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi}\frac{(x-x_0)^2}{r^3}\,dz\,d\varphi$$

$$= v_x(x_0,y_0,z_0,t_0)\lim_{\eta\to 0}\eta^2\int\limits_{-h}^{+h}\frac{dz}{(\sqrt{z^2+\eta^2})^3}\int\limits_{0}^{2\pi}\cos^2\varphi\,d\varphi$$

$$= v_x(x_0,y_0,z_0,t_0)2\pi\lim_{\eta\to 0}\int\limits_{0}^{\frac{h}{\eta}}\frac{d\zeta}{(\sqrt{\zeta^2+1})^3}$$

$$= 2\pi v_x(x_0,y_0,z_0,t_0)\int\limits_{0}^{\infty}\frac{d\zeta}{(\sqrt{1+\zeta^2})^3},$$

where $z=\eta\zeta$. We have

$$\int\limits_{0}^{\infty}\frac{d\zeta}{(\sqrt{1+\zeta^2})^3} = \int\limits_{0}^{\infty}\frac{d}{d\zeta}\left(\frac{\zeta}{\sqrt{1+\zeta^2}}\right)d\zeta = \left.\frac{\zeta}{\sqrt{1+\zeta^2}}\right|_{0}^{\infty} = 1.$$

Finally,

$$\lim_{\eta\to 0}\iiint\limits_{\Omega_{h,\eta}}(\overrightarrow{v},\overrightarrow{w}^I)\Big|_{t=t_0}\,d\Omega = 2\pi v_x(x_0,y_0,z_0,t_0). \tag{115}$$

We note that it was important in our process that we dealt with a cylinder and evaluated the limit as the radius tends to zero. Taking other surfaces or using other methods of limit passage, we could obtain quite different results.

11 Continuation of Computation of v_x and v_y

Let us compute the limit

$$\lim_{\eta\to 0}\int\limits_{0}^{t_0}\left[\iint\limits_{S_{h,\eta}}(pw_n^I + q^I v_n)\,dS\right]dt.$$

We set

$$\begin{aligned}
&\iint\limits_{S_{h,\eta}} pw_n^{ijk}\,dS = K^{ijk}, &&\iint\limits_{S_{h,\eta}} q^{ijk}v_x\cos nx\,dS = L_x^{ijk},\\
&\iint\limits_{S_{h,\eta}} q^{ijk}v_y\cos ny\,dS = L_y^{ijk}, &&\iint\limits_{S_{h,\eta}} q^{ijk}v_z\cos nz\,dS = L_z^{ijk}.
\end{aligned}\tag{116}$$

Introducing the notation

$$\lim_{\eta\to 0} K^{ijk} = k^{ijk}, \qquad \lim_{\eta\to 0} L_y^{ijk} = l_y^{ijk},$$
$$\lim_{\eta\to 0} L_x^{ijk} = l_x^{ijk}, \qquad \lim_{\eta\to 0} L_z^{ijk} = l_z^{ijk}, \tag{117}$$

from (95) we obtain

$$\lim_{\eta\to 0} \int_0^{t_0} \left[\iint_{S_{h,\eta}} (pw_n^I + q^I v_n)\, dS \right] dt$$

$$= \int_0^{t_0} [(k^{111} + k^{112} - k^{121} - k^{122})\, dt + \int_0^{t_0} [(l_x^{111} + l_x^{113} - l_x^{121} - l_x^{123})$$
$$+ (l_y^{111} + l_y^{113} - l_y^{121} - l_y^{123}) + (l_z^{111} + l_z^{113} - l_z^{121} - l_z^{123})]\, dt. \tag{118}$$

We need to compute k^{ijk}, l_x^{ijk}, l_y^{ijk}, l_z^{ijk}. We begin with l^{ijk}. We note that

$$\lim_{\eta\to 0} L_x^{ijk} = \lim_{\eta\to 0} \iint_{S_{h,\eta}} v_x|_{x_0,y_0,z_0} q^{ijk} \cos nx\, dS$$

and

$$\lim_{\eta\to 0} L_y^{ijk} = \lim_{\eta\to 0} \iint_{S_{h,\eta}} v_y|_{x_0,y_0,z_0} q^{ijk} \cos ny\, dS,$$
$$\lim_{\eta\to 0} L_z^{ijk} = \lim_{\eta\to 0} \iint_{S_{h,\eta}} v_z|_{x_0,y_0,z_0} q^{ijk} \cos nz\, dS. \tag{119}$$

Indeed, for example,

$$\lim_{\eta\to 0} \int_{-h}^{+h} \int_0^{2\pi} (v_x - v_x|_{x_0,y_0,z_0}) q^{ijk} \varrho \cos nx \Big|_{\varrho=\eta} dz\, d\varphi = 0$$

because everywhere, except for a neighborhood of the point $z = z_0$, the integrand $q^{ijk}\varrho$ tends to zero and, in the neighborhood of this point, the integral is as small as desired because $v_x - v_x|_{x_0,y_0,z_0}$ is small.

The quantities q^{111} and q^{113} have the factor $x - x_0$, and quantities q^{121} and q^{123} have the factor $y - y_0$. The remaining factors are independent of the angle φ in cylindrical coordinates. Whence it follows that among the integrals l^{ijk}, only the integrals l_x^{111}, l_x^{113}, l_y^{121}, l_y^{123} do not vanish and

$$l_x^{121} = l_z^{121} = l_x^{123} = l_z^{123} = l_y^{111} = l_y^{113} = l_z^{111} = l_z^{113} = 0. \tag{120}$$

Obviously, we have

$$\begin{aligned} l_y^{123} &= -v_y(x_0, y_0, z_0, t) \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{\partial \Phi^{12}}{\partial t} \eta \cos ny \, dz \, d\varphi \\ &= -v_y(x_0, y_0, z_0, t) a_1(\tau), \end{aligned}$$

$$\begin{aligned} l_y^{121} &= -v_y(x_0, y_0, z_0, t) \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{\partial^3 \Phi^{12}}{\partial t^3} \eta \cos ny \, dz \, d\varphi \\ &= -v_y(x_0, y_0, z_0, t) a_2(\tau), \end{aligned}$$

$$\begin{aligned} l_x^{113} &= -v_x(x_0, y_0, z_0, t) \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{\partial \Phi^{11}}{\partial t} \eta \cos nx \, dz \, d\varphi \\ &= -v_x(x_0, y_0, z_0, t) a_3(\tau), \end{aligned}$$

$$\begin{aligned} l_x^{111} &= -v_x(x_0, y_0, z_0, t) \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{\partial^3 \Phi^{11}}{\partial t^3} \eta \cos nx \, dz \, d\varphi \\ &= -v_x(x_0, y_0, z_0, t) a_4(\tau), \end{aligned}$$

where $\tau = t - t_0$. It is clear that

$$a_2(\tau) = a_1''(\tau) \quad \text{and} \quad a_4(\tau) = a_3''(\tau).$$

Differentiating $a_1(\tau)$ with respect to τ and replacing the variables x and y, which does not change the value of the integral, we obtain

$$\begin{aligned} a_3(\tau) &= a_1'(\tau), \\ a_2(\tau) &= a_1''(\tau), \\ a_4(\tau) &= a_2'(\tau) = a_1'''(\tau). \end{aligned} \tag{121}$$

Thus, it is necessary to compute

$$a_1(\tau) = \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{\partial \Phi^{12}}{\partial t} \eta \cos ny \, dz \, d\varphi$$

$$= \lim_{\eta\to 0} \int\limits_{-h}^{+h} \int\limits_{0}^{2\pi} \frac{(y - y_0)^2}{\varrho^2 r} \frac{\varrho(t - t_0)}{r} J_0' \left(\frac{\varrho\tau}{r} \right) \Bigg|_{\varrho=\eta} dz \, d\varphi. \tag{122}$$

Making the change of integration variables in the last integral and setting $\dfrac{\varrho}{r} = \zeta$, we obtain[5]

$$\zeta^2 = \frac{\varrho^2}{\varrho^2 + z^2}, \quad \frac{z^2}{\varrho^2} = \frac{1}{\zeta^2} - 1, \quad \frac{dz}{\varrho} = -\frac{d\zeta}{\zeta^2\sqrt{1-\zeta^2}},$$
$$a_1(\tau) = 2\pi\tau \int_0^1 \frac{J_0'(\zeta\tau)}{\sqrt{1-\zeta^2}}\, d\zeta. \tag{123}$$

By (110), we have

$$a_1(\tau) = 2\pi(\cos\tau - 1). \tag{124}$$

Then, we have

$$\begin{aligned} l_y^{123} &= -\, v_y(x_0, y_0, z_0, t) 2\pi(\cos\tau - 1), \\ l_y^{121} &= \quad v_y(x_0, y_0, z_0, t) 2\pi\cos\tau, \\ l_x^{113} &= \quad v_x(x_0, y_0, z_0, t) 2\pi\sin\tau, \\ l_x^{111} &= -\, v_y(x_0, y_0, z_0, t) 2\pi\sin\tau, \end{aligned} \tag{125}$$

$$l_x^{111} + l_x^{113} - l_y^{121} - l_y^{123} = -2\pi v_y(x_0, y_0, z_0, t). \tag{126}$$

Let us compute the integrals k^{ijk}. For this purpose, we slightly transform them. We set

$$K^{ijk} = \iint\limits_{S_{h,\eta}} \Bigg[p\Big|_{x_0,y_0,z_0,t} + (x - x_0)\frac{\partial p}{\partial x}\Big|_{x_0,y_0,z_0,t} + (y - y_0)\frac{\partial p}{\partial y}\Big|_{x_0,y_0,z_0,t}$$

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[5] The integrals

$$A_y = \int_{-h}^{+h}\int_0^{2\pi} \frac{(y-y_0)^2}{\varrho^2 r}\, \frac{\varrho(t-t_0)}{r} J_0'\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta} dz\, d\varphi,$$

$$A_x = \int_{-h}^{+h}\int_0^{2\pi} \frac{(x-x_0)^2}{\varrho^2 r}\, \frac{\varrho(t-t_0)}{r} J_0'\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta} dz\, d\varphi$$

coincide and $\varrho^2 = (x-x_0)^2 + (y-y_0)^2$. Therefore,

$$A_y = 2\pi\tau \int_0^h \frac{\zeta^2}{\varrho} J_0'(\tau\zeta)\Big|_{\varrho=\eta} dz = 2\pi\tau \int_{\eta/\sqrt{\eta^2+h^2}}^1 \frac{J_0'(\tau\zeta)}{\sqrt{1-\zeta^2}}\, d\zeta. \text{ – } \mathit{Ed.}$$

$$+(z-z_0)\frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t}\Bigg] w_n^{ijk}dS + \iint\limits_{S_{h,\eta}} \Bigg[(p-p_0) - \frac{\partial p}{\partial x}\bigg|_{x_0,y_0,z_0,t}(x-x_0)$$

$$-\frac{\partial p}{\partial y}\bigg|_{x_0,y_0,z_0,t}(y-y_0) - \frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t}(z-z_0)\Bigg] w_n^{ijk}dS. \qquad (127)$$

We prove that the second term of this formula tends to zero as $\eta \to 0$.

Indeed, in a neighborhood of the point x_0, y_0, z_0 the integrand does not exceed $r\delta(\eta)$, where $\delta(\eta)$ tends to zero as $\eta \to 0$. On the other hand, $(w_n^{ijk}\varrho)$ tends to zero everywhere outside this neighborhood and does not exceed $\dfrac{A}{r}$ anywhere.

Consequently, the integral over the neighborhood of x_0, y_0, z_0 does not exceed the quantity

$$2\pi \int \delta(\eta)\frac{dz}{r} = 4\pi\delta(\eta),$$

and it is as small as desired over the remaining part. Therefore, it remains to compute the first term in (127).

It is easy to verify that

$$\iint\limits_{S_{h,\eta}} p\bigg|_{x_0,y_0,z_0,t} w_n^{ijk}dS = p\bigg|_{x_0,y_0,z_0,t}\iint\limits_{S_{h,\eta}} w_n^{ijk}dS = 0. \qquad (128)$$

This follows from the fact that all w_n^{ijk} are odd functions in one of the variables $x-x_0$, $y-y_0$, $z-z_0$ and have a common odd exponent with respect to all these variables.

Let

$$\begin{aligned} K_x^{ijk} &= \iint\limits_{S_{h,\eta}} (x-x_0)\frac{\partial p}{\partial x}\bigg|_{x_0,y_0,z_0,t} w_n^{ijk}dS, \\ K_y^{ijk} &= \iint\limits_{S_{h,\eta}} (y-y_0)\frac{\partial p}{\partial y}\bigg|_{x_0,y_0,z_0,t} w_n^{ijk}dS, \qquad (129) \\ K_z^{ijk} &= \iint\limits_{S_{h,\eta}} (z-z_0)\frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t} w_n^{ijk}dS \end{aligned}$$

and

$$\lim_{\eta\to 0} K_x^{ijk} = k_x^{ijk}, \quad \lim_{\eta\to 0} K_y^{ijk} = k_y^{ijk}, \quad \lim_{\eta\to 0} K_z^{ijk} = k_z^{ijk}.$$

Then, we have

$$k_x^{ijk} = \frac{\partial p}{\partial x}\bigg|_{x_0,y_0,z_0,t} b_x^{ijk}, \quad k_y^{ijk} = \frac{\partial p}{\partial y}\bigg|_{x_0,y_0,z_0,t} b_y^{ijk}, \quad k_z^{ijk} = \frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t} b_z^{ijk},$$

where

$$b_x^{ijk} = \lim_{\eta\to 0} \iint\limits_{S_{h,\eta}} (x - x_0) w_n^{ijk} dS,$$

$$b_y^{ijk} = \lim_{\eta\to 0} \iint\limits_{S_{h,\eta}} (y - y_0) w_n^{ijk} dS,$$

$$b_z^{ijk} = \lim_{\eta\to 0} \iint\limits_{S_{h,\eta}} (z - z_0) w_n^{ijk} dS.$$

On the areas $z = \pm h$ of the surface of the cylinder $S_{h,\eta}$ the limits of the integrals w_z^{ijk} are equal to zero.

Hence we need to compute the integrals

$$\int_{-h}^{+h}\int_0^{2\pi} (x - x_0) \left[\frac{(x - x_0)}{\varrho} w_x^{ijk} + \frac{(y - y_0)}{\varrho} w_y^{ijk} \right] \varrho\, dz\, d\varphi.$$

We have

$$b_x^{111} = \lim_{\eta\to 0} \int_{-h}^{+h}\int_0^{2\pi} \left[(x - x_0)^2 \frac{\partial^3 \Phi^{11}}{\partial x \partial t^2} + (y - y_0)(x - x_0) \frac{\partial^3 \Phi^{11}}{\partial y \partial t^2} \right] dz\, d\varphi,$$

$$b_x^{112} = \lim_{\eta\to 0} \int_{-h}^{+h}\int_0^{2\pi} \left[(x - x_0)^2 \frac{\partial^2 \Phi^{11}}{\partial y \partial t} - (y - y_0)(x - x_0) \frac{\partial^2 \Phi^{11}}{\partial x \partial t} \right] dz\, d\varphi,$$

$$b_x^{121} = \lim_{\eta\to 0} \int_{-h}^{+h}\int_0^{2\pi} \left[(x - x_0)^2 \frac{\partial^3 \Phi^{12}}{\partial x \partial t^2} + (y - y_0)(x - x_0) \frac{\partial^3 \Phi^{12}}{\partial y \partial t^2} \right] dz\, d\varphi,$$

$$b_x^{122} = \lim_{\eta\to 0} \int_{-h}^{+h}\int_0^{2\pi} \left[(x - x_0)^2 \frac{\partial^2 \Phi^{12}}{\partial y \partial t} - (y - y_0)(x - x_0) \frac{\partial^2 \Phi^{12}}{\partial x \partial t} \right] dz\, d\varphi.$$

Consequently,

$$b_x^{111} = \lim_{\eta\to 0} \Bigg(\int_{-h}^{+h}\int_0^{2\pi} \left[\frac{3(x - x_0)^2 \varrho^2}{r^5} - \frac{(x - x_0)^2}{r^3} \right] J_0\left(\frac{\varrho\tau}{r}\right)\Bigg|_{\varrho=\eta} dz\, d\varphi$$

$$+ \int_{-h}^{+h}\int_0^{2\pi} \frac{(x - x_0)^2 (z - z_0)^2}{r^5} \left(\frac{\varrho\tau}{r}\right)^2 J_0\left(\frac{\varrho\tau}{r}\right)\Bigg|_{\varrho=\eta} dz\, d\varphi$$

$$+\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi}\frac{1}{r^5}[4\varrho^2(x-x_0)^2-2r^2(x-x_0)^2]\frac{\varrho\tau}{r}J_0'\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta}dz\,d\varphi\Bigg).$$

Applying the transformation mentioned above, we obtain[6]

$$b_x^{111}=2\pi\Bigg[\int\limits_0^1\frac{-\zeta+3\zeta^3}{\sqrt{1-\zeta^2}}J_0(\zeta\tau)\,d\zeta+\tau^2\int\limits_0^1\frac{\zeta^3-\zeta^5}{\sqrt{1-\zeta^2}}J_0(\zeta\tau)\,d\zeta$$

$$+\tau\int\limits_0^1\frac{-2\zeta^2+4\zeta^4}{\sqrt{1-\zeta^2}}J_0'(\zeta\tau)\,d\zeta\Bigg]$$

and, in view of (110),

$$b_x^{111}=2\pi\cos\tau. \tag{130}$$

Further, $b_x^{112}=0$ since the integrand is an odd function of $y-y_0$. In the same way, we have $b_x^{121}=0$. Let us compute b_x^{122}:

$$b_x^{122}=\lim_{\eta\to0}\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi}\frac{(x-x_0)^2}{\varrho^2 r}\left(\frac{\varrho\tau}{r}\right)J_0'\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta}dz\,d\varphi$$

$$=\lim_{\eta\to0}\tau\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi}\frac{(x-x_0)^2}{\varrho r^2}J_0'\left(\frac{\varrho\tau}{r}\right)dz\,d\varphi$$

$$=2\pi\tau\int\limits_0^1\frac{J_0'(\zeta\tau)\,d\zeta}{\sqrt{1-\zeta^2}}=2\pi(\cos\tau-1). \tag{131}$$

Thus,

$$k_x^{111}+k_x^{112}-k_x^{121}-k_x^{122}=2\pi\frac{\partial p}{\partial x}\Big|_{x_0,y_0,z_0,t}. \tag{132}$$

All k_z^{ijk} also vanish. This fact is true because in the computation of the corresponding b_z^{ijk} the integrand is an odd function of $(z-z_0)$.

In the integrals b_y^{111} and b_y^{122}, the integrand is an odd function of $y-y_0$. Hence these integrals also vanish.

Let us compute the integrals b_y^{112} and b_y^{121}. For b_y^{112} we have

$$b_y^{112}=\lim_{\eta\to0}\int\limits_{-h}^{+h}\int\limits_{0}^{2\pi}[(x-x_0)(y-y_0)w_x^{112}+(y-y_0)^2w_y^{112}]\,dz\,d\varphi$$

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[6] See p. 317. – *Ed.*

$$= \lim_{\eta\to 0}\int_{-h}^{+h}\int_{0}^{2\pi}\left[\frac{3(x-x_0)^2(y-y_0)^2}{\varrho r^4} - \frac{[3(x-x_0)^2-r^2](y-y_0)^2}{\varrho r^4}\right]$$

$$\times \frac{\varrho\tau}{r} J_0\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta} dz\, d\varphi = 2\pi\tau\int_0^1 \frac{\zeta J_0(\zeta\tau)\, d\zeta}{\sqrt{1-\zeta^2}} = 2\pi\sin\tau. \tag{133}$$

Further,

$$b_y^{121} = \lim_{\eta\to 0}\int_{-h}^{+h}\int_{0}^{2\pi}[(x-x_0)(y-y_0)w_x^{121} + (y-y_0)^2 w_y^{121}]\, dz\, d\varphi$$

$$= \lim_{\eta\to 0}\int_{-h}^{+h}\int_{0}^{2\pi}\Bigg\{\left[\frac{3(x-x_0)^2(y-y_0)^2}{\varrho r^4} + \frac{[3(y-y_0)^2-r^2](y-y_0)^2}{\varrho r^4}\right]\frac{\varrho\tau}{r}J_0\left(\frac{\varrho\tau}{r}\right)$$

$$-\frac{z^2(y-y_0)^2}{\varrho r^4}\left(\frac{\varrho\tau}{r}\right)^2 J_0'\left(\frac{\varrho\tau}{r}\right)\Bigg\}\Bigg|_{\varrho=\eta} dz\, d\varphi$$

$$= \lim_{\eta\to 0}\int_{-h}^{+h}\int_{0}^{2\pi}\Bigg\{\left[\frac{3\varrho(y-y_0)^2}{r^4} - \frac{(y-y_0)^2}{\varrho r^2}\right]\frac{\varrho\tau}{r}J_0\left(\frac{\varrho\tau}{r}\right)$$

$$-\frac{z^2(y-y_0)^2}{\varrho r^4}\left(\frac{\varrho\tau}{r}\right)^2 J_0'\left(\frac{\varrho\tau}{r}\right)\Bigg\}\Bigg|_{\varrho=\eta} dz\, d\varphi$$

$$= \lim_{\eta\to 0} 2\pi\int_0^h \tau\left(\frac{3\varrho^4}{r^5} - \frac{\varrho^2}{r^3}\right) J_0\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta} dz$$

$$- \lim_{\eta\to 0} 2\pi\int_0^h \tau^2\left(\frac{\varrho^3}{r^4} - \frac{\varrho^5}{r^6}\right) J_0'\left(\frac{\varrho\tau}{r}\right)\Big|_{\varrho=\eta} dz$$

$$= 2\pi\left\{\tau\int_0^1 \frac{3\zeta^3-\zeta}{\sqrt{1-\zeta^2}} J_0(\zeta t)\, d\zeta + \tau^2\int_0^1 \frac{\zeta^4-\zeta^2}{\sqrt{1-\zeta^2}} J_0'(\zeta t)\, d\zeta\right\},$$

or, by (110),

$$b_y^{121} = 2\pi\sin\tau, \tag{134}$$

i.e.,

$$k_y^{112} - k_y^{121} = 0. \tag{135}$$

Combining the above arguments, we have

$$k^{111} + k^{112} - k^{121} - k^{122} = 2\pi\frac{\partial p}{\partial x}\Big|_{x_0,y_0,z_0,t}.$$

Returning to (112) and introducing the notation

$$\lim_{\eta\to 0}\iiint\limits_{\Omega_{h,\eta}} F\,d\Omega = \text{P. V. c.}\iiint\limits_{\Omega} F\,d\Omega,$$

we obtain

$$2\pi v_x(x_0,y_0,z_0,t_0) - 2\pi\int\limits_0^{t_0}\left[-v_y+\frac{\partial p}{\partial x}\right]\Big|_{x_0,y_0,z_0,t}\,dt$$

$$= \text{P. V. c.}\iiint\limits_{\Omega}(\vec{v},\vec{w}^{I})|_{t=0}\,d\Omega$$

$$+\int\limits_0^{t_0}\text{P. V. c.}\iiint\limits_{\Omega}\left[(\vec{w}^{I},\vec{F})+q^{I}g\right]\,d\Omega\,dt. \tag{136}$$

Using the first equation in system (1), we see that

$$v_y - \frac{\partial p}{\partial x} = \frac{\partial v_x}{\partial t} - F_x.$$

Substituting this expression into (136) and making a simple transformation, we obtain

$$v_x(x_0,y_0,z_0,t_0) = \frac{1}{2}v_x(x_0,y_0,z_0,0) + \text{P. V. c.}\frac{1}{4\pi}\iiint\limits_{\Omega}(\vec{v},\vec{w}^{I})\Big|_{t=0}\,d\Omega$$

$$+\int\limits_0^{t_0}\left\{\frac{1}{2}F_x\Big|_{x_0,y_0,z_0,t} + \frac{1}{4\pi}\text{P. V. c.}\iiint\limits_{\Omega}\left[(\vec{w}^{I},\vec{F})+q^{I}g\right]\,d\Omega\right\}dt. \tag{137}$$

To compute the value of $v_y(x_0,y_0,z_0,t_0)$, we should apply the same arguments to $\vec{w}^{II}$. It is easy to see that the same result is obtained from (137) with y replaced by $-y$ and v_y replaced by $-v_y$. Interchanging the variables x and y, we obtain the required result. Finally, we have

$$v_y(x_0,y_0,z_0,t_0) = \frac{1}{2}v_y(x_0,y_0,z_0,0) + \text{P. V. c.}\frac{1}{4\pi}\iiint\limits_{\Omega}(\vec{v},\vec{w}^{II})|_{t=0}\,d\Omega$$

$$+\int\limits_0^{t_0}\left\{\frac{1}{2}F_y\Big|_{x_0,y_0,z_0,t} + \frac{1}{4\pi}\text{P. V. c.}\iiint\limits_{\Omega}\left[(\vec{w}^{II},\vec{F})+q^{II}g\right]\,d\Omega\right\}dt. \tag{138}$$

12 Computation of v_z

Let us compute v_z. For this purpose, we again apply (71) to the unknown solution and the function $\overrightarrow{w}^{III}$ in the same domain $\Omega_{h,\eta}$. We obtain

$$\iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^{III})|_{t=t_0}\, d\Omega - \iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^{III})|_{t=0}\, d\Omega$$

$$- \int\limits_0^{t_0} \Big[\iint\limits_{S_{h,\eta}} (p w_n^{III} + q^{III} v_n) dS \Big]\, dt$$

$$= \int\limits_0^{t_0} \Big\{ \iiint\limits_{\Omega_{h,\eta}} [(\overrightarrow{w}^{III}, \overrightarrow{F}) + q^{III} g]\, d\Omega \Big\}\, dt. \tag{139}$$

We pass to the limit as $h \to 0$ and find

$$\lim_{h\to 0} \iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^{III})|_{t=t_0}\, d\Omega \tag{140}$$

and

$$\lim_{h\to 0} \int\limits_0^{t_0} \Big[\iint\limits_{S_{h,\eta}} (p w_n^{III} + q^{III} v_n)\, dS \Big]\, dt. \tag{141}$$

We note that for $t = t_0$ the components of the vector $\overrightarrow{w}^{III}$ take the values

$$w_x^{III} = \frac{\partial^2 \frac{1}{r}}{\partial x \partial z}, \quad w_y^{III} = \frac{\partial^2 \frac{1}{r}}{\partial y \partial z}, \quad w_z^{III} = \frac{\partial^2 \frac{1}{r}}{\partial z^2}. \tag{142}$$

Using (142), we obtain

$$\iiint\limits_{\Omega_{h,\eta}} (\overrightarrow{v}, \overrightarrow{w}^{III})|_{t=t_0}\, d\Omega = \iiint\limits_{\Omega_{h,\eta}} \left(\overrightarrow{v}, \operatorname{grad} \frac{\partial \frac{1}{r}}{\partial z} \right) d\Omega = - \iint\limits_{S_{h,\eta}} v_n \frac{\partial \frac{1}{r}}{\partial z}\, dS,$$

where n is the inward normal to the surface $S_{h,\eta}$ bounding $\Omega_{h,\eta}$. The last integral can be written as

$$\iint\limits_{S_{h,\eta}} v_n \frac{\partial \frac{1}{r}}{\partial z}\, dS = - \int\limits_{-h}^{h} \int\limits_0^{2\pi} \left(v_x \frac{x - x_0}{\varrho} + v_y \frac{y - y_0}{\varrho} \right) \frac{z}{r^3} \eta \Big|_{\varrho=\eta} dz\, d\varphi$$

$$- \int\limits_0^{2\pi} \int\limits_0^{\eta} v_z \Big|_{z=z_0+h} \frac{h}{r^3} \varrho\, d\varrho\, d\varphi - \int\limits_0^{2\pi} \int\limits_0^{\eta} v_z \Big|_{z=z_0-h} \frac{h}{r^3} \varrho\, d\varrho\, d\varphi.$$

The first term in this formula tends to zero because the integrand is bounded as $h \to 0$, and the integration domain is eliminated. The second and third terms can be represented in the form

$$\int_0^{2\pi}\int_0^{\eta} v_z\Big|_{x_0,y_0,z_0,t} \frac{h}{r^3}\varrho\,d\varrho\,d\varphi + \int_0^{2\pi}\int_0^{\eta} v_z\Big|_{x_0,y_0,z_0,t} \frac{h}{r^3}\varrho\,d\varrho\,d\varphi$$

$$+\int_0^{2\pi}\int_0^{\eta}\left(v_z\Big|_{x,y,z_0+h,t} - v_z\Big|_{x_0,y_0,z_0,t}\right)\frac{h}{r^3}\varrho\,d\varrho\,d\varphi$$

$$+\int_0^{2\pi}\int_0^{\eta}\left(v_z\Big|_{x,y,z_0-h,t} - v_z\Big|_{x_0,y_0,z_0,t}\right)\frac{h}{r^3}\varrho\,d\varrho\,d\varphi.$$

In the last two integrals, the integrand is as small as desired because the function v_z is continuous. The first two integrals are $2v_z(x_0,y_0,z_0,t_0)\omega$, where ω is the solid angle which would be formed by observing the upper and lower bases of the cylinder from the origin. Whence it follows that

$$\lim_{h\to 0}\iint\limits_{S_{h,\eta}} v_n \frac{\partial\frac{1}{r}}{\partial z}\,dS = -4\pi v_z(x_0,y_0,z_0,t_0). \tag{143}$$

In this case, the value of the limit is twice larger than in the case when we compute the limit with respect to η.

Let us compute the limit of integral (141). We use the above notation. As above, we have

$$\lim_{h\to 0}\int_0^{t_0}\left[\iint\limits_{S_{h,\eta}}(pw_n^{III} + q^{III}v_n)\,dS\right]dt = \int_0^{t_0}(k_z^{31} + k_z^{33} + l_z^{31} + l_z^{33})\,dt, \tag{144}$$

where

$$l_z^{3j} = \lim_{h\to 0}\iint\limits_{S_{h,\eta}} q^{3j}v_z\cos nz\,dS, \quad k_z^{3j} = \lim_{h\to 0}\iint\limits_{S_{h,\eta}} pw_z^{3j}\cos nz\,dS.$$

The limits of the integrals l_x^{ij} and l_y^{ij} are zeros. Indeed, on the upper and lower bases v_x and v_y do not participate, and the integrals over the lateral surfaces tend to zero. Thus, we need to compute l^{31}, l^{33}, k^{31}, and k^{33}.

For l^{33} we have[7]

[7] S. L. Sobolev uses the definition of the function Φ_{III} and the property of the Bessel function

$$-\xi J_0(\xi) \equiv \xi J_0''(\xi) + J_0'(\xi). \text{ – } \textit{Ed.}$$

$$l^{33} = \lim_{h\to 0} \int_0^{2\pi} \int_0^{\eta} [v_z|_{z_0+h} - v_z|_{z_0-h}] \frac{\partial \Phi_{III}}{\partial t} \varrho \, d\varrho \, d\varphi$$

$$= -4\pi \lim_{h\to 0} \int_0^{\eta} v_z|_{x_0,y_0,z_0,t} \left[\frac{|z-z_0|}{\varrho r^2} \frac{\varrho(t-t_0)}{r} J_0\left(\frac{\varrho(t-t_0)}{r}\right)\right] \varrho \, d\varrho$$

$$= -4\pi v_z|_{x_0,y_0,z_0,t} \lim_{\eta\to 0} h\tau \int_0^{h} \frac{\varrho}{r^3} J_0\left(\frac{\varrho\tau}{r}\right) d\varrho.$$

We make the change of variables by setting $\dfrac{\varrho}{r} = \zeta$,

$$\zeta = \frac{\varrho}{\sqrt{\varrho^2+h^2}}, \quad \varrho^2 = \frac{h^2\zeta^2}{1-\zeta^2}, \quad \varrho = \frac{h\zeta}{\sqrt{1-\zeta^2}}, \quad r = \frac{h}{\sqrt{1-\zeta^2}},$$

$$\frac{h\varrho \, d\varrho}{r^3} = \frac{\zeta \, d\zeta}{\sqrt{1-\zeta^2}}.$$

Then,

$$l^{33} = -4\pi v_z(x_0,y_0,z_0,t)\tau \int_0^1 \frac{\zeta J_0(\zeta\tau)\, d\zeta}{\sqrt{1-\zeta^2}} = -4\pi v_z(x_0,y_0,z_0,t)\sin\tau. \quad (145)$$

Obviously,

$$l^{31} = \lim_{h\to 0} \int_0^{2\pi} \int_0^{\eta} [v_z|_{z_0+h} - v_z|_{z_0-h}] \frac{\partial^3 \Phi_{III}}{\partial t^3} \varrho \, d\varrho \, d\varphi$$

$$= \frac{d^2 l^{33}}{d\tau^2} = 4\pi v_z(x_0,y_0,z_0,t)\sin\tau \quad (146)$$

and

$$l^{31} + l^{33} = 0. \quad (147)$$

To complete our calculations it remains to compute k^{31} and k^{33}.
We set

$$k^{33} = \lim_{h\to 0} \iint_{S_{h,\eta}} \left[p\Big|_{x_0,y_0,z_0,t} + (x-x_0)\frac{\partial p}{\partial x}\Big|_{x_0,y_0,z_0,t} + (y-y_0)\frac{\partial p}{\partial y}\Big|_{x_0,y_0,z_0,t}\right.$$

$$\left. +(z-z_0)\frac{\partial p}{\partial z}\Big|_{x_0,y_0,z_0,t}\right] w_n \, dS + \iint_{S_{h,\eta}} \left[p - p\Big|_{x_0,y_0,z_0,t} - (x-x_0)\frac{\partial p}{\partial x}\Big|_{x_0,y_0,z_0,t}\right.$$

$$-(y-y_0)\frac{\partial p}{\partial y}\bigg|_{x_0,y_0,z_0,t} - (z-z_0)\frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t}\Bigg] w_n\,dS.$$

The limit of the second integral is zero since the integrand contains a factor that does not exceed $r\delta(h)$ in a neighborhood of the point x_0, y_0, z_0, where $\delta(h)\to 0$, and the values of w_n at $z-z_0=h$ and $z-z_0=-h$ differ by only the sign.

In the first integral, only the following term can have a nonzero limit,

$$\iint\limits_{S_{h,\eta}} (z-z_0)\frac{\partial p}{\partial z} w_n\,dS.$$

The remaining terms vanish since the values of w_n at $z=\pm h$ differ only by sign and the integrals over the lateral surfaces are equal to zero because this function multiplied by $(x-x_0)$ and $(y-y_0)$ is odd.

Let us compute

$$k_z^{33} = \lim_{h\to 0}\iint\limits_{S_{h,\eta}} (z-z_0)\frac{\partial p}{\partial z} w_n^{33}\,dS$$

and

$$k_z^{31} = \lim_{h\to 0}\iint\limits_{S_{h,\eta}} (z-z_0)\frac{\partial p}{\partial z} w_n^{31}\,dS.$$

After a simple transformation, we obtain

$$k_z^{33} = 4\pi\frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t} c(\tau),$$

where

$$c(\tau) = \lim_{h\to 0}\int\limits_0^\eta \left[\frac{h}{r^3}\frac{\varrho\tau}{r}J_0'\left(\frac{\varrho\tau}{r}\right) + \frac{h^3}{\varrho^2 r^3}\left(\frac{\varrho\tau}{r}\right)^2 J_0\left(\frac{\varrho\tau}{r}\right)\right]\varrho\,d\varrho$$

$$= \tau\int\limits_0^1 \frac{\zeta^2 J_0'(\zeta\tau)\,d\zeta}{\sqrt{1-\zeta^2}} + \tau^2\int\limits_0^1 \frac{(\zeta-\zeta^3)J_0(\zeta\tau)\,d\zeta}{\sqrt{1-\zeta^2}}.$$

By (110), we find $k_z^{33}=0$.

Further,

$$k_z^{31} = 4\pi\frac{\partial p}{\partial z}\bigg|_{x_0,y_0,z_0,t}\frac{d^2 c(\tau)}{d\tau^2} = 0.$$

Consequently,

$$\lim_{h\to 0}\int\limits_0^t \Bigg[\iint\limits_{S_{h,\eta}} (pw_n^{III} + q^{III}v_n)\,dS\Bigg]\,dt = 0. \tag{148}$$

Returning to (139), we introduce the notation

$$\lim_{h\to 0}\iiint\limits_{\Omega_{h,\eta}} \vec{F}\,d\Omega = \text{P. V. d.}\iiint\limits_{\Omega} \vec{F}\,d\Omega. \tag{149}$$

Passing to the limit in (139), we obtain

$$4\pi v_z(x_0,y_0,z_0,t_0) - \text{P. V. d.}\iiint\limits_{\Omega} (\vec{v},\vec{w}^{III})\Big|_{t=0}\,d\Omega$$

$$= \int\limits_0^t \left\{\text{P. V. d.}\iiint\limits_{\Omega} [(\vec{w}^{III},\vec{F}) + q^{III}g]\,d\Omega\right\}dt$$

or

$$v_z(x_0,y_0,z_0,t_0) = \frac{1}{4\pi}\text{P. V. d.}\iiint\limits_{\Omega} (\vec{v},\vec{w}^{III})\Big|_{t=0}\,d\Omega$$

$$+\frac{1}{4\pi}\int\limits_0^t \left\{\text{P. V. d.}\iiint\limits_{\Omega} [(\vec{w}^{III},\vec{F}) + q^{III}g]\,d\Omega\right\}dt. \tag{150}$$

Our problem is completely solved.

13 The Cauchy Problem for a Fourth-Order Equation

The equation[8]

$$\frac{\partial^2 \Delta u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = \Phi \tag{151}$$

is of independent interest. We consider the Cauchy problem for this equation in an unbounded domain, i.e., we look for a solution of this equation such that

$$u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = u_1. \tag{152}$$

The problem has a number of interesting properties, and we indicate how this problem can be solved explicitly. For this purpose, we construct a formula similar to the Green formula. We have

$$w\left(\frac{\partial^2 \Delta u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2}\right) - u\left(\frac{\partial^2 \Delta w}{\partial t^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\equiv \frac{\partial}{\partial t}\left[w\frac{\partial \Delta u}{\partial t} - \Delta u\frac{\partial w}{\partial t}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial t^2} - u\frac{\partial^3 w}{\partial x\partial t^2}\right] + \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\frac{\partial^2 w}{\partial t^2} - u\frac{\partial^3 w}{\partial y\partial t^2}\right]$$

[8] Equation (151) is called the nonhomogeneous Sobolev equation. – *Ed.*

$$+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}\left(\frac{\partial^2 w}{\partial t^2}+w\right)-u\frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial t^2}+w\right)\right]. \tag{153}$$

Integrating this formula over the volume $\varOmega$ bounded by the surface S, we obtain

$$\int\limits_0^{t_0}\left\{\iiint\limits_{\varOmega}[wLu-uLw]\,d\varOmega\right\}dt=\iiint\limits_{\varOmega}\left[w\frac{\partial\Delta u}{\partial t}-\Delta u\frac{\partial w}{\partial t}\right]\,d\varOmega\Big|_{t=t_0}$$
$$-\iiint\limits_{\varOmega}\left[w\frac{\partial\Delta u}{\partial t}-\Delta u\frac{\partial w}{\partial t}\right]\,d\varOmega\Big|_{t=0}$$
$$+\int\limits_0^{t_0}\left\{\iint\limits_S\left[u\frac{\partial}{\partial n}\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial t^2}\frac{\partial u}{\partial n}+\left(u\frac{\partial w}{\partial z}-w\frac{\partial u}{\partial z}\right)\cos nz\right]\,dS\right\}dt. \tag{154}$$

Formula (154) was derived under the assumption that the domain $\varOmega$ is bounded. This formula is also valid if $\varOmega$ contains infinity, if, for example, the functions u, w, $\dfrac{\partial^2 w}{\partial t^2}$ decay at infinity as $\dfrac{1}{R}$, the first-order derivatives with respect to coordinates as $\dfrac{1}{R^2}$, the second-order derivatives with respect to coordinates as $\dfrac{1}{R^3}$.

This formula can be proved by the simple limit passage.

Let u be the required solution of equation (151). We set

$$w=\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right) \tag{155}$$

and apply (154) to the volume $\varOmega_{h,\eta}$ obtained by eliminating from the space the cylinder $S_{h,\eta}$ with radius η and height $2h$ around the point x_0, y_0, z_0. Passing to the limit as $\eta\to 0$, we obtain

$$\int\limits_0^{t_0}\left\{\iiint\limits_{\varOmega}\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\varPhi\,d\varOmega\right\}dt$$
$$=\iiint\limits_{\varOmega}\left\{\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\frac{\partial\Delta u}{\partial t}-\Delta u\frac{\partial}{\partial t}\left[\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\right]\right\}\Big|_{t=t_0}d\varOmega$$
$$-\iiint\limits_{\varOmega}\left\{\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\frac{\partial\Delta u}{\partial t}-\Delta u\frac{\partial}{\partial t}\left[\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\right]\right\}\Big|_{t=0}d\varOmega$$
$$+\lim_{\eta\to 0}\int\limits_0^{t_0}\Bigg[\iint\limits_{S_{h,\eta}}\Bigg\{u\frac{\partial}{\partial n}\frac{\partial^2}{\partial t^2}\left[\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\right]-\frac{\partial^2}{\partial t^2}\left[\frac{1}{\varrho}\varXi\left(\varrho\frac{t-t_0}{r}\right)\right]\frac{\partial u}{\partial n}$$

$$+\left[u\frac{\partial}{\partial z}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]-\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\frac{\partial u}{\partial z}\right]\cos nz\Big\}dS\Big]\,dt. \tag{156}$$

In this case, the limit is independent of the method of contracting the surface $S_{h,\eta}$ since all integrals exist not only in the sense of the principal value.

We compute each term on the right side of (156). Obviously, for $t=t_0$,

$$\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)=0,\quad \frac{\partial}{\partial t}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]=\frac{1}{r}J_0(0)=\frac{1}{r}.$$

Thus, the first term on the right side of (156) takes the form

$$-\iiint\limits_{\Omega}\frac{\Delta u}{r}d\Omega=4\pi u(x_0,y_0,z_0,t_0).$$

The second term can be written as

$$-\iiint\limits_{\Omega}\left[-\Delta u_1\frac{1}{\varrho}\Xi\left(\varrho\frac{t_0}{r}\right)-\Delta u_0\frac{1}{r}J_0\left(\varrho\frac{t_0}{r}\right)\right]d\Omega.$$

We show that the limit of the last integral is equal to zero. It is obvious that all the terms containing $\cos nz$ vanish under the limit passage, since $\cos nz$ differs from zero only on the small walls of the cylinder. It is not hard to see that

$$\lim_{\eta\to 0}\iint\limits_{S_{h,\eta}}\frac{\partial u}{\partial n}\frac{\partial^2}{\partial t^2}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]dS=0. \tag{157}$$

Indeed, (157) holds because $\dfrac{\partial u}{\partial n}$ is bounded and

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]$$

is of the order $\dfrac{1}{r}$.

Let us compute the limit of the remaining term. It is easy to verify that

$$\lim_{\eta\to 0}\int\limits_0^{t_0}\left\{\iint\limits_{S_{h,\eta}}u\frac{\partial}{\partial n}\frac{\partial^2}{\partial t^2}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]dS\right\}dt$$

$$=\int\limits_0^{t_0}u(x_0,y_0,z_0,t)\lim_{\eta\to 0}\iint\limits_{S_{h,\eta}}\frac{\partial^3}{\partial\varrho\partial t^2}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right]dS\,dt$$

and

$$\lim_{\eta\to 0}\iint\limits_{S_{h,\eta}} \frac{\partial^3}{\partial\varrho\partial t^2}\left[\frac{1}{\varrho}\Xi\left(\varrho\frac{t-t_0}{r}\right)\right] dS = \lim_{\eta\to 0}\frac{\partial}{\partial t}\iint\limits_{S_{h,\eta}}\frac{\partial}{\partial\varrho}\left\{\frac{1}{r}J_0\left(\frac{\varrho\tau}{r}\right)\right\} dS$$

$$= \lim_{\eta\to 0}\frac{\partial}{\partial t}\left\{4\pi\int\limits_0^h \varrho\left[-\frac{\varrho}{r^3}J_0\left(\frac{\varrho\tau}{r}\right) - \frac{\varrho^2\tau}{r^4}J_0'\left(\frac{\varrho\tau}{r}\right) + \frac{\tau}{r^2}J_0'\left(\frac{\varrho\tau}{r}\right)\right] dz\right\}$$

$$= 4\pi\frac{\partial}{\partial t}\left\{-\int\limits_0^1 \frac{\zeta J_0(\zeta\tau)\, d\zeta}{\sqrt{1-\zeta^2}} + \tau\int\limits_0^1 \frac{(1-\zeta^2)J_0'(\zeta\tau)\, d\zeta}{\sqrt{1-\zeta^2}}\right\} = 4\pi\frac{\partial(-1)}{\partial t} = 0,$$

which is required[9].

After these remarks, we can pass to the limit in (156). Transferring the known terms to one side and dividing by 4π, we obtain the final result

$$u(x_0, y_0, z_0, t_0) = -\frac{1}{4\pi}\iiint\limits_{-\infty}^{+\infty}\left\{\frac{1}{r}J_0\left(\varrho\frac{t_0}{r}\right)\Delta u_0 + \frac{1}{\varrho}\Xi\left(\frac{\varrho t_0}{r}\right)\Delta u_1\right\} d\Omega$$

$$-\frac{1}{4\pi}\int\limits_0^{t_0}\left\{\iiint\limits_{\Omega}\frac{1}{\varrho}\Xi\left(\varrho\frac{t_0-t}{r}\right)\Phi\, d\Omega\right\} dt. \qquad (158)$$

Formula (158) gives a representation for u at the point (x_0, y_0, z_0, t_0). It remains to verify that we actually obtained the solution of equation (151) with conditions (152).

14 Verification of the Obtained Solution

We first note that it suffices to verify the existence of a solution under the homogeneous conditions. Indeed, it is obvious that the function $v = u - u_0 - tu_1$ satisfies the new nonhomogeneous equation with homogeneous initial conditions.

Under some restrictions on Φ the function u defined by (158) has continuous spatial derivatives and time-derivatives up to the required order.

We prove that the function u satisfies equation (151). Let $\psi(x, y, z, t)$ be an arbitrary infinitely differentiable function vanishing outside V_ψ. This function satisfies the identity

$$\psi(x_0, y_0, z_0, t_0) = -\frac{1}{4\pi}\int\limits_T^{t_0}\left\{\iiint\limits_{\Omega}\frac{1}{\varrho}\Xi\left(\varrho\frac{t_0-t}{r}\right)L\psi\, d\Omega\right\} dt, \qquad (159)$$

if any point of the domain V_ψ has a coordinate t less than T.

[9] S. L. Sobolev uses a change of variables (123). – *Ed.*

This identity can be obtained from (158) by the change $t_1 = T - t$.

Multiplying both sides of (159) by $\Phi(x_0, y_0, z_0, t_0)$ and integrating over the whole (x_0, y_0, z_0, t_0)-space, we obtain

$$\iiiint \psi(x_0, y_0, z_0, t_0)\Phi(x_0, y_0, z_0, t_0)\, d\Omega_0\, dt_0$$
$$= -\frac{1}{4\pi} \underset{T>t>t_0}{\int \cdots \int} \frac{1}{\varrho} \Xi \left(\varrho \frac{t_0 - t}{r} \right) L\psi(x, y, z, t)\Phi(x_0, y_0, z_0, t_0)\, d\Omega\, d\Omega_0\, dt\, dt_0.$$

We transform this integral in the double integral by integration with respect to x_0, y_0, z_0, t_0 inside the domain and performing the exterior integration with respect to x, y, z, t. We obtain

$$\iiiint \psi(x_0, y_0, z_0, t_0)\Phi(x_0, y_0, z_0, t_0)\, d\Omega_0\, dt_0$$
$$= \iiiint L\psi(x, y, z, t)u(x, y, z, t)\, d\Omega\, dt$$

or, after integrating by parts,

$$\iiiint \psi(x_0, y_0, z_0, t_0)[Lu - \Phi]\, d\Omega_0\, dt_0 = 0. \tag{160}$$

This integral identity holds for any ψ. Whence it follows that

$$Lu = \Phi,$$

which is required.

15 Some Qualitative Consequences of the Obtained Formulas

As we see, the solution of all our problems is closely connected with the function

$$V = \frac{1}{r} J_0 \left(t \frac{\varrho}{r} \right) = \frac{1}{r} J_0(t \sin\theta). \tag{161}$$

Using this function, as well as some other functions of the same type, we have constructed the general solution of all these problems.

We trace how the function V changes with time. We consider a sphere with constant radius. On this sphere, the function V at each time moment depends only on the polar angle θ. The argument of the Bessel function J_0 changes from 0 to t on this sphere. With the course of time, more and more waves generated by the maxima and minima of the Bessel function will be placed in the interval between the pole and the equator of the sphere. Waves appear at the equator and move towards the pole so that they accumulate on the sphere but do not disappear. Hence more and more short waves will be formed from long waves.

According to our reckoning, such a formation of short waves from long waves is of interest.

References

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10. On Motion of a Symmetric Top with a Cavity Filled with Fluid*

S. L. Sobolev

Chapter 1 The General Theory of a Symmetric Top

1 The Motion Equations and Boundary Conditions

1^0. We consider a heavy top rotating with an angular velocity ω around its axis.

Let the foot of the top be fixed. We place the coordinate origin at this fixed point. Let the z^*-axis be directed vertically upward, let the x^*-axis and the y^*-axis of the fixed coordinate system be located in the horizontal plane. Let the top have a cavity filled with a fluid.

We assume that the shape of the cavity is symmetric with respect to the axis of the top. Both for the top itself and for the cavity the axis is the axis of symmetry of order k, where $k > 2$; in other words, if we rotate the top around the axis by the angle $\dfrac{2\pi}{k}$, then it will match itself.

We denote by M_1 the mass of the shell of the top, by M_2 the mass of the fluid, and by ϱ the density of the fluid. Let x_2, y_2, z_2 be the coordinate axes related to the top; moreover, let the symmetry axis of the top coincide with the z_2-axis, and let the coordinate origin be located at the fixed point. We denote by A_1 the moment of inertia of the shell with respect to the x_2-axis and the y_2-axis, and by A_2 the moment of inertia of the fluid with respect to the same axis.

* Zh. Prikl. Mekh. Tekhn. Fiz., **3**, 20–55 (1960)

The author finished this work in 1943. However, it was not published at the time. Since then a similar type of questions has caused a number of studies. In the author's work [1] and in a number of papers of other authors, spectral problems for a system of equations of similar type, boundary value problems, etc. were studied. However, this branch of mechanics is still attractive, and possibly the publication of one of the first papers on these questions is of interest.

Similarly, C_1 and C_2 denote the moments of inertia of the shell and the fluid with respect to the z_2-axis. The distances from the point of support to the centers of mass of the shell and the fluid are l_1 and l_2, respectively.

We denote by X^*, Y^*, $\sqrt{1 - X^{*2} - Y^{*2}}$ the coordinates of the unit vector in the direction from the point of support along the axis of the top; $\mathbf{u}^*$ is the vector of the velocity of the fluid, which we assume to be ideal, and p^* is a pressure inside. We denote by S the surface of the cavity filled with the fluid, and by V the volume it occupies.

We study only motions close to the uniform rotation of the top around a vertical line. Let us clarify what it means.

Let

$$\begin{aligned} &\mathbf{u}^{(0)} = \omega \mathbf{k} \times \mathbf{r}, \\ &p^{(0)} = \varrho \frac{\omega^2 (r^2 - z^{*2})}{2} - \varrho g z^* + p_0 = \varrho \frac{\omega^2 (x^{*2} + y^{*2})}{2} - \varrho g z^* + p_0, \qquad (1.1) \\ &X^{(0)} = Y^{(0)} = 0, \end{aligned}$$

where $\mathbf{r}$ is the coordinate vector, and $\mathbf{k}$ is the unit vector along the z^*-axis. The quantities $\mathbf{u}^{(0)}$, $p^{(0)}$, $X^{(0)}$, $Y^{(0)}$ describe this uniform rotation. We are interested only in the differences

$$\mathbf{u} - \mathbf{u}^{(0)}, \quad p^* - p^{(0)}, \quad X^* - X^{(0)}, \quad Y^* - Y^{(0)}. \qquad (1.2)$$

Further, we look for quantities (1.2).

The problem, so stated, is the linear problem of the theory of partial differential equations.

Our goal is to find conditions under which motion (1.1) of such top is stable.

2^0. The present work is divided into three chapters. In Chap. 1 we consider the general theory of motion of the top with a symmetric cavity filled with fluid. In Chap. 2 we study the theory of motion of the top with a cavity shaped as an ellipsoid of rotation. Chap. 3 is devoted to the top with a cylindrical cavity. Let us present briefly the content of each chapter.

The general solution of the problem on vibrations of the filled top, as well as the solution of any linear problem on vibrations with an infinite number of degrees of freedom, can be represented differently in the form of a sum of finite or infinite number of terms, each of which is, in turn, a solution of the problem.

Sometimes the study of these terms can be simpler than the study of the general solution.

In Chap. 1 for any symmetric top we divide each solution of the problem, which is naturally to assumed to be real, into the sum of k distinct complex solutions, which are simpler to study. These solutions are distinguished by the transformation which they experience under the rotation of the top by the angle $\dfrac{2\pi}{k}$.

Among these k solutions, there are only two such when, together with the fluid, the shell also vibrates. These two solutions are complex conjugate. Thus, it is sufficient to study only one of them; there is no need to consider the remaining $k-1$ solutions.

Thus, the solution which we study later has simpler properties than the general solution.

If the cavity filled with the fluid is a body of rotation, then the problem is even simpler, because in this case one can isolate a solution such that in the polar coordinates r, θ, z its dependence on the angle θ is especially simple.

The quantities describing the motion of the top with a cavity can be considered in a certain phase space $\{R\}$, which in this case is infinite-dimensional, because of the infinite number of degrees of freedom. In this case the equations of motion of the fluid within a shell are written in the form

$$\frac{dR}{dt} = iBR + R_0, \tag{1.3}$$

where B is a linear operator, R is an unknown element or a vector of the phase space. The solution of this equation, satisfying the condition $R = R^{(0)}$ for $t = 0$, is given by the formula

$$R = e^{iBt}R^{(0)} + \int\limits_0^t e^{iB(t-t_1)}R_0(t_1)\,dt_1. \tag{1.4}$$

The spectral theory of operators allows us to represent this solution by using the resolvent of the operator B,

$$\Gamma_\lambda = (\lambda E - B)^{-1}. \tag{1.5}$$

It is proved that the operator B is bounded. Hence its resolvent is regular near $\lambda = \infty$. In this case

$$e^{iBt}R_0 = \frac{1}{2\pi i}\int\limits_C e^{i\lambda t}\Gamma_\lambda R_0\,d\lambda, \tag{1.6}$$

where C is an arbitrary sufficiently large contour.

The remaining of Chap. 1 is devoted to the study of the operator B and its resolvent.

In the phase space $\{R\}$ we construct a Hermitian form $Q(R_1, R_2)$ whose behavior is determined by the quantity

$$L = C_1 + C_2 - A_1 - A_2 - \frac{K}{\omega^2}, \qquad K = g(l_1M_1 + l_2M_2). \tag{1.7}$$

Here K is the value of the tilting moment of the force of gravity referred to the angle of deflection of the z_2-axis from the vertical.

If L is positive, then the Hermitian form Q is positive definite; if L is negative, then it divides into the difference of a positive form and one negative term in the complementary space. The relation

$$Q(BR_1, R_2) = Q(R_1, BR_2) \tag{1.8}$$

holds.

The operator B is self-conjugate with respect to the form Q. If $L > 0$, then the operator Q is the so-called Hermitian operator, and its entire spectrum is concentrated on the real axis. In this case, the operator $\exp(iBt)$ is bounded.

If $L < 0$, and we are interested mostly in this case, then one can establish the existence of at most one pair of complex eigenvalues of B. This case is essentially new in operator theory.

The presence or the absence of these complex eigenvalues is connected with the presence or the absence of solutions of the problem in question with the factor $\exp(\pm i\sigma t \pm \tau t)$, i.e., with the question on stability of the solution of the problem. This question is studied in detail in the following chapters.

3^0. As noted above, if the cavity is a body of rotation, then the problem is somewhat simplified. In this case the phase space $\{R\}$ can be divided into the sum of three mutually complementary subspaces:

$$\{R\} = \{R\}_1 + \{\overline{R}\}_1 + \{R\}_2.$$

Here the spaces $\{R\}_1$ and $\{\overline{R}\}_1$ contain those motions of the fluid where the pressure p has the form $e^{\mp i\theta}p(r, z)$, and $\{R\}_2$ contains those where the pressure is orthogonal to $e^{\mp i\theta}$. The operator B keeps all these spaces invariant. All motions of the fluid are also divided into the sum of motions in $\{R\}_1$, $\{\overline{R}\}_1$ and in $\{R\}_2$. Here the motions in $\{R\}_2$ are such that the shell of the top is not involved in them.

An even bigger simplification occurs when the cavity is an ellipsoid of rotation. In this case the phase space $\{R\}_1$, in turn, decomposes into a certain three-dimensional space $\{R\}_3$ and complementary infinite-dimensional space $\{R\}^3$:

$$\{R\}_1 = \{R\}_3 \oplus \{R\}^3.$$

In this case the shell is included only in motions of $\{R\}_3$.

Therefore, the question on stability of the motion of the top with an ellipsoidal cavity is reduced to the study of the behavior of roots of a certain equation of third order depending on parameters. In particular, when the top shell is weightless and the top itself is mounted at the center of gravity, this question has been already studied in the literature.

The qualitative analysis shows that the character of the motion of a top with an ellipsoidal cavity is determined by four dimensionless parameters, namely, by relations between the numbers

$$A, \quad A_2^{(0)}, \quad C_1, \quad C_2, \quad \frac{K}{\omega^2} = \nu, \tag{1.9}$$

where $A = A_1 + A_2$, $A_2^{(0)}$ is the main equatorial moment of inertia of the fluid.

Suppose that A, $A_2^{(0)}$, C_1, and C_2 are fixed. Let us vary the angular velocity ω (or, which is the same, the tilting moment K). The domain of change of ν can be divided in this case into four parts:

$$\begin{aligned} &(\mathrm{I}) \quad -\infty < \nu < \nu_1, \qquad &&(\mathrm{III}) \quad \nu_2 < \nu < \nu_3, \\ &(\mathrm{II}) \quad \nu_1 < \nu < \nu_2, \qquad &&(\mathrm{IV}) \quad \nu_3 < \nu < \infty, \end{aligned} \tag{1.10}$$

where ν_1, ν_2, and ν_3 are the roots of a certain cubic equation. In the first and third domains the motion is stable, while in the second and forth domains it is unstable. If this cubic equation has the only root, then the domain of change of ν is divided only into two parts. If we suppose that $A = A_2^0$, $C_1 = 0$, i.e., if we study the top with a weightless shell rotating around the center of gravity, then the stability of its motion occurs when $c < a$ or $c > 3a$, where c is the semiaxis of the ellipsoid along the axis of rotation, and a is another axis.

This result is known in the literature.

4^0. In the case of a cylindrical cavity the problem is solved by expanding the desired solution and the resolvent into a series of a special type. It turns out to be possible to construct the function D with the property that its roots are the eigenvalues of the operator. This function has an essential singularity, on the interval of the real axis $-2\omega < \lambda < 2\omega$, and it is regular in the remaining part of the plane.

The study reveals the following result. Unlike the top with an ellipsoidal cavity, changing the parameter ν, there exist an infinite set of intervals, where the top loses its stability. Moreover, the values of imaginary parts of the roots are either large or small. If we engage such a top, and start reducing the velocity of rotations, then we pass through the domain of agitated motions many times.

Based on this, we can conclude that if we wish to calm the motion, we need to shape the cavity in the form of an ellipsoid of rotation. We calculate a particular example of a top with an ellipsoidal cavity. In the case when the moments of inertia of the shell are significantly larger than the moment of inertia of the fluid, and the cavity is sufficiently long, ($c > 3a$), we give an approximate formula for determining the necessary value of the angular velocity

$$q > 2\sqrt{KA^{(0)}}, \qquad A^{(0)} = A - A_2^{(0)} \frac{2a^2}{c^2 - a^2}, \tag{1.11}$$

where q is a moment of momentum of the top around the axis.

2 Mathematical Statement of the Problem

1^0. The equations of motion of the shell, which we consider to be a solid body with a fixed point, if we assume the deviations from the uniform rotation to be small, have the form

$$\begin{aligned} A_1 \overset{..}{X}{}^* + C_1\omega \overset{.}{Y}{}^* - M_{y^*} &= 0, \\ A_1 \overset{..}{Y}{}^* - C_1\omega \overset{.}{X}{}^* + M_{x^*} &= 0, \end{aligned} \tag{2.1}$$

where M_{x^*} and M_{y^*} are the components of moments of all forces acting at the shell.

There are three possible types of these forces: the weight of the shell, the pressure of the fluid, and exterior forces. The force of gravity is applied at the point with the coordinates

$$l_1 X^*, \quad l_1 Y^*, \quad l_1\sqrt{1 - X^{*2} - Y^{*2}}$$

and directed along the negative z, therefore, its moments are

$$-gl_1 M_1 Y^*, \qquad gl_1 M_1 X^*. \tag{2.2}$$

If we calculate the moments of forces of the pressure of the fluid at the shell, then we have for them the following expressions, respectively,

$$\begin{aligned} M_{x^*}(p^*) &= \iint_S p^*[y^* \cos nz^* - z^* \cos ny^*]\, dS, \\ M_{y^*}(p^*) &= \iint_S p^*[z^* \cos nx^* - x^* \cos nz^*]\, dS. \end{aligned} \tag{2.3}$$

Assuming that the moments of exterior forces are equal to $M_{x^*}^{(0)}$ and $M_{y^*}^{(0)}$, respectively, we have

$$\begin{aligned} A_1 \overset{..}{X}{}^* + C_1\omega \overset{.}{Y}{}^* - gl_1 M_1 X^* - M_{y^*}(p^*) - M_{y^*}^{(0)} &= 0, \\ A_1 \overset{..}{Y}{}^* - C_1\omega \overset{.}{X}{}^* - gl_1 M_1 Y^* + M_{x^*}(p^*) + M_{x^*}^{(0)} &= 0. \end{aligned} \tag{2.4}$$

The expressions for the force of pressure can be obtained from the equations of hydrodynamics. These equations have the form

$$\frac{d\mathbf{u}^*}{dt} + \frac{1}{\varrho}\operatorname{grad} p^* = \mathbf{F} - g\mathbf{k}, \quad \operatorname{div} \mathbf{u}^* = 0, \tag{2.5}$$

where $\mathbf{F}$ is a vector of external mass forces, g is the acceleration of gravity.

2^0. Let us introduce the moving system of coordinates x, y, z, such that its origin coincides with the moving point all the time, the z-axis remains parallel to the z^*-axis, and the x-axis and the y-axis revolve around the z-axis with angular velocity ω. The absolute particle acceleration of fluid is the sum of the relative, transfer, and Coriolis accelerations. We have

$$\frac{d\mathbf{u}^*}{dt} = \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}'\nabla)\mathbf{u}' + (-\omega^2 x\mathbf{i} - \omega^2 y\mathbf{j}) - 2\omega(\mathbf{u}' \times \mathbf{k}), \tag{2.6}$$

where $\mathbf{u}'$ is the vector of relative motion in the coordinates x, y, z.

Also assume that

$$p^* = -g\varrho z + \frac{\omega^2 \varrho(x^2+y^2)}{2} + p'. \tag{2.7}$$

Then, neglecting small terms, equation (2.5) can be written in the form

$$\begin{aligned} &\frac{\partial u_x'}{\partial t} - 2\omega u_y' + \frac{1}{\varrho}\frac{\partial p'}{\partial x} = F_x, \qquad \frac{\partial u_z'}{\partial t} + \frac{1}{\varrho}\frac{\partial p'}{\partial z} = F_z, \\ &\frac{\partial u_y'}{\partial t} + 2\omega u_x' + \frac{1}{\varrho}\frac{\partial p'}{\partial y} = F_y, \qquad \frac{\partial u_x'}{\partial x} + \frac{\partial u_y'}{\partial y} + \frac{\partial u_z'}{\partial z} = 0. \end{aligned} \tag{2.8}$$

Substituting (2.7) into (2.3) and (2.4), we have with accuracy up to the terms of higher order

$$\begin{aligned} M_{x^*}(p^* - p') &= \iiint\limits_V [-g\varrho y^* - \omega^2 \varrho y^* z^*]\, dv, \\ M_{y^*}(p^* - p') &= \iiint\limits_V [g\varrho x^* + \omega^2 \varrho x^* z^*]\, dv, \end{aligned} \tag{2.9}$$

but

$$\begin{gathered} \iiint\limits_V \varrho y^*\, dv = l_2 M_2 Y^*, \qquad \iiint\limits_V \varrho x^*\, dv = l_2 M_2 X^*, \\ \iiint\limits_V z^*\, dv = l_2, \quad \iiint\limits_V y^* z^*\, dv = A^*_{2y,z}, \quad \iiint\limits_V x^* z^*\, dv = A^*_{2x,z}, \end{gathered} \tag{2.10}$$

where $A^*_{2y,z}$ and $A^*_{2x,z}$ are the components of the tensor of the moments of the inertia of the volume V with respect to the axis x^*, y^*, z^*. These components can be represented in the form

$$\begin{aligned} A^*_{2x,z} &= \frac{C_2}{2}(\cos x_2 x^* \cos x_2 z^* + \cos y_2 x^* \cos y_2 z^*) \\ &+ \left(A_2 - \frac{C_2}{2}\right) \cos z_2 x^* \cos z_2 z^* = (A_2 - C_2) \cos z_2 x^* \cos z_2 z^*, \\ A^*_{2y,z} &= (A_2 - C_2) Y^*. \end{aligned} \tag{2.11}$$

Thus, we obtain

$$\begin{aligned} &A_1 \ddot{X}^* + C_1 \omega \dot{Y}^* - K_2 X^* - M_{y^*}(p') - M^{(0)}_{y^*} = 0, \\ &A_1 \ddot{Y}^* - C_1 \omega \dot{X}^* - K_2 Y^* + M_{x^*}(p') + M^{(0)}_{x^*} = 0, \end{aligned} \tag{2.12}$$

where $K_2 = g(l_1M_1 + l_2M_2) + \omega^2(A_2 - C_2)$.

To write the boundary conditions more conveniently, we pass from the coordinates X^* and Y^* to others. Let

$$X^* + iY^* = Z^*. \tag{2.13}$$

Multiplying the second equation in (2.12) by i and adding it to the first one, we obtain

$$A_1 \ddot{Z}^* - C_1\omega i \dot{Z}^* - K_2Z^* + 2iN^*(p') + 2iN^{(0)} = 0, \tag{2.14}$$

where

$$2N^*(p') = M_{x^*}(p') + iM_{y^*}(p'), \qquad 2N^{(0)} = M^{(0)}_{x^*} + iM^{(0)}_{y^*}. \tag{2.15}$$

Let us introduce the notation

$$\mu^* = z^*(\cos nx^* + i\cos ny^*) - (x^* + iy^*)\cos nz^*.$$

Then, $2N^*(p') = i \iint\limits_S \mu^* p'\,dS$.

Also, let $Z^* = e^{i\omega t}Z$. Obviously, the real and imaginary parts of the complex number $Z = X + iY$ give the value of deviation of the unit vector of the top axis in the coordinates x, y, z. We write equation (2.14) as

$$A_1 \ddot{Z} - (C_1 - 2A_1)\omega i \dot{Z} + L\omega^2 Z + ie^{-i\omega t}2[N^*(p') + N^{(0)}_*] = 0, \tag{2.16}$$

where, as indicated above, $L = C_1 + C_2 - A_1 - A_* - \dfrac{K}{\omega^2}$. Obviously,

$$e^{-i\omega t}N^*(p') = N(p'), \qquad e^{-i\omega t}N^{(0)}_* = N^{(0)}.$$

Here,

$$2N(p') = i \iint\limits_S \mu p'\,dS, \qquad \mu = z(\cos nx + i\cos ny) - (x + iy)\cos nz.$$

3^0. Obviously, the normal component of the velocity $\mathbf{u}'$ on the surface S must coincide with the normal component of the velocity of the corresponding point of the surface S in the system x^*, y^*, z^*.

Denoting by $\mathbf{w}$ the vector of the transfer velocity S in the fixed system, we obtain

$$w_x = \dot{X}\, z, \quad w_y = \dot{Y}\, z, \quad w_z = -\dot{X}\, x - \dot{Y}\, y. \tag{2.17}$$

Hence,

$$u'_n|_S = \dot{X}\,(z\cos nx - x\cos nz) + \dot{Y}\,(z\cos ny - y\cos nz). \tag{2.18}$$

Condition (2.18) makes it possible to solve the problem stated. For the sake of convenience let us express condition (2.18) in terms of $\dot{Z}$; we obtain

$$u'_n|_S = \frac{1}{2}(\dot{Z}\,\overline{\mu} + \dot{\overline{Z}}\,\mu). \tag{2.19}$$

4⁰. The symmetry of the cavity occupied by the fluid allows splitting the general solution of the problem into several particular solutions, similar to how in the domain with an axial symmetry of infinite order the solution can be expanded into a Fourier series in the cyclic coordinate. However, in the case when there is only a symmetry of a finite order, this expansion is unrealizable, and we use a different approach.

Let φ be a certain function defined in the domain V of the real variables x, y, z. Let us denote the complex coordinates as

$$x + iy = \zeta, \qquad x - iy = \overline{\zeta} \tag{2.20}$$

and assume that ζ and $\overline{\zeta}$ are independent variables varying along a corresponding two-dimensional manifold.

Using the function $\varphi(x, y)$, let us introduce k new functions $\varphi_{(s)}(\zeta, \overline{\zeta})$, $s = 0, 1, \dots, k-1$,

$$\varphi_{(s)}(\zeta, \overline{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1} e^{s2\pi il/k}\varphi(\zeta e^{2\pi il/k}, \overline{\zeta}e^{-2\pi il/k}). \tag{2.21}$$

Let us point out some simple properties of the symbol $\varphi_{(s)}$. Thus, we have

$$\sum_{s=0}^{k-1}\varphi_{(s)}(\zeta, \overline{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1}\varphi(\zeta e^{2\pi il/k}, \overline{\zeta}e^{-2\pi il/k})\left(\sum_{s=0}^{k-1} e^{s2\pi il/k}\right).$$

However,

$$\sum_{s=0}^{k-1}\left(e^{2\pi il/k}\right)^s = \frac{(e^{2\pi il/k})^k - 1}{e^{2\pi il/k} - 1} = \begin{cases} 0, & l \not\equiv 0 \pmod k, \\ k, & l = 0 \pmod k. \end{cases}$$

Hence,

$$\sum_{s=0}^{k-1}\varphi_{(s)}(\zeta, \overline{\zeta}) = \varphi(\zeta, \overline{\zeta}). \tag{2.22}$$

The function φ_s possesses the distinctive periodicity

$$\begin{aligned}\varphi_{(s)}(\zeta e^{2\pi i/k}, \overline{\zeta}e^{-2\pi i/k}) &= \frac{1}{k}\sum_{l=0}^{k-1} e^{s2\pi il/k}\varphi(\zeta e^{2\pi i(l+1)/k}, \overline{\zeta}e^{-2\pi i(l+1)/k}) \\ &= e^{-s2\pi i/k}\varphi_{(s)}(\zeta, \overline{\zeta}).\end{aligned} \tag{2.23}$$

Obviously,

$$\varphi_{(s_1)} = \varphi_{(s_2)} \quad \text{for} \quad s_1 \equiv s_2 \pmod k.$$

If φ is a real function, then

$$\varphi_{(s_1)} = \overline{\varphi_{(s_2)}} \quad \text{for} \quad s_2 \equiv k - s_1 \pmod k. \tag{2.24}$$

Formula (2.22) produces the required splitting of the solution into a finite number of the terms $k = 0, 1, \ldots, k-1$. Strictly speaking, differentiation of φ with respect to ζ or $\overline{\zeta}$ makes no sense, however, if we assume

$$\frac{\partial \varphi}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}\right), \qquad \frac{\partial \varphi}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y}\right), \tag{2.25}$$

then all the usual differentiation formulas remain valid.

Using this fact and differentiating (2.21), we obtain

$$\frac{\partial}{\partial \zeta}\varphi_{(s)}(\zeta, \overline{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1} e^{(s+1)2\pi i l/k}\frac{\partial \varphi}{\partial \zeta}(\zeta e^{2\pi i l/k}, \overline{\zeta} e^{-2\pi i l/k}) = \left(\frac{\partial \varphi}{\partial \zeta}\right)_{(s+1)}.$$

Then,

$$\frac{\partial}{\partial \overline{\zeta}}\varphi_{(s)}(\zeta, \overline{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1} e^{(s-1)2\pi i l/k}\frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta e^{2\pi i l/k}, \overline{\zeta} e^{-2\pi i l/k}) = \left(\frac{\partial \varphi}{\partial \overline{\zeta}}\right)_{(s-1)},$$

i.e.,

$$\frac{\partial}{\partial \zeta}\varphi_{(s)} = \left(\frac{\partial \varphi}{\partial \zeta}\right)_{(s+1)}, \qquad \frac{\partial}{\partial \overline{\zeta}}\varphi_{(s)} = \left(\frac{\partial \varphi}{\partial \overline{\zeta}}\right)_{(s-1)}. \tag{2.26}$$

Let us also emphasize the formula

$$\sum_{s=0}^{k-1} e^{s2\pi i t/k}\varphi_{(s)}(\zeta, \overline{\zeta}) = \sum_{s=0}^{k-1} \varphi(\zeta e^{-2\pi i t/k}, \overline{\zeta} e^{2\pi i t/k}), \tag{2.27}$$

which follows from the transformation

$$\sum_{s=0}^{k-1} e^{s2\pi i t/k}\varphi_{(s)}(\zeta, \overline{\zeta}) = \frac{1}{k}\sum_{l=0}^{k-1} \varphi(\zeta e^{2\pi i l/k}, \overline{\zeta} e^{-2\pi i l/k})\left(\sum_{s=0}^{k-1} e^{s2\pi i (t+l)/k}\right).$$

5⁰. Let us introduce in equation (2.8) the variables ζ and $\overline{\zeta}$. We have

$$\frac{\partial p'}{\partial x} = \frac{\partial p'}{\partial \zeta} + \frac{\partial p'}{\partial \overline{\zeta}}, \qquad \frac{\partial p'}{\partial y} = i\left(\frac{\partial p'}{\partial \zeta} - \frac{\partial p'}{\partial \overline{\zeta}}\right). \tag{2.28}$$

Assuming

$$u'_x + iu'_y = u'_\zeta, \quad u'_x - iu'_y = u'_{\overline{\zeta}}, \quad F_x + iF_y = F_\zeta, \quad F_x - iF_y = F_{\overline{\zeta}},$$

multiplying the second equation from the left set of (2.8) by i, adding and subtracting from the first one, we obtain

$$\begin{aligned}
&\frac{\partial u'_\zeta}{\partial t} + 2\omega i u'_\zeta + \frac{2}{\varrho}\frac{\partial p'}{\partial \overline{\zeta}} = F_\zeta, \qquad && \frac{\partial u'_z}{\partial t} + \frac{1}{\varrho}\frac{\partial p'}{\partial z} = F_z,\\
&\frac{\partial u'_{\overline{\zeta}}}{\partial t} - 2\omega i u'_{\overline{\zeta}} + \frac{2}{\varrho}\frac{\partial p'}{\partial \zeta} = F_{\overline{\zeta}}, \qquad && \frac{\partial u'_\zeta}{\partial \zeta} + \frac{\partial u'_{\overline{\zeta}}}{\partial \overline{\zeta}} + \frac{\partial u'_z}{\partial z} = 0.
\end{aligned} \tag{2.29}$$

Let us apply to equations (2.29) the operation (s). We obtain

$$\begin{aligned}
&\frac{\partial u'_{\zeta,(s-1)}}{\partial t} + 2\omega i u'_{\zeta,(s-1)} + \frac{2}{\varrho}\frac{\partial p'_{(s)}}{\partial \overline{\zeta}} = F_{\zeta,(s-1)},\\
&\frac{\partial u'_{\overline{\zeta},(s+1)}}{\partial t} - 2\omega i u'_{\overline{\zeta},(s+1)} + \frac{2}{\varrho}\frac{\partial p'_{(s)}}{\partial \zeta} = F_{\overline{\zeta},(s+1)},\\
&\frac{\partial u'_{z,(s)}}{\partial t} + \frac{1}{\varrho}\frac{\partial p'_{(s)}}{\partial z} = F_{z,(s)},\\
&\frac{\partial u'_{\zeta,(s-1)}}{\partial \zeta} + \frac{\partial u'_{\overline{\zeta},(s+1)}}{\partial \overline{\zeta}} + \frac{\partial u'_{z,(s)}}{\partial z} = 0, \quad s = 0, 1, \dots, k-1.
\end{aligned} \tag{2.30}$$

Thus, the system of equations also splits; instead of one system (2.8) we now have k different systems relating $u_{\zeta,(s-1)}$, $u_{\overline{\zeta},(s+1)}$, $p'_{(s)}$, $u_{z,(s)}$.

Let us study now the boundary conditions, and also the operator $N(p')$. Replacing in conditions (2.19) u_x and u_y by their expressions, we obtain

$$\begin{aligned}
u'_n|_S &= u'_x \cos nx + u'_y \cos ny + u'_z \cos nz|_S\\
&= \frac{1}{2}u'_\zeta(\cos nx - i\cos ny) + \frac{1}{2}u'_{\overline{\zeta}}(\cos nx + i\cos ny) + u'_z \cos nz|_S\\
&= \frac{1}{2}u'_\zeta\overline{\lambda}_1 + \frac{1}{2}u'_{\overline{\zeta}}\lambda_1 + u'_z\cos nz|_S = \frac{1}{2}\,\dot{Z}\,\overline{\mu} + \frac{1}{2}\,\dot{\overline{Z}}\,\mu,
\end{aligned} \tag{2.31}$$

where

$$\lambda_1 = \cos nx + i\cos ny. \tag{2.32}$$

Obviously, we have

$$\begin{aligned}
\lambda_1(\zeta e^{2\pi i/k}, \overline{\zeta}e^{-2\pi i/k}) &= e^{2\pi i/k}\lambda_1(\zeta,\overline{\zeta}),\\
\overline{\lambda}_1(\zeta e^{2\pi i/k}, \overline{\zeta}e^{-2\pi i/k}) &= e^{-2\pi i/k}\overline{\lambda}_1(\zeta,\overline{\zeta}),\\
\mu(\zeta e^{2\pi i/k}, \overline{\zeta}e^{-2\pi i/k}) &= e^{2\pi i/k}\mu(\zeta,\overline{\zeta}),\\
\overline{\mu}(\zeta e^{2\pi i/k}, \overline{\zeta}e^{-2\pi i/k}) &= e^{-2\pi i/k}\overline{\mu}(\zeta,\overline{\zeta}).
\end{aligned} \tag{2.33}$$

Hence,

$$\lambda_1 = \lambda_{1,(-1)}, \quad \overline{\lambda}_1 = \overline{\lambda}_{1,(1)}, \quad \mu = \mu_{(-1)}, \quad \overline{\mu} = \overline{\mu}_{(1)}. \tag{2.34}$$

Applying the operation (s) to both parts of (2.31), we obtain

$$\begin{aligned} &\frac{1}{2} u'_{\zeta,(s-1)} \overline{\lambda}_1 + \frac{1}{2} u'_{\overline{\zeta},(s+1)} \lambda_1 + u'_{z,(s)} \cos nz|_S = \frac{1}{2}\, \dot{Z}\, \overline{\mu}_{(s)} + \frac{1}{2}\, \dot{\overline{Z}}\, \mu_{(s)} \\ &\qquad = \begin{cases} \frac{1}{2}\, \dot{Z}\, \overline{\mu}, & s \equiv 1 \pmod{k}, \\ \frac{1}{2}\, \dot{\overline{Z}}\, \mu, & s \equiv -1 \pmod{k}, \\ 0, & s \not\equiv \pm 1 \pmod{k}. \end{cases} \end{aligned} \tag{2.35}$$

Substituting into $N(p')$ instead of p' its expression[1], we also obtain

$$2N(p') = i \sum_{s=0}^{k-1} \iint_S p'_{(s)} \mu \, dS. \tag{2.36}$$

However,

$$\iint_S p'_{(s)} \mu \, dS = e^{2\pi i (1-s)/k} \iint_S p'_{(s)} \mu \, dS. \tag{2.37}$$

One easily gets convinced in this fact by rotating the coordinate axis. Therefore, all integrals (2.36) are equal to zero, except for the one where $s = 1$, and, hence,

$$2N(p') = i \iint_S p'_{(1)} \mu \, dS, \qquad 2\overline{N}(\overline{p'}) = -i \iint_S p'_{(-1)} \overline{\mu} \, dS. \tag{2.38}$$

Thus, the boundary conditions for systems (2.30) appear to be also split and mutually independent.

Systems (2.30) for $s \equiv \pm 1 \pmod{k}$ have the boundary conditions for $u'_{\zeta,(s-1)}$, $u'_{\overline{\zeta},(s+1)}$, $u_{z,(s)}$ nonhomogeneous of the same type as (2.31); for $s \not\equiv \pm 1 \pmod{k}$ the conditions are homogeneous.

The boundary conditions on (2.30) for $s = 1$ include $\dot{Z}$, which is connected, thus, to the quantities

$$u'_{\zeta,(0)}, \quad u'_{\overline{\zeta},(2)}, \quad u'_{z,(1)}, \quad p'_{(1)}. \tag{2.39}$$

On the other side, the quantity Z satisfies the equation

$$A_1 \ddot{Z} - B_1 \omega i \dot{Z} + L\omega^2 Z + 2iN(p'_{(1)}) + 2iN^{(0)} = 0, \qquad B_1 = C_1 - 2A_1, \tag{2.40}$$

[1] S. L. Sobolev uses a formula of type (2.22). – *Ed.*

which also involves $p'_{(1)}$. Thus, variables (2.39) and Z are related through the system of equations. Similarly, in system (2.30) for $s = -1$ the boundary conditions are related to $\overline{Z}$. The equation

$$A_1 \ddot{\overline{Z}} + B_1 \omega i \dot{\overline{Z}} + L\omega^2 \overline{Z} - 2i\overline{N}(p'_{(-1)}) - 2i\overline{N}^{(0)} = 0 \qquad (2.41)$$

presents another connection between $\overline{Z}$ and the functions $u'_{\zeta,(-2)}$, $u'_{\overline{\zeta},(0)}$, $u'_{z,(-1)}$, $p'_{(-1)}$.

6^0. If the cavity filled with fluid has the shape of a body of rotation, then the quantities $p'_{(1)}$ and $p'_{(-1)}$ have a particularly simple form. Obviously, in this case conditions (2.23) hold for any k. Suppose that $p'_{(1)}$ can be expanded into the series

$$p'_{(1)} = \sum_{l=-\infty}^{\infty} e^{il\theta} p_{l,(1)}.$$

Substituting this expression into formula (2.23), we obtain

$$\sum_{l=-\infty}^{\infty} e^{il(\theta+2\pi/k)} p_{l,(1)} = e^{-2\pi i/k} \sum_{l=-\infty}^{\infty} e^{li\theta} p_{l,(1)}.$$

Hence,

$$\sum_{l=-\infty}^{\infty} (e^{2\pi il/k} - e^{2\pi i/k}) p_{l,(1)} e^{li\theta} = 0,$$

and then either $e^{2\pi il/k} = e^{-2\pi i/k}$ or $p_{l,(1)} = 0$.

Therefore, $l \equiv -1 (\operatorname{mod} k)$.

Thus, in the expansion of $p'_{(1)}$ only the terms containing $e^{-i\theta}$, $e^{(k-1)i\theta}$, $e^{(2k-1)i\theta}$, etc. can appear. If it occurs for all k, then $p'_{(1)}$ contains the factor $e^{-i\theta}$.

Similarly, $p'_{(-1)}$ contains only the terms $e^{i\theta}$, $e^{(k+1)i\theta}$, $e^{(2k+1)i\theta}$, etc., and, if it occurs for all k, then $p'_{(-1)}$ is divisible by $e^{i\theta}$.

7^0. In the following sections we will study the solutions of (2.30), but now let us also present formulas for certain expressions, having a definite physical meaning. Let us denote

$$\begin{gathered} u'_{\overline{\zeta},(s+1)} + u'_{\zeta,(s-1)} = v_{x,[s]}, \qquad u'_{\overline{\zeta},(s+1)} - u'_{\zeta,(s-1)} = -iv_{y,[s]}, \\ 2u'_{z,(s)} = v_{z,[s]}, \qquad 2p'_{(s)} = p_{[s]}, \qquad F_{\overline{\zeta},(s+1)} + F_{\zeta,(s-1)} = F_{x,[s]}, \qquad (2.42) \\ F_{\overline{\zeta},(s+1)} - F_{\zeta,(s-1)} = iF_{y,[s]}, \qquad 2F_{z,(s)} = F_{z,[s]}. \end{gathered}$$

Obviously,

$$u'_x = \frac{1}{2} \sum_{s=0}^{k-1} v_{x,[s]} = \frac{1}{4} \sum_{s=0}^{k-1} (v_{x,[s]} + \overline{v}_{x,[s]}),$$

$$
\begin{aligned}
u'_y &= \frac{1}{2}\sum_{s=0}^{k-1} v_{y,[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(v_{y,[s]} + \overline{v}_{y,[s]}),\\
u'_z &= \frac{1}{2}\sum_{s=0}^{k-1} v_{z,[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(v_{z,[s]} + \overline{v}_{z,[s]}),\\
F_x &= \frac{1}{2}\sum_{s=0}^{k-1} F_{x,[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(F_{x,[s]} + \overline{F}_{x,[s]}),\\
F_y &= \frac{1}{2}\sum_{s=0}^{k-1} F_{y,[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(F_{y,[s]} + \overline{F}_{y,[s]}),\\
F_z &= \frac{1}{2}\sum_{s=0}^{k-1} F_{z,[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(F_{z,[s]} + \overline{F}_{z,[s]}),\\
p' &= \frac{1}{2}\sum_{s=0}^{k-1} p_{[s]} = \frac{1}{4}\sum_{s=0}^{k-1}(p_{[s]} + \overline{p}_{[s]}).
\end{aligned}
\tag{2.43}
$$

The equations for $v_{[s]}$ are obtained from equations (2.30) by addition and subtraction.

We obtain

$$
\begin{aligned}
&\frac{\partial v_{x,[s]}}{\partial t} - 2\omega v_{y,[s]} + \frac{1}{\varrho}\frac{\partial p_{[s]}}{\partial x} = F_{x,[s]}, \quad \frac{\partial v_{z,[s]}}{\partial t} + \frac{1}{\varrho}\frac{\partial p_{[s]}}{\partial z} = F_{z,[s]},\\
&\frac{\partial v_{y,[s]}}{\partial t} + 2\omega v_{x,[s]} + \frac{1}{\varrho}\frac{\partial p_{[s]}}{\partial y} = F_{y,[s]}, \quad \frac{\partial v_{x,[s]}}{\partial x} + \frac{\partial v_{y,[s]}}{\partial y} + \frac{\partial v_{z,[s]}}{\partial z} = 0,
\end{aligned}
\tag{2.44}
$$

and further

$$
\begin{aligned}
&v_{x,[s]}\cos nx + v_{y,[s]}\cos ny + v_{z,[s]}\cos nz|_S\\
&\qquad = \begin{cases} \dot{Z}\,\overline{\mu}, & s \equiv 1 \pmod k,\\[1ex] \dot{\overline{Z}}\,\mu, & s \equiv -1 \pmod k,\\[1ex] 0, & s \not\equiv \pm 1 \pmod k. \end{cases}
\end{aligned}
\tag{2.45}
$$

These equations coincide with (2.8), however, they have different boundary conditions. Together with (2.40) and (2.41) they give the complete system of the relations.

At first we studied only the real values of $\mathbf{u}'$, X, Y, and p. In this case all functions for $s \equiv -1 \pmod k$ are simply conjugate to functions for $s \equiv 1 \pmod k$.

The solutions for $s \not\equiv \pm 1 \pmod k$ are not interesting for us, because the shell is not involved in these motions. Therefore, later we are going to study only the case when $s = 1$.

8^0. If we suppose that the vector $\mathbf{F}_{[s]}$ on the right side of (2.44) is such that its components are continuous up to the boundary of the domain and have continuous derivatives inside, then this vector can be split into the sum of two terms such that

$$\mathbf{F}_{[s]} = \mathbf{\Phi} + \mathbf{\Psi}, \tag{2.46}$$

where

$$\operatorname{div} \mathbf{\Psi} = 0, \quad \operatorname{rot} \mathbf{\Phi} = 0, \quad \Psi_n|_S = 0. \tag{2.47}$$

Next, we can assume that

$$\mathbf{\Phi} = \operatorname{grad} \Xi + i\varphi\omega \operatorname{grad} \overline{\chi}, \qquad \iint\limits_S \Xi \mu \, dS = 0, \tag{2.48}$$

and the function χ is determined from the conditions

$$\Delta\chi = 0, \qquad \left.\frac{\partial \chi}{\partial n}\right|_S = \mu, \tag{2.49}$$

which are compatible in view of the equality

$$\iint\limits_S \mu \, dS = 0. \tag{2.50}$$

In correspondence to (2.46) and (2.48), the general solution of the problem can be split into two solutions in respect to two cases:

$$\text{(I)} \;\; \Xi = 0, \qquad \text{(II)} \;\; \varphi = 0, \; N^{(0)} = 0, \; \mathbf{\Psi} = 0. \tag{2.51}$$

For case (II) the particular solution of the problem, as is easy to verify directly, has the form

$$\mathbf{v}_{[1]} = 0, \quad Z = 0, \quad p_{[1]} = \varrho \Xi. \tag{2.52}$$

Then, further we can restrict ourselves for case (I).

3 Vector Space and the Operator for a Derivative

1^0. Assume that the function χ has first-order derivatives and $|\operatorname{grad} \chi|^2$ is integrable over V. Let

$$\iiint\limits_V \left(\frac{\partial \chi}{\partial x}\frac{\partial \overline{\chi}}{\partial x} + \frac{\partial \chi}{\partial y}\frac{\partial \overline{\chi}}{\partial y} + \frac{\partial \chi}{\partial z}\frac{\partial \overline{\chi}}{\partial z} \right) dv$$

$$= \iint\limits_S \chi \frac{\partial \overline{\chi}}{\partial n} \, dS = \iint\limits_S \overline{\chi} \frac{\partial \chi}{\partial n} \, dS = 2\varkappa^2. \tag{3.1}$$

Set

$$\dot{Z} = i\omega W. \tag{3.2}$$

Equation (2.40) can be rewritten in this case in the form

$$A_1 \dot{W} - B_1 \omega i W - L\omega i Z + \frac{1}{\omega} N(p_{[1]}) + \frac{2}{\omega} N^{(0)} = 0. \tag{3.3}$$

Let us multiply the first three equations in (2.44) by $\frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial y}, \frac{\partial \chi}{\partial z}$, respectively. Then, we sum these expressions, and integrate over V. Hence,

$$\iiint\limits_V \left(F_{x,[1]} \frac{\partial \chi}{\partial x} + F_{y,[1]} \frac{\partial \chi}{\partial y} + F_{z,[1]} \frac{\partial \chi}{\partial z} \right) dv$$

$$= \iiint\limits_V \left[\frac{\partial}{\partial t} \left(v_{x,[1]} \frac{\partial \chi}{\partial x} + v_{y,[1]} \frac{\partial \chi}{\partial y} + v_{z,[1]} \frac{\partial \chi}{\partial z} \right) - 2\omega v_{y,[1]} \frac{\partial \chi}{\partial x} \right.$$

$$\left. + 2\omega v_{x,[1]} \frac{\partial \chi}{\partial y} + \frac{1}{\varrho} \left(\frac{\partial p_{[1]}}{\partial x} \frac{\partial \chi}{\partial x} + \frac{\partial p_{[1]}}{\partial y} \frac{\partial \chi}{\partial y} + \frac{\partial p_{[1]}}{\partial z} \frac{\partial \chi}{\partial z} \right) \right] dv$$

$$= \iint\limits_S \left[\chi \frac{\partial}{\partial t} (v_{x,[1]} \cos nx + v_{y,[1]} \cos ny + v_{z,[1]} \cos nz) + \frac{1}{\varrho} p_{[1]} \frac{\partial \chi}{\partial n} \right] dS$$

$$-2\omega \iiint\limits_V \left(v_{y,[1]} \frac{\partial \chi}{\partial x} - v_{x,[1]} \frac{\partial \chi}{\partial y} \right) dv \tag{3.4}$$

or, in view of condition $(2.45)^2$,

$$2i\omega \dot{W} \varkappa^2 - \frac{2i}{\varrho} N(p_{[1]}) = 2\omega \iiint\limits_V \left(v_{y,[1]} \frac{\partial \chi}{\partial x} - v_{x,[1]} \frac{\partial \chi}{\partial y} \right) dv + 2i\omega \varphi \varkappa^2. \tag{3.5}$$

Now, let

$$\mathbf{v}_{[1]} = i\omega W \operatorname{grad} \overline{\chi} + \mathbf{v}, \qquad p_{[1]} = i\omega \varrho (\varphi - \dot{W}) \overline{\chi} + p, \tag{3.6}$$

$$\iiint\limits_V \left(\frac{\partial \overline{\chi}}{\partial y} \frac{\partial \chi}{\partial x} - \frac{\partial \overline{\chi}}{\partial x} \frac{\partial \chi}{\partial y} \right) dv = iE. \tag{3.7}$$

Then

$$-\frac{1}{\omega} N(p) = i\varrho\omega W E - i\varrho \iiint\limits_V \left(v_y \frac{\partial \chi}{\partial x} - v_x \frac{\partial \chi}{\partial y} \right) dv. \tag{3.8}$$

Equation (3.8) together with (3.3) gives

[2] S. L. Sobolev uses formulas (2.38), (2.49). – *Ed.*

$$(A_1 + \varrho \varkappa^2)\,\dot{W} - (B_1 + \varrho E)\omega i W - L\omega i Z$$
$$-\varrho i \iiint\limits_V \left(v_x \frac{\partial \chi}{\partial y} - v_y \frac{\partial \chi}{\partial x} \right) dv - \varrho \varphi \varkappa^2 + \frac{2}{\omega} N^{(0)} = 0. \tag{3.9}$$

Returning to the system in question, we obtain

$$\begin{aligned}
&\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_n|_S = 0,\\
&\frac{\partial v_x}{\partial t} - 2\omega v_y + \frac{1}{\varrho}\frac{\partial p}{\partial x} - 2i\omega^2 W \frac{\partial \overline{\chi}}{\partial y} = \Psi_x,\\
&\frac{\partial v_y}{\partial t} + 2\omega v_x + \frac{1}{\varrho}\frac{\partial p}{\partial y} + 2i\omega^2 W \frac{\partial \overline{\chi}}{\partial x} = \Psi_y,\\
&\frac{\partial v_z}{\partial t} + \frac{1}{\varrho}\frac{\partial p}{\partial z} = \Psi_z.
\end{aligned} \tag{3.10}$$

We call the system of equations (3.9), (3.10), and (3.2) the system D. Together with this system we will sometimes consider the system $\overline{D}$.

Applying the operator div to the left sides of equations (3.10), we obtain

$$2\omega \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) + \frac{1}{\varrho} \Delta p = 0. \tag{3.11}$$

Same equations (3.10) give us

$$\begin{aligned}
\left.\frac{\partial p}{\partial n}\right|_S &= -2\varrho\omega(v_x \cos ny - v_y \cos nx)\\
&- 2i\omega^2 \varrho \left(\frac{\partial \overline{\chi}}{\partial x} \cos ny - \frac{\partial \overline{\chi}}{\partial y} \cos nx \right) W.
\end{aligned} \tag{3.12}$$

Condition (3.12) does not contradict (3.11). As we see, the function p is completely defined (up to a constant term) by the definition of $\mathbf{v}$ and W. Setting

$$\begin{aligned}
&\Delta p_0 = 2\omega\varrho \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right), && \left.\frac{\partial p_0}{\partial n}\right|_S = 2\varrho\omega(v_y \cos nx - v_x \cos ny),\\
&\Delta \overline{\chi}_1 = 0, && \left.\frac{\partial \overline{\chi}_1}{\partial n}\right|_S = -2\omega \left(\frac{\partial \overline{\chi}}{\partial y} \cos nx - \frac{\partial \overline{\chi}}{\partial x} \cos ny \right),
\end{aligned} \tag{3.13}$$

we have

$$p = p_0 - i\omega\varrho W \overline{\chi}_1. \tag{3.14}$$

Whence we see that the knowledge of values of $\mathbf{v}$, Z, W, $\mathbf{\Psi}$, $N^{(0)}$, and φ at a moment of time allows us to compute $\dfrac{d\mathbf{v}}{dt}$, $\dfrac{dZ}{dt}$, and $\dfrac{dW}{dt}$.

2^0. Let us introduce new notation. We say that the system $\{Z, W, \mathbf{v}\}$ is the element of the vector space $\{R\}$, where Z, W are complex numbers, $\mathbf{v}$ is a vector function in the domain V and satisfies certain additional conditions. We denote by R this element.

For $\mathbf{v}$ we make two assumptions:

(a)

$$\iiint\limits_V (v_x\overline{v}_x + v_y\overline{v}_y + v_z\overline{v}_z)\,dv = \|v\|^2 < \infty;$$

(b) for any arbitrary function ψ with continuous derivatives in the domain V, the following identity holds:

$$\iiint\limits_V \left(\frac{\partial\psi}{\partial x}v_x + \frac{\partial\psi}{\partial y}v_y + \frac{\partial\psi}{\partial z}v_z\right)\,dv = 0. \tag{3.15}$$

The set of vectors $\mathbf{v}$ satisfying conditions (a) and (b) forms a Hilbert space H. Let us prove the lemma.

Lemma 3.1. *For any element $\mathbf{v}$ from H, there exists another element $\mathbf{v}_\varepsilon$ such that $\|\mathbf{v}_\varepsilon - \mathbf{v}\| < \varepsilon$. The function $\mathbf{v}_\varepsilon$ has continuous derivatives of any order inside V and is equal identically to zero outside some domain V_η, where V_η is the inner domain of V, such that the distance of each of its points to the boundary of V exceeds η.*

Proof. To prove this lemma, it suffices to take as $\mathbf{v}_\varepsilon$ the function

$$\mathbf{v}_\varepsilon(P) = \frac{1}{C}\iiint\limits_{V_{2\eta}} \exp\frac{\eta^2}{r^2-\eta^2}\mathbf{v}(P')\,dv_{P'}, \quad r(P,P') < \eta, \tag{3.16}$$

where P' is a varying point, $r = r(P, P')$ is the distance from P' to P, C is a certain constant. Function (3.16) is the so-called average function for $\mathbf{v}$, and its properties were studied in detail by the author in [2]. □

3^0. The equations of system D can be written in the form

$$\dot{R} - R_0 = iBR, \tag{3.17}$$

where R_0 denotes the element of $\{R\}$ with the components

$$\left(0, \frac{\varrho\varphi\varkappa^2 - (2/\omega)N^{(0)}}{A_1 + \varrho\varkappa^2}, \mathbf{\Psi}\right), \tag{3.18}$$

and B is a linear operator defined for all such elements of this space that $\mathbf{v}$ has continuous derivatives.

For the sake of convenience, let us write out again the formulas defining this operator.

If we let $BR = R_1$, where $R_1 = \{Z_1, w_1, \mathbf{v}_1\}$, then

$$\begin{aligned} v_{1x} &= -2\omega i v_y + \frac{i}{\varrho}\frac{\partial p}{\partial x} + 2\omega^2 W \frac{\partial \overline{\chi}}{\partial y}, \\ v_{1y} &= 2\omega i v_x + \frac{i}{\varrho}\frac{\partial p}{\partial y} - 2\omega^2 W \frac{\partial \overline{\chi}}{\partial x}, \\ v_{1z} &= \frac{i}{\varrho}\frac{\partial p}{\partial z}, \end{aligned} \tag{3.19}$$

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_n|_S = 0, \tag{3.20}$$

$$(A_1 + \varrho\varkappa^2) w_1 = (B_1 + \varrho E)\omega W + L\omega Z + \varrho \iiint\limits_V \left(v_x \frac{\partial \chi}{\partial y} - v_y \frac{\partial \chi}{\partial x} \right) dv, \tag{3.21}$$

$$Z_1 = \omega W. \tag{3.22}$$

Besides equation (3.17) we also consider the corresponding homogeneous equation

$$\dot{R} = iBR. \tag{3.23}$$

Let us prove that the operator B is bounded, and, hence, it can be extended to the whole space. From equations (3.19) it follows that

$$4\omega^2(v_x\overline{v}_x + v_y\overline{v}_y) - 4i\omega^3\overline{W}\left(v_x\frac{\partial\chi}{\partial x} + v_y\frac{\partial\chi}{\partial y}\right) + 4i\omega^3 W\left(\overline{v}_x\frac{\partial\overline{\chi}}{\partial x} + \overline{v}_y\frac{\partial\overline{\chi}}{\partial y}\right)$$
$$+4\omega^4 W\overline{W}\left(\frac{\partial\chi}{\partial x}\frac{\partial\overline{\chi}}{\partial x} + \frac{\partial\chi}{\partial y}\frac{\partial\overline{\chi}}{\partial y}\right) = (v_{1x}\overline{v}_{1x} + v_{1y}\overline{v}_{1y} + v_{1z}\overline{v}_{1z})$$
$$+\frac{1}{\varrho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \overline{p}}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial \overline{p}}{\partial y} + \frac{\partial p}{\partial z}\frac{\partial \overline{p}}{\partial z}\right) + \frac{1}{\varrho}\left[\left(v_{1,x}\frac{\partial \overline{p}}{\partial x} + v_{1,y}\frac{\partial \overline{p}}{\partial y} + v_{1,z}\frac{\partial \overline{p}}{\partial z}\right)\right.$$
$$\left. - \left(\overline{v}_{1,x}\frac{\partial p}{\partial x} + \overline{v}_{1,y}\frac{\partial p}{\partial y} + \overline{v}_{1,z}\frac{\partial p}{\partial z}\right)\right].$$

Hence, integrating over V,

$$\|\mathbf{v}_1\|^2 + \frac{1}{\varrho^2}\iiint\limits_V (\operatorname{grad} p \cdot \operatorname{grad} \overline{p})\, dv = 4\omega^2 \iiint\limits_V (v_x\overline{v}_x + v_y\overline{v}_y)\, dv$$
$$-4i\omega^3\overline{W}\iiint\limits_V \left(v_x\frac{\partial\chi}{\partial x} + v_y\frac{\partial\chi}{\partial y}\right) dv + 4i\omega^3 W\iiint\limits_V \left(\overline{v}_x\frac{\partial\overline{\chi}}{\partial x} + \overline{v}_y\frac{\partial\overline{\chi}}{\partial y}\right) dv$$
$$+4\omega^4 W\overline{W}\iiint\limits_V \left(\frac{\partial\chi}{\partial x}\frac{\partial\overline{\chi}}{\partial x} + \frac{\partial\chi}{\partial y}\frac{\partial\overline{\chi}}{\partial y}\right) dv. \tag{3.24}$$

If we assume that $W\overline{W} = \|W\|^2$, $Z\overline{Z} = \|Z\|^2$, then, using the Schwarz–Bunyakovskii inequality, from (3.24) we have

$$\|\mathbf{v}_1\|^2 \leq 4\omega^2[\|\mathbf{v}\|^2 + 2\sqrt{2}\varkappa\omega\|\mathbf{v}\|\,\|W\| + 2\varkappa^2\omega^2\|W\|^2],$$

or

$$\|\mathbf{v}_1\| \leq 2\omega[\|\mathbf{v}\| + \sqrt{2}\varkappa\|W\|]. \tag{3.25}$$

If we approach now to the given vector R by using R_ε such that $\mathbf{v}_\varepsilon$ has continuous derivatives, then for R_ε we can evaluate $iBR_\varepsilon = R_{\varepsilon_1}$.

Making ε_1, ε_2 go to zero, we can see that $\|\mathbf{v}_{1,\varepsilon_1} - \mathbf{v}_{1,\varepsilon_2}\| \to 0$, and, hence, there exists the limit $\mathbf{v}_\varepsilon$, as $\varepsilon \to 0$, from H, i.e., $\lim\limits_{\varepsilon\to 0} iBR_\varepsilon \in \{R\}$, as required.

4^0. Let us compose the complex bilinear form on two elements $R^{(1)}$ and $R^{(2)}$ of the space $\{R\}$,

$$Q(R^{(1)}, R^{(2)}) = (A_1 + \varrho\varkappa^2)w^{(1)}\overline{w}^{(2)} + L\omega Z^{(1)}\overline{Z}^{(2)}$$
$$+\frac{\varrho}{2\omega^2}\iiint\limits_V \left(v_x^{(1)}\overline{v}_x^{(2)} + v_y^{(1)}\overline{v}_y^{(2)} + v_z^{(1)}\overline{v}_z^{(2)}\right) dv. \tag{3.26}$$

Obviously,

$$Q(R^{(1)}, R^{(2)}) = Q(\overline{R}^{(2)}, \overline{R}^{(1)}), \quad Q(\lambda R^{(1)}, R^{(2)}) = \lambda Q(R^{(1)}, R^{(2)}). \tag{3.27}$$

The form $Q(R^{(1)}, R^{(2)})$ is said to be the inner product of $R^{(1)}$ and $R^{(2)}$. Let us prove the formula

$$Q(BR^{(1)}, R^{(2)}) = Q(R^{(1)}, BR^{(2)}). \tag{3.28}$$

It is convenient to say that the operator B is generalized Hermitian in a sense that it is defined on the entire space $\{R\}$ and satisfies condition (3.28). By condition, we have

$$Q(BR^{(1)}, R^{(2)}) = L\omega w^{(1)}\overline{Z}^{(2)} + L\omega Z^{(1)}\overline{w}^{(2)} + (B_1 + \varrho E)\omega w^{(1)}\overline{w}^{(2)}$$
$$+\varrho\overline{w}^{(2)}\iiint\limits_V \left(\frac{\partial\chi}{\partial y}v_x^{(1)} - \frac{\partial\chi}{\partial x}v_y^{(1)}\right) dv + \varrho\overline{w}^{(1)}\iiint\limits_V \left(\frac{\partial\overline{\chi}}{\partial y}\overline{v}_x^{(2)} - \frac{\partial\overline{\chi}}{\partial x}\overline{v}_y^{(2)}\right) dv$$
$$+\frac{\varrho}{2\omega^2}\iiint\limits_V \left\{\left[\overline{v}_x^{(2)}\left(2\omega v_y^{(1)} - \frac{1}{\varrho}\frac{\partial p^{(1)}}{\partial x}\right) + \overline{v}_y^{(2)}\left(-2\omega v_x^{(1)} - \frac{1}{\varrho}\frac{\partial p^{(1)}}{\partial y}\right)\right.\right.$$
$$\left. -\overline{v}_z^{(2)}\frac{1}{\varrho}\frac{\partial p^{(1)}}{\partial z}\right] + \left[v_x^{(1)}\left(2\omega\overline{v}_y^{(2)} - \frac{1}{\varrho}\frac{\partial\overline{p}^{(2)}}{\partial x}\right)\right.$$
$$\left.\left. +v_y^{(1)}\left(-2\omega\overline{v}_x^{(2)} - \frac{1}{\varrho}\frac{\partial\overline{p}^{(2)}}{\partial y}\right) - v_z^{(1)}\frac{1}{\varrho}\frac{\partial\overline{p}^{(2)}}{\partial z}\right]\right\} dv, \tag{3.29}$$

and in view of (3.15)

$$Q(BR^{(1)}, R^{(2)}) = L\omega(w^{(1)}\overline{Z}^{(2)} + \overline{w}^{(2)}Z^{(1)}) + (B_1 + \varrho E)\omega w^{(1)}\overline{w}^{(2)}$$

$$+\varrho\left[\overline{w}^{(2)} \iiint\limits_V \left(\frac{\partial\chi}{\partial y}v_x^{(1)} - \frac{\partial\chi}{\partial x}v_y^{(1)}\right) dv \right.$$

$$\left. +\overline{w}^{(1)} \iiint\limits_V \left(\frac{\partial\overline{\chi}}{\partial y}\overline{v}_x^{(2)} - \frac{\partial\overline{\chi}}{\partial x}\overline{v}_y^{(2)}\right) dv\right]. \tag{3.30}$$

Therefore, $Q(BR^{(1)}, R^{(2)}) = Q(\overline{B}\,\overline{R}^{(2)}, \overline{R}^{(1)})$, i.e., formula (3.28) is proved.

5^0. Proved formulas (3.28) and (3.25) imply a number of important consequences. First, the convergence of the series

$$e^{iBt}R^{(0)} = \sum_{k=0}^{\infty} \frac{t^k(iB)^k}{k!}R^{(0)} \tag{3.31}$$

follows from them.

Indeed, let us agree to denote by the symbol $\|R\|$ the value

$$\|R\| = \max\{\|w\|, \|Z\|, \|\mathbf{v}\|\} \tag{3.32}$$

and call it the norm R.

Each term of series (3.31) is defined for any element $R^{(0)}$. From inequality (3.25) it follows that it is possible to indicate a number M such that $\|BR\| \leq M\|R\|$; the operator satisfying this condition is called, as is known, bounded.

From the boundedness of the operator B it follows that

$$\|(iB)^kR\| \leq M^k\|R\|, \tag{3.33}$$

and, hence, the convergence of series (3.31).

The sum of series (3.31) is an analytic function of the variable t, satisfying equation (3.23) and the initial conditions $R = R^{(0)}$ for $t = 0$.

Let us note that from (3.30) we have

$$\frac{d}{dt}Q(e^{iBt}R_1, e^{iBt}R_2) = 0 \quad \text{or} \quad Q(e^{iBt}R_1, e^{iBt}R_2) = \text{const}. \tag{3.34}$$

The behavior of the unknown solution depends substantially on the sign of the constant L. Let us examine the cases when $L > 0$ and $L < 0$, separately.

In the case $L = 0$ the system splits, because in this case W, $\mathbf{v}$ are not at all related to Z.

In the first case, the space $\{R\}$ forms a complex Hilbert space with the inner product $Q(R_1, R_2)$, since from $Q(R, R) = 0$ the equality $R = 0$ follows.

In this case the operator B is Hermitian. For this case, the existing detailed spectral theory of operators in the Hilbert space gives the complete answer on all the questions we can state. Without presenting this theory in detail, let us just note the main fact that the motion in this case is stable; indeed, in view of (3.34) the value $Q(\exp i\beta t R_1, \exp i\beta t R_2)$ remains bounded, and therefore for any initial conditions, the values $\|w\|$, $\|Z\|$, $\|\mathbf{v}\|$ for function (3.31) cannot increase without bound, and if the initial value $Q(R_1, R_2)$ is sufficiently small, then these values will be arbitrarily small as well.

4 The Study of the Resolvent in the Non-Hilbert Case

1^0. It remains to consider the case $L < 0$. To find the solution of our problem explicitly, let us again use the spectral theory of operators.

Let us consider the operator equation

$$(\lambda E - B)R = R_0, \tag{4.1}$$

where R_0 is an arbitrary element of $\{R\}$, and λ is a complex number.

We prove that this equation has a solution in the plane of the complex variable λ everywhere except on the interval $-2\omega < \lambda < 2\omega$ of the real axis, and possibly at certain other isolated values, i.e.,

$$R = \Gamma_\lambda R_0, \tag{4.2}$$

where the right side is meromorphic everywhere in the plane of λ outside the interval $|\lambda| < 2\omega$ of the real axis.

The operator Γ_λ is bounded and $\|\Gamma_\lambda\|$ has a finite estimate for any domain not containing the values mentioned.

The quadratic form

$$J(R_1, R_2) = Q(R_1, \overline{R}_2), \tag{4.3}$$

symmetric with respect to the arguments $Q(R_1, \overline{R}_2) = Q(R_2, \overline{R}_1)$, is called the inner product of two elements R_1 and R_2.

Two elements R_1 and R_2 are called orthogonal in the Fredholm sense if

$$J(R_1, R_2) = 0.$$

Operator A^* is called adjoint to A if

$$J(AR_1, R_2) = J(R_1, A^* R_2).$$

We say that for equation (4.1) the Fredholm theory is valid in a certain domain O of the plane λ, if:

Condition 4.1. For any λ from O an alternative holds:
either (a): equation (4.1) has a unique solution for any R_0,
or (b): the corresponding homogeneous equation has nontrivial solution.
Case (b) can occur only for isolated values of λ.

Condition 4.2. For any λ the number of linearly independent solutions of the equations

$$(\lambda E - B)R = 0, \tag{4.4}$$

$$(\lambda E - B^*)R = 0 \tag{4.5}$$

is finite and the same.

Condition 4.3. The solutions of equation (4.4) corresponding to some λ from O are orthogonal in the Fredholm sense to all solutions of the adjoint homogeneous equations corresponding to another λ from O.

Condition 4.4. The necessary and sufficient condition for equation (4.1) to be solvable is that R_0 be orthogonal in the Fredholm sense to all solutions of homogeneous adjoint equation (4.5) with the same λ.

The following theorem proved by J. Radon holds (we state it in somewhat modified form).

Theorem 4.1. *If the given operator B can be represented in the form $B = B_1 + B_2$, where the resolvent B_1 is regular in a certain domain O of the plane λ, and the operator B_2 is compact, then the Fredholm theory holds for B in the domain O.*

Remark 4.1. Condition 4.3 and the first part of Condition 4.4 (the necessity of the orthogonality condition) are valid for all values independently of whether λ belongs to the domain O or not.

Remark 4.2. Due to J. Radon, if the operator B_1 has a bounded resolvent in some norm, then the operator B in any subdomain of O, not containing isolated singularities, also has a bounded resolvent.

Let us introduce the operator P_1 of the projection of the vector R into the subspace H. The components of P_1B are $(0, 0, \mathbf{v})$. Let us express the operator B in the form

$$B = P_1BP_1 + [(E - P_1)B + P_1B(E - P_1)] = B_1 + B_2. \tag{4.6}$$

The operator $B_1 = P_1BP_1$ is bounded, moreover, $\|B_1\| \leq 2\omega$, and Hermitian, which can be seen directly from formula (3.28). Hence its resolvent is regular (and, thus, bounded) for $|\lambda| > 2\omega$ and everywhere outside the real axis.

The operator $B_2 = B - B_1$ is compact, because the set of its values is of finite dimension: if the image of its action on some vector is $(Z', w', \mathbf{v}')$, then[3]

$$\mathbf{v}' = \alpha \mathrm{grad}\, \overline{\chi}_1 + \beta \mathrm{grad}\, \overline{\chi}.$$

By the Radon theorem, outside the interval $|\lambda| < 2\omega$ of the real axis the Fredholm theory is valid for B, as required. Outside this interval and the singularities the resolvent is bounded.

It is also useful to note that, by the inequality $\|BR\| \le M\|R\|$, the series for the resolvent

$$\varGamma_\lambda = \frac{E}{\lambda} + \frac{B}{\lambda^2} + \frac{B^2}{\lambda^3} + \cdots \tag{4.7}$$

converges for $|\lambda| > M$. Hence, in a neighborhood of the point at infinity, $\varGamma_\lambda$ is a regular function vanishing at $\lambda = \infty$.

2^0. Let us apply the Fredholm theory to the study of equation (4.1). First, let us see how we can represent the operator adjoint to B. By definition,

$$J(BR_1, R_2) = J(R_1, B^* R_2). \tag{4.8}$$

Using expression (4.3) for J in terms of Q, we obtain

$$Q(BR_1, \overline{R}_2) = Q(R_1, \overline{B^* R_2}).$$

On the other side, $Q(BR_1, \overline{R}_2) = Q(R_1, B\overline{R}_2)$; hence,

$$B\overline{R}_2 = \overline{B^* R_2} \quad \text{or} \quad B^* R_2 = \overline{B\overline{R}_2}. \tag{4.9}$$

Suppose that for certain λ_0 the equality $BR_0 = \lambda_0 R_0$ is valid, i.e., λ_0 is an eigenvalue of the operator B, and R_0 is the corresponding eigenvector. By the above proved, λ_0 is also the eigenvalue for the adjoint equation, i.e., there exists a vector R_1 such that

$$B^* R_1 = \lambda_0 R_1. \tag{4.10}$$

By the definition of the operator B^* we have $\overline{B\overline{R}_1} = \lambda_0 R_1$; hence, conjugating both parts, we have

$$B\overline{R}_1 = \overline{\lambda}_0 \overline{R}_1. \tag{4.11}$$

Therefore, in this case the vector $\overline{R}_1$ is also the eigenvector of the operator B corresponding to the eigenvalue $\overline{\lambda}_0$.

The opposite statement is proved in absolutely the same way. $\overline{R}_0$ is the eigenvector of the operator B^*, corresponding to the number $\overline{\lambda}_0$. Indeed,

$$\overline{BR_0} = \overline{\lambda}_0 \overline{R}_0 = B^* \overline{R}_0,$$

as required. However, from this the orthogonality of $\overline{R}_0$ to all eigenvectors of the operator B, except those corresponding to the eigenvalue $\overline{\lambda}_0$, follows, i.e.,

[3] See formulas (3.14), (3.19). – *Ed.*

$$J(R_i, \overline{R}_0) = Q(R_i, R_0) = 0 \quad \text{for} \quad \lambda_i \neq \overline{\lambda}_0. \tag{4.12}$$

If λ_0 is not real, then

$$Q(R_0, R_0) = 0. \tag{4.13}$$

3^0. We now prove the following assertion.

Lemma 4.1. *There exist no two linearly independent vectors R_1 and R_2 such that*

$$Q(R_1, R_1) = 0, \quad Q(R_2, R_2) = 0, \quad Q(R_1, R_2) = 0. \tag{4.14}$$

Proof. To prove it, let us compose the expression

$$\alpha\overline{\alpha}Q(R_1, R_1) + (\alpha\overline{\beta} + \beta\overline{\alpha})Q(R_1, R_2) + \beta\overline{\beta}Q(R_2, R_2) = 0.$$

The left side of this expression can be transformed into

$$Q(\alpha R_1 + \beta R_2, \alpha R_1 + \beta R_2). \tag{4.15}$$

Hence form (4.15) is identically zero for any α and β. Obviously, $Z_1 \neq 0$ and $Z_2 \neq 0$, otherwise from (4.14) it would follow that one of the vectors R_1 and R_2 is identically zero. Let $\alpha = Z_2$, $\beta = -Z_1$, then $\alpha R_1 + \beta R_2$ has the component $Z = 0$. In this case from (4.15) we have that $\alpha R_1 + \beta R_2$ becomes identically zero. Hence R_1 and R_2 are linearly dependent. □

The theorem follows immediately from this lemma.

4^0. Let us study spectral properties of the operator B.

Theorem 4.2. *The operator B cannot have more than two complex conjugate eigenvalues.*

Proof. Let λ_1 and λ_2 be eigenvalues, $\lambda_1 \neq \lambda_2$, and let R_1 and R_2 be the eigenvectors, respectively. Then,

$$Q(R_1, R_2) = 0, \quad Q(R_1, R_1) = 0, \quad Q(R_2, R_2) = 0.$$

Hence R_1 and R_2 are linearly dependent. However, in this case λ_1 cannot differ from λ_2. □

Let us study now the question about multiple characteristic numbers. Consider the equations

$$(\lambda E - B)R = 0, \quad (\lambda E - B)^2 R = 0, \quad \ldots, \quad (\lambda E - B)^k R = 0. \tag{4.16}$$

The number of linearly independent solutions of all equations (4.16) is said to be the multiplicity of an eigenvalue.

Theorem 4.3. *The multiplicity of a complex eigenvalue of the operator B cannot exceed* 1*.*

Proof. Indeed, if the multiplicity of a complex characteristic number is 2, then either there exist two linearly independent vectors, or the equation $(\lambda E - B)^2 R = 0$ has the solution R_1, different from the solution R_0.

The first is impossible. However, if the second holds, then $(\lambda E - B)R_1$ is a solution of the equation $(\lambda E - B)R = 0$, and, hence,

$$(\lambda E - B)R_1 = \alpha R_0.$$

Let R_2 be the solution of the adjoint equation. Then by the Fredholm solvability condition (4.12),

$$J(R_0, R_2) = Q(R_0, \overline{R}_2) = 0. \tag{4.17}$$

By (4.17), R_0 and $\overline{R}_2$ satisfy the conditions $Q(R_0, \overline{R}_2) = 0,\ Q(\overline{R}_2, \overline{R}_2) = 0$. We come to a contradiction, thus proving our statement. □

Theorem 4.4. *The operator B cannot have more than one nonsimple real eigenvalue.*

Proof. Let

$$(\lambda E - B)R_1 = \alpha R_0, \qquad (\lambda E - B)R_0 = 0, \tag{4.18}$$

and λ is real. Obviously, $\overline{R}_0$ is the solution of the adjoint equation. The necessary condition of solvability of the first of equations (4.18), valid, as mentioned before, also for real λ, yields

$$J(R_0, \overline{R}_0) = Q(R_0, R_0) = 0.$$

If we combine this with the condition $Q(R_0', R_0) = 0$, where R_0' is an eigenvector corresponding to another eigenvalue, then $Q(R_0', R_0')$ cannot be zero. The theorem is proved. □

Theorem 4.5. *The equation $(\lambda E - B)^4 R = 0$ cannot have solutions different from the solutions of $(\lambda E - B)^3 R = 0$.*

Proof. Indeed, suppose that

$$\begin{aligned} &(\lambda E - B)R_3 = R_2, \qquad (\lambda E - B)R_2 = R_1,\\ &(\lambda E - B)R_1 = R_0, \qquad (\lambda E - B)R_0 = 0. \end{aligned} \tag{4.19}$$

Then the equality holds

$$(\lambda E - B)^2 R_3 = R_1. \tag{4.20}$$

The vector $\overline{R}_1$ is a solution of the equation adjoint to $(\lambda E - B)^2 R = 0$. The Fredholm orthogonality conditions give

$$J(R_0, \overline{R}_0) = J(R_1, \overline{R}_0) = J(R_1, \overline{R}_1) = 0$$

or

$$Q(R_0, R_0) = Q(R_1, R_0) = Q(R_1, R_1) = 0,$$

which is impossible. The theorem is proved. □

5^0. From these theorems the corollaries on the impossibility of existence of certain special solutions of the equation

$$\dot{R} = iBR$$

follow.

Indeed, for each eigenvalue λ_0 and eigenvector R_0 there is, as is easy to see, a particular solution of this equation of the form

$$R = e^{i\lambda_0 t} R_0, \tag{4.21}$$

and for each system of equations

$$(\lambda E - B)R_k = R_{k-1}, \ (\lambda E - B)R_{k-1} = R_{k-2}, \ \ldots, \ (\lambda E - B)R_0 = 0, \tag{4.22}$$

there is a particular solution of the form

$$R = e^{i\lambda_0 t}\left(\frac{t^k}{k!}R_0 - \frac{t^{k-1}}{(k-1)!}R_1 + \cdots + (-1)^k R_k\right). \tag{4.23}$$

Thus, the following is proved:

I. The solutions of form (4.21) can exist for no more than two complex conjugated values, one for each.

II. The solutions of form (4.23) cannot exist for complex λ_0.

III. The solutions of form (4.23) cannot exist more than for one real λ_0.

IV. The solutions of form (4.23) cannot have $k > 2$.

5 Representation of a Solution in Terms of the Resolvent

The operator e^{iBt} constructed in the form of series (3.31) can be written as follows:

$$e^{iBt}R^{(0)} = -\frac{1}{2\pi i}\int\limits_C e^{i\lambda t}\Gamma_\lambda R^{(0)}\, d\lambda, \quad \Gamma_\lambda = (\lambda E - B)^{-1}, \tag{5.1}$$

where C is a sufficiently large contour containing all singularities of Γ_λ.

Let us prove that the required solution of equation (3.17), satisfying the condition $R = R^{(0)}$ at $t = 0$, can be represented in the form

$$R(t) = e^{iBt}R^{(0)} + \int\limits_0^t e^{iB(t-t_1)}R_0(t_1)\, dt_1. \tag{5.2}$$

First, let us prove the lemma.

Lemma 5.1. *The operator* $-\frac{1}{2\pi i}\int\limits_C \Gamma_\lambda\, d\lambda$ *is an identity operator.*

Proof. We have

$$(\lambda E - B)\varGamma_\lambda R = R \quad \text{or} \quad \lambda\varGamma_\lambda R - B\varGamma_\lambda R = R. \tag{5.3}$$

Integrating the last equality over the contour C, we obtain

$$-\frac{1}{2\pi i}\int\limits_C \varGamma_\lambda R\,d\lambda + \frac{1}{2\pi i}\int\limits_C \frac{B\varGamma_\lambda}{\lambda}R\,d\lambda = -\frac{1}{2\pi i}\int\limits_C \frac{d\lambda}{\lambda}R = R. \tag{5.4}$$

To establish the lemma, it suffices to show that

$$\int\limits_C \frac{\varGamma_\lambda}{\lambda}R\,d\lambda = 0. \tag{5.5}$$

This equality immediately follows from the fact that, near $\lambda = \infty$, $\varGamma_\lambda$ is a regular function tending to zero. □

Representation (5.2) follows directly from Lemma 5.1.

Indeed, assuming $t = 0$, we have $R(0) = R^{(0)}$.

Let us also note that

$$\frac{d}{dt}e^{iBt} = -\frac{1}{2\pi i}\int\limits_C i\lambda e^{i\lambda t}\varGamma_\lambda\,d\lambda. \tag{5.6}$$

Therefore,

$$\frac{dR}{dt} - iBR = -\frac{1}{2\pi}\int\limits_C e^{i\lambda t}(\lambda E - B)\varGamma_\lambda R^{(0)}\,d\lambda$$

$$-\int\limits_0^t \frac{1}{2\pi}\int\limits_C e^{i\lambda(t-t_1)}(\lambda E - B)\varGamma_\lambda R_0(t_1)\,d\lambda\,dt_1 + R_0(t).$$

However $(\lambda E - B)\varGamma_\lambda = E$. Hence,

$$\frac{dR}{dt} - iBR = \frac{-1}{2\pi}\int\limits_C e^{i\lambda t}\,d\lambda R^{(0)}$$

$$-\int\limits_0^t \left\{\frac{1}{2\pi}\int\limits_C e^{i\lambda(t-t_1)}\,d\lambda\right\} R_0(t_1)\,dt_1 + R_0(t) = R_0(t),$$

which is required.

Let us pass now to the study of the question about stability of the motion.

We recall a well-known fact from the Fredholm theory.

At the point λ_0, which is an eigenvalue of the operator B of multiplicity 1, the expansion of the function $\varGamma_\lambda R_0$ in powers of $\lambda - \lambda_0$ has the form

$$\varGamma_\lambda R_0 = \frac{J(R_0, R_1)}{\lambda - \lambda_0} R_* + \cdots, \tag{5.7}$$

where R_* is an eigenvector, and R_1 is an eigenvector of the adjoint operator such that

$$J(R_*, R_1) = 1. \tag{5.8}$$

The omitted terms in formula (5.7) contain nonnegative powers of $(\lambda - \lambda_0)$.

The solution of the problem on vibrations of the top with a cavity is called roughly unstable with the exponent ε_0, if Z increases as $e^{\varepsilon_0 t}$ or faster. If, on the contrary, Z increases slower than $e^{\varepsilon_0 t}$, then we say that the motion up to exponent ε_0 is roughly stable. Let us now prove the theorem.

Theorem 5.1. *If the resolvent $\varGamma_\lambda$ has a pair of complex conjugated poles at the points*

$$\lambda = \sigma \pm i\tau, \tag{5.9}$$

and $R_0(t)$ increases slower than $e^{\varepsilon_0 t}$, then the solution of the problem is roughly stable up to any exponent $\varepsilon_0 > \tau$ and roughly unstable with the exponent $\varepsilon_0 \leq \tau$.

Proof. For the proof let us come back to our formula (5.2).

The integral over the contour C can be transformed in this case into an integral over the contour C_1, located arbitrarily close to the real axis, by adding two residues at the poles of the resolvent.

Thus, we obtain

$$e^{iBt} R^{(0)} = -\frac{1}{2\pi i} \int\limits_{C_1} e^{i\lambda t} \varGamma_\lambda R^{(0)}\, d\lambda + e^{i\lambda_0 t} J(R^{(0)}, R_1) R_* + e^{i\overline{\lambda}_0 t} J(R^{(0)}, \overline{R}_*) \overline{R}_1.$$

By estimating each term, we have

$$\left\| -\frac{1}{2\pi i} \int\limits_{C_1} e^{i\lambda t} \varGamma_\lambda R^{(0)}\, d\lambda \right\| \leq e^{\tau t} \frac{1}{2\pi} \int\limits_{C_1} \|\varGamma_\lambda R^{(0)}\|\, d\lambda,$$

however,

$$\|\varGamma_\lambda R^{(0)}\| \leq K_{C_1} \|R^{(0)}\|,$$

where the constant K_{C_1} depends possibly on the contour C_1. Similarly, one also estimates the second term of formula (5.2). The theorem is proved. $\square$

Chapter 2 Top with Cavity Shaped as an Ellipsoid of Rotation

6 The Derivation of Main Relations

1^0. Let us return to system D and consider its properties in the particular case when the domain filled with fluid is an ellipsoid of rotation with the axes a, a, and c.

In this case we have the parametric equation of the surface S

$$z = c\sin\varphi + l_2, \qquad x + iy = a\cos\varphi e^{i\theta}. \tag{6.1}$$

Here l_2 is the z coordinate of the center of gravity for the ellipsoid. Let us also compute $\cos nx$ and $\cos ny$. We obtain

$$\cos nx + i\cos ny = \frac{c\cos\varphi e^{i\theta}}{\sqrt{a^2\sin^2\varphi + c^2\cos^2\varphi}}, \tag{6.2}$$

$$\cos nz = \frac{a\sin\varphi}{\sqrt{a^2\sin^2\varphi + c^2\cos^2\varphi}}, \tag{6.3}$$

and, hence[4],

$$\mu = (c^2 - a^2)e^{i\theta}\frac{\sin\varphi\cos\varphi}{\sqrt{a^2\sin^2\varphi + c^2\cos^2\varphi}} + l_2 c e^{i\theta}\frac{\cos\varphi}{\sqrt{a^2\sin^2\varphi + c^2\cos^2\varphi}}. \tag{6.4}$$

Finally, we are ready to compute the function χ. Indeed, let

$$\chi = re^{i\theta}[m(z - l_2) + l_2] = (x + iy)[(m(z - l_2) + l_2)],$$

$$\overline{\chi} = (x - iy)\left(mz + \frac{2a^2}{c^2 + a^2}l_2\right), \qquad m = \frac{c^2 - a^2}{c^2 + a^2}. \tag{6.5}$$

It is not difficult to get convinced that

$$\left.\frac{d\chi}{dn}\right|_S = \mu. \tag{6.6}$$

Indeed,

$$\frac{\partial\chi}{\partial x} = (z - l_2)m + l_2, \quad \frac{\partial\chi}{\partial y} = i[(z - l_2)m + l_2], \quad \frac{\partial\chi}{\partial z} = (x + iy)m,$$

$$\left.\frac{\partial\chi}{\partial n}\right|_S = (z - l_2)m(\cos nx + i\cos ny)$$

---

[4] By definition, $\mu = z(\cos nx + i\cos ny) - (x + iy)\cos nz$. – *Ed.*

$$+(x+iy)m\cos nz + l_2(\cos nx + i\cos ny)\Big|_S = \mu.$$

Let us compute the values

$$(iB)^k(Z, w, 0). \tag{6.7}$$

In view of the fact that $\dfrac{\partial^2 \chi}{\partial z^2} = 0$, the result of computations is an element of a certain three-dimensional subspace $\{R\}^3$ independent of A, C_1, K, and ω. Let us establish this proposition in a more general form.

Consider the vector $\mathbf{v}^*$,

$$v_x^* = (z-l_2)a^2 m\xi, \quad v_y^* = -i(z-l_2)a^2 m\xi, \quad v_z^* = -(x-iy)c^2 m\xi, \tag{6.8}$$

where m is defined in (6.5). It is easy to verify that

$$\operatorname{div}\mathbf{v}^* = 0, \quad v_n|_S = 0. \tag{6.9}$$

Relations (6.9) are verified elementary, for example,

$$v_n|_S = m[a^2(z-l_2)(\cos nx - i\cos ny) - c^2(x-iy)\cos nz]\xi|_S = 0.$$

We define the subspace $\{R\}^3$ as the subspace consisting of the elements of the form $(Z, w, i\omega\xi\mathbf{v}^*)$.

Let v_1 be an element of this subspace. Let us compute iBv_1.

Substituting it into equation (3.10), we obtain

$$i\omega(z-l_2)a^2 m(\dot{\xi} + 2\omega i\xi) + \frac{1}{\varrho}\frac{\partial p}{\partial x} - 2\omega^2 w[(z-l_2)m + l_2] = 0,$$

$$\omega(z-l_2)a^2 m(\dot{\xi} + 2\omega i\xi) + \frac{1}{\varrho}\frac{\partial p}{\partial y} + 2i\omega^2 w[(z-l_2)m + l_2] = 0,$$

$$-i\omega(x-iy)c^2 m\,\dot{\xi} + \frac{1}{\varrho}\frac{\partial p}{\partial z} = 0.$$

Hence p must be of the form

$$p = -i(x-iy)(z-l_2)\omega\varrho m(a^2\,\dot{\xi} + 2\omega i a^2\xi + 2\omega i w) + 2\omega^2\varrho w l_2(x-iy)$$

$$= i(x-iy)(z-l_2)\omega c^2\varrho m\,\dot{\xi} + 2\omega^2\varrho w l_2(x-iy). \tag{6.10}$$

This is possible if

$$(c^2+a^2)\,\dot{\xi} + 2\omega i a^2\xi + 2\omega i w = 0. \tag{6.11}$$

Let us evaluate now the quantities

$$\varkappa^2, \quad E, \quad \text{and} \quad \iiint\limits_V \left(v_x^*\frac{\partial\chi}{\partial y} - v_y^*\frac{\partial\chi}{\partial x}\right) dv.$$

First, we obtain

$$\iiint\limits_V (z-l_2)^2 r\,dz\,dr\,d\theta = 2\pi \int\limits_0^c z^2 a^2 \left(1-\frac{z^2}{c^2}\right) dz$$

$$= 2\pi \int\limits_0^c \left(z^2a^2 - \frac{a^2 z^4}{c^2}\right) dz = \frac{4}{15}\pi a^2 c^3,$$

$$\iiint\limits_V r^2\, r\,dz\,dr\,d\theta = \pi \int\limits_0^c a^4 \left(1-\frac{z^2}{c^2}\right)^2 dz = \frac{8}{15}\pi a^4 c.$$

Hence,

$$C_2 = \frac{8}{15}\pi a^4 c\varrho. \tag{6.12}$$

In view of

$$\iiint\limits_V (z-l_2)\,dv = 0,$$

we obtain

$$\iiint\limits_V z^2\, r\,dz\,dr\,d\theta$$

$$= \iiint\limits_V [(z-l_2)^2 + 2l_2(z-l_2) + l_2^2]\,dv = \frac{4}{15}\pi a^2 c^3 + l_2^2 V, \tag{6.13}$$

$$A_2 = l_2^2 M_2 + \frac{4}{15}\pi\varrho a^2 c(a^2+c^2) = l_2^2 M_2 + A_2^{(0)}, \tag{6.14}$$

$$A_2 - C_2 = l_2^2 M_2 + \frac{4}{15}\pi\varrho a^2 c(c^2-a^2).$$

Next,

$$2\varkappa^2 = \iiint\limits_V \left(\frac{\partial\chi}{\partial x}\frac{\partial\overline{\chi}}{\partial x} + \frac{\partial\chi}{\partial y}\frac{\partial\overline{\chi}}{\partial y} + \frac{\partial\chi}{\partial z}\frac{\partial\overline{\chi}}{\partial z}\right) dv = m^2 \iiint\limits_V [2(z-l_2)^2 + r^2]\,dv$$

$$+2l_2^2 \iiint\limits_V dv + 4l_2 m \iiint\limits_V (z-l_2)\,dv = 2l_2^2 V + \frac{8}{15}\pi\varrho a^2 c\frac{(c^2-a^2)^2}{c^2+a^2},$$

i.e.,

$$\varkappa^2\varrho = l_2^2 M_2 + \frac{4}{15}\pi\varrho a^2 c\frac{(c^2-a^2)^2}{c^2+a^2}, \tag{6.15}$$

$$E = -i \iiint\limits_V \left(\frac{\partial\chi}{\partial x}\frac{\partial\overline{\chi}}{\partial y} - \frac{\partial\chi}{\partial y}\frac{\partial\overline{\chi}}{\partial x}\right) dv = -2 \iiint\limits_V m^2 (z-l_2)^2 r\,dz\,dr\,d\theta$$

$$-2l_2^2 \iiint\limits_V dv = -\frac{8}{15}\pi m^2 a^2 c^3 - 2l_2^2 V$$

or

$$\varrho E = -\frac{8}{15}\pi \varrho a^2 c^3 m^2 - 2l_2^2 M_2. \tag{6.16}$$

Finally,

$$\iiint\limits_V \left(v_x^* \frac{\partial \chi}{\partial y} - v_y^* \frac{\partial \chi}{\partial x} \right) dv$$

$$= i \iiint\limits_V a^2 m^2 2(z - l_2)^2 \, dv = i\frac{8}{15}\pi a^4 c^3 m^2. \tag{6.17}$$

Substituting these data into (3.9), we have

$$\left(A_1 + l_2^2 M_2 + \frac{4\pi\varrho}{15} a^2 c \frac{(c^2 - a^2)^2}{c^2 + a^2} \right) \dot{w} - \left(B_1 - 2l_2^2 M_2 - \frac{8\pi\varrho}{15} a^2 c^3 m^2 \right) \omega i w$$

$$+ \left(\frac{g}{\omega^2}(l_1 M_1 + l_2 M_2) + A_1 + A_2 - C_1 - C_2 \right) \omega i Z + \frac{8\pi\varrho}{15} a^4 c^3 m^2 \omega i \xi = 0,$$

or, assuming

$$A_1 + l_2^2 M_2 = A^*, \qquad B_1 - 2l_2^2 M_2 = C_1 - 2A^* = B,$$

$$A_1 + A_2 = A, \qquad C_1 + C_2 = C,$$

we obtain[5]

$$\left(A^* + \frac{4\pi\varrho}{15} a^2 c \frac{(c^2 - a^2)^2}{c^2 + a^2} \right) \dot{w} - \left(B - \frac{8\pi\varrho}{15} a^2 c^3 m^2 \right) \omega i w$$

$$-L\omega i Z + \frac{8\pi\varrho}{15} a^4 c^3 m^2 \omega i \xi = 0. \tag{6.18}$$

Equations (6.11), (6.18), and

$$\dot{Z} - i\omega w = 0 \tag{6.19}$$

determine the motion of our ellipsoidal top. The nature of stability of the motion is determined by the presence or absence of complex roots in the determinant of the system

$$\varDelta(\lambda) = \begin{vmatrix} a_{11} & -L\omega & a_{13} \\ -\omega & \lambda & 0 \\ 2\omega & 0 & a_{33} \end{vmatrix} = 0,$$

$$a_{11} = \left(A^* + \frac{4\pi\varrho}{15} a^2 c \frac{(c^2 - a^2)^2}{c^2 + a^2} \right) \lambda - \left(B - \frac{8\pi\varrho}{15} a^2 c^3 m^2 \right) \omega,$$

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[5] The quantity L is defined by formula (1.7). – *Ed.*

$$a_{13} = \frac{8\pi\varrho}{15}a^4c^3m^2, \qquad a_{33} = (c^2+a^2)\lambda + 2a^2\omega$$

or

$$\Delta(\lambda) = [(c^2+a^2)\lambda + 2a^2\omega]\left[\left(A^* + \frac{4\pi\varrho}{15}a^2c\frac{(c^2-a^2)^2}{c^2+a^2}\right)\lambda^2\right.$$
$$\left. - \left(B - \frac{8\pi\varrho}{15}a^2c^3m^2\right)\omega\lambda - L\omega^2\right] - \frac{16\pi\varrho}{15}a^4c^3m^2\omega^2\lambda = 0.$$

If all three roots of $\Delta(\lambda)$ are real and simple, then the motion is stable; if among the roots only one is real, then it is unstable. Suppose

$$L = L^* + C_2 - A_2^{(0)} = L^* - \frac{4\pi\varrho}{15}a^2c(c^2-a^2).$$

Then

$$L^* = C_1 - A_1 - l_2^2M_2 - (g/\omega^2)(l_1M_1 + l_2M_2).$$

In this case the quantity $\Delta(\lambda)$ can be rewritten as follows:

$$\Delta(\lambda) = [(c^2+a^2)\lambda + 2a^2\omega][A^*\lambda^2 - B\omega\lambda - L^*\omega^2]$$
$$+\frac{4\pi\varrho}{15}a^2cm[(c^2+a^2)\lambda + 2a^2\omega][(c^2-a^2)\lambda^2 + 2c^2m\omega\lambda + (c^2+a^2)\omega^2]$$
$$-\frac{16\pi\varrho}{15}a^4c^3m^2\omega^2\lambda = [(c^2+a^2)\lambda + 2a^2\omega][A^*\lambda^2 - B\omega\lambda - L^*\omega^2]$$
$$+\frac{4\pi\varrho}{15}a^2cm[(c^2+a^2)(c^2-a^2)\lambda^3 + 2(c^2+a^2)(c^2-a^2)\lambda^2\omega$$
$$+(c^2+a^2)^2\lambda^2\omega + 2a^2(c^2+a^2)\omega^3]$$
$$= [(c^2+a^2)\lambda + 2a^2\omega][A^*\lambda^2 - B\omega\lambda - L^*\omega^2]$$
$$+\frac{4\pi\varrho}{15}a^2c(c^2-a^2)(\lambda+\omega)[(c^2-a^2)\lambda^2 + (c^2-a^2)\lambda\omega + 2a^2\omega^2]. \tag{6.20}$$

It is convenient to note in some cases that

$$A^*\lambda^2 - B\omega\lambda - L^*\omega^2 = A^*\lambda^2 - (C_1 - 2A^*)\lambda\omega - (C_1 - A^*)\omega^2 + K$$
$$= A^*(\lambda+\omega)^2 - C_1\omega(\lambda+\omega) + K.$$

Thus,

$$\Delta(\lambda) = [(c^2+a^2)\lambda + 2a^2\omega][A^*(\lambda+\omega)^2 - C_1\omega(\lambda+\omega) + K]$$
$$+\frac{4\pi\varrho}{15}a^2c(c^2-a^2)(\lambda+\omega)[(c^2-a^2)\lambda^2 + (c^2-a^2)\lambda\omega + 2a^2\omega^2]. \tag{6.21}$$

2^0. Let us study separately the case $K = 0$, i.e., we assume that the top moves around the center of gravity. Then we obtain two equations for λ,

$$\lambda + \omega = 0,$$

$$[(c^2+a^2)\lambda+2a^2\omega][A^*(\lambda+\omega)-C_1\omega]$$
$$+\frac{4\pi\varrho}{15}a^2c(c^2-a^2)[(c^2-a^2)\lambda^2+(c^2-a^2)\lambda\omega+2a^2\omega^2]=0. \tag{6.22}$$

The root of the first of these equations $\lambda=-\omega$ gives the eigenvector $(Z,0,0)$, which corresponds to $Z^*=\text{const}$, i.e., to the latent position of the deviating top. The roots of the second equation in (6.22) give vibrations around the new state of equilibrium.

If we assume $A^*=C_1=0$, which means that the shell is weightless, then for λ we obtain the equation

$$(c^2-a^2)\lambda^2+(c^2-a^2)\lambda\omega+2a^2\omega^2=0. \tag{6.23}$$

Its roots are

$$\lambda=\frac{-(c^2-a^2)\pm\sqrt{(c^2-a^2)(c^2-9a^2)}}{2(c^2-a^2)}\omega=-\frac{\omega}{2}\pm\sqrt{\frac{c^2-9a^2}{c^2-a^2}}\,\frac{\omega}{2}. \tag{6.24}$$

The motion is stable for $c>3a$ or for $c<a$. If $a<c<3a$, then the motion is unstable.

If $\varrho=0$, then for λ we again have two equations

$$(c^2+a^2)\lambda+2a^2\omega=0,\qquad A^*(\lambda+\omega)^2-C_1\omega(\lambda+\omega)+K=0. \tag{6.25}$$

The root of the first equation gives us the eigenvector $(0,0,\mathbf{v}^*)$. The roots of the second equation give the usual precessional-nutational motion of the shell.

3^0. In the general form the equation $\Delta(\lambda)=0$, where $\Delta(\lambda)$ is from (6.21), can be studied graphically. For this, let us rewrite it in the form

$$\frac{A^*}{\gamma}\left(\frac{\lambda}{\omega}+1\right)^2-\frac{C_1}{\gamma}\left(\frac{\lambda}{\omega}+1\right)+\frac{\nu}{\gamma}$$
$$+\frac{\left(\frac{\lambda}{\omega}+1\right)\left[(c^2-a^2)\frac{\lambda^2}{\omega^2}+(c^2-a^2)\frac{\lambda}{\omega}+2a^2\right]}{(c^2+a^2)\frac{\lambda}{\omega}+2a^2}\,\frac{c^2-a^2}{c^2}=0. \tag{6.26}$$

Here

$$\nu=\frac{K}{\omega^2},\quad \gamma=A_2^{(0)}-\frac{1}{2}C_2=\frac{4\pi\varrho}{15}a^2c^3. \tag{6.27}$$

The function on the left side of (6.26) can be easily constructed, if we specify the ratio $\dfrac{c}{a}$, or, which is the same, the quantity m defined by (6.5). Let

$$\varphi\left(\frac{\lambda}{\omega}\right)=-\frac{\left(\frac{\lambda}{\omega}+1\right)\left(m\frac{\lambda^2}{\omega^2}+m\frac{\lambda}{\omega}+1-m\right)}{\frac{\lambda}{\omega}+1-m}\,\frac{2m}{m+1}. \tag{6.28}$$

Let us construct the curves $\varphi = \varphi\left(\frac{\lambda}{\omega}\right)$ for different values of m and separately draw on a transparency the parabola

$$y = \frac{A^*}{\gamma}\left(\frac{\lambda}{\omega}+1\right)^2 - \frac{C_1}{\gamma}\left(\frac{\lambda}{\omega}+1\right) + \frac{\nu}{\gamma}. \tag{6.29}$$

Let us place this transparency on the graph $\varphi = \varphi\left(\frac{\lambda}{\omega}\right)$. The motion is stable if the parabola intersects the curve in three points; if the parabola intersects the curve in one point, then the motion is unstable.

4^0. Instead of studying equation (6.21) graphically, we can do it analytically by using the theory of algebraic equations. It is known that the necessary and sufficient condition for all three roots of the equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \tag{6.30}$$

to be real, has the form (D is the discriminant of the equation)

$$D = a_1^2a_2^2 + 18a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 \geq 0.$$

The substitution

$$\frac{\lambda}{\omega} + 1 - m = t \tag{6.31}$$

reduces equation (6.26) to the form

$$a_0t^3 + a_1t^2 + (a_2+\nu)t + a_3 = 0. \tag{6.32}$$

By equating to zero the discriminant of this equation, we obtain

$$\begin{gathered}4a_0\nu^3 + (12a_0a_2 - a_1^2)\nu^2 + (12a_0a_2^2 - 2a_1^2a_2 - 18a_0a_1a_3)\nu \\ +(4a_0a_2^3 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 27a_0^2a_3^2 + 4a_1^3a_3) = 0.\end{gathered} \tag{6.33}$$

This equation can have either three real roots ν_1, ν_2, ν_3 or only one. In the first case the interval of change of ν splits into four pieces,

$$-\infty < \nu < \nu_1, \quad \nu_1 < \nu < \nu_2, \quad \nu_2 < \nu < \nu_3, \quad \nu_3 < \nu < \infty. \tag{6.34}$$

In the first and third intervals the motion of the top is stable; in the second and fourth intervals it is unstable.

If (6.30) has one root, then there will be only two intervals

$$-\infty < \nu < \nu_1, \qquad \nu_1 < \nu < \infty.$$

In the first interval the motion is stable, in the second interval it is unstable.

5^0. To compute the value ν_3 in the case when the product A^*C_1 is large, compared with γ, and also m is not very small, we can replace $\varphi\left(\frac{\lambda}{\omega}\right)$ by its expansion in powers of $\left(\frac{\lambda}{\omega}+1\right)$. Here, neglecting the terms $\left(\frac{\lambda}{\omega}+1\right)^3$ and higher, we obtain the equation

$$\left(\frac{A^*}{\gamma}+\frac{2m-1}{m^2}\right)\left(\frac{\lambda}{\omega}+1\right)^2-\left(\frac{C_1}{\gamma}+\frac{1-m}{m}\right)\left(\frac{\lambda}{\omega}+1\right)+\frac{\nu}{\gamma}=0 \quad (6.35)$$

or

$$\left[A^*+4\pi\varrho a^2c(c^2+a^2)\frac{c^2-3a^2}{c^2-a^2}\right]\left(\frac{\lambda}{\omega}+1\right)^2$$
$$-(C_1+8\pi\varrho a^4c)\left(\frac{\lambda}{\omega}+1\right)+\nu=0, \quad (6.36)$$

or

$$\left[A^*+A_2^{(0)}\left(1-\frac{2a^2}{c^2-a^2}\right)\right]\left(\frac{\lambda}{\omega}+1\right)^2-(C_1+C_2)\left(\frac{\lambda}{\omega}+1\right)+\nu=0,$$

i.e.,

$$\left[A-A_2^{(0)}\frac{2a^2}{c^2-a^2}\right]\left(\frac{\lambda}{\omega}+1\right)^2-C\left(\frac{\lambda}{\omega}+1\right)+\nu=0.$$

In this case the condition for the roots to be real is

$$\frac{K}{\omega^2}<\frac{C^2}{4A^{(0)}}, \quad (6.37)$$

where

$$A^{(0)}=A-A_2^{(0)}\frac{2a^2}{c^2-a^2}.$$

Hence,

$$\omega>2\frac{\sqrt{A^{(0)}K}}{C}. \quad (6.38)$$

This condition is very similar to the one used usually in computations.

Suppose now that the shell of the top was first moving with the angular velocity $\omega^{(0)}$, and the fluid was stationary. According to the law of conservation moment of momentum $C_1\omega^{(0)}=C\omega$, hence, for the initial angular velocity, we obtain the approximate condition

$$\omega^{(0)}>2\frac{\sqrt{A^{(0)}K}}{C_1}. \quad (6.39)$$

Chapter 3 Top with Cylindrical Cavity

7 Computation of a Resolvent

1⁰. In the case when the cavity is a body of rotation, it is convenient to introduce different variables. We chose as the independent variables instead of x and y the coordinates r and θ, where

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{7.1}$$

The unknowns are

$$Z, w, v_r, v_\theta, v_z, \tag{7.2}$$

where

$$v_z = 2iu_{z,(1)}e^{i\theta}, \quad v_r = iu_{\zeta,(0)} + ie^{2i\theta}u_{\overline{\zeta},(2)}, \quad v_\theta = iu_{\zeta,(0)} - ie^{2i\theta}u_{\overline{\zeta},(2)}. \tag{7.3}$$

Here the operations indicated by the indexes in the parenthesis are defined by (2.21)–(2.26). The quantities

$$-\frac{i}{2}v_r e^{-i\theta}, \quad \frac{1}{2}v_\theta e^{-i\theta}, \quad -\frac{i}{2}v_z e^{-i\theta}$$

are the components on the r-axis, the θ-axis, and the z-axis of the velocity vector, if the components of this vector in the Cartesian coordinates are v_x, v_y, v_z.

Let us write out equations for the new unknowns. It is convenient to use for this purpose formulas (2.30). In this case the new unknowns depend only on t and $r = \sqrt{\zeta\overline{\zeta}}$.

Assuming for simplicity that $F_\zeta = F_{\overline{\zeta}} = 0$ and performing necessary computations, we obtain the homogeneous system

$$\begin{aligned} &\frac{\partial v_r}{\partial t} + 2\omega i v_\theta - \frac{i}{\varrho}\frac{\partial p}{\partial r} = 0, \\ &\frac{\partial v_\theta}{\partial t} + 2\omega i v_r - \frac{i}{\varrho r}\frac{\partial p}{\partial \theta} = 0, \\ &\frac{\partial v_z}{\partial t} - \frac{i}{\varrho}\frac{\partial p}{\partial z} = 0, \end{aligned} \tag{7.4}$$

$$\frac{\partial (rv_r)}{\partial r} - v_\theta + r\frac{\partial v_z}{\partial z} = 0. \tag{7.5}$$

Let us consider the boundary conditions and equations for Z. Let $\mu = e^{-i\theta}\mu^*$.

Then,

$$[v_r \cos nr + v_z \cos nz]_S = i\ \dot{Z}\ \overline{\mu}^{**} = -\omega w \overline{\mu}^{**}, \tag{7.6}$$

$$N^{**}(p) = \frac{i}{2} \iint\limits_S p\mu^{**}\, dS, \tag{7.7}$$

$$Z - i\omega w = 0, \quad A_1\ \dot{w}\ - B_1 \omega i w - L\omega i Z - \frac{1}{\omega} N^{**}(p) = 0. \tag{7.8}$$

2⁰. In the case when the domain V is a cylinder of radius b with height $2h$ and center at the point $(0, 0, l_2)$, the function

$$\mu^{**} = \overline{\mu}^{**} = \begin{cases} z & \text{for} \quad r = b, \\ \mp r & \text{for} \quad z - l_2 = \pm h. \end{cases} \tag{7.9}$$

Equations for determining the resolvent have the form

$$\begin{aligned} \lambda v_r + 2\omega v_\theta - \frac{1}{\varrho}\frac{\partial p}{\partial r} &= v_{r_0}, \\ \lambda v_\theta + 2\omega v_r - \frac{1}{\varrho r}\frac{\partial p}{\partial \theta} &= v_{\theta_0}, \\ \lambda v_z - \frac{1}{\varrho}\frac{\partial p}{\partial z} &= v_{z_0}, \end{aligned} \tag{7.10}$$

$$\lambda Z - \omega w = Z_0, \quad (A_1\lambda - B_1\omega)w - L\omega Z + \frac{i}{\omega} N^{**}(p) = A w_0, \tag{7.11}$$

$$v_r = -\omega Z w \ \text{ for } \ r = b, \qquad v_z = \omega r w \ \text{ for } \ z = \pm h. \tag{7.12}$$

We should add equation (7.5) here. From equations (7.5), (7.12), and (7.10) we can find the expression for $N^{**}(p)$ in terms of w, v_0, and w_0. Substituting the expression into (7.11), we obtain a system of equations on Z and w. If we know w, in turn, it is easy to find $\mathbf{v}$.

The equality to zero of the determinant of the system obtained on Z and w, gives us the formal condition of existence of fundamental frequencies, or eigenvalues of the operator B. Let

$$\begin{aligned} v_{r_0} &= -\omega w_0 z + \omega \sum c_{l,0}(r) \sin\frac{(2l+1)\pi z'}{2h}, \\ v_{\theta_0} &= -\omega w_0 z + \omega \sum g_{l,0}(r) \sin\frac{(2l+1)\pi z'}{2h}, \\ v_{z_0} &= \omega w_0 r + \omega \sum b_{l,0}(r) \cos\frac{(2l+1)\pi z'}{2h}, \end{aligned} \tag{7.13}$$

where $z' = z - l_2$.

We search for the solution in the form

$$v_r = -\omega w z + \omega \sum c_l(r) \sin \frac{(2l+1)\pi z'}{2h},$$

$$v_\theta = -\omega w z + \omega \sum g_l(r) \sin \frac{(2l+1)\pi z'}{2h}, \tag{7.14}$$

$$v_z = \omega w r + \omega \sum b_l(r) \cos \frac{(2l+1)\pi z'}{2h},$$

$$\frac{p}{\varrho} = (\omega\lambda w - \omega w_0) r z + \omega \sum a_l(r) \sin \frac{(2l+1)\pi z'}{2h}$$

$$-2\omega[(\lambda+\omega)w - w_0] l_2 r. \tag{7.15}$$

In this case equation of continuity (7.5) takes the form

$$[r c_{l,0}(r)]' - g_{l,0}(r) - r\frac{(2l+1)\pi}{2h} b_{l,0}(r) = 0, \tag{7.16}$$

$$[r c_l(r)]' - g_l(r) - r\frac{(2l+1)\pi}{2h} b_l(r) = 0. \tag{7.17}$$

Let us substitute (7.14) and (7.15) into equation (7.10). Using

$$z = z' + l_2 = \sum_{l=0}^{\infty} \frac{8h(-1)^l}{\pi^2(2l+1)^2} \sin \frac{(2l+1)\pi z'}{2h} + l_2, \tag{7.18}$$

we obtain

$$\begin{gathered}
\sum \omega \left\{ \lambda c_l + 2\omega g_l - a_l' + \frac{8h(-1)^l}{\pi^2(2l+1)^2}(2w_0 - 2(\lambda+\omega)w) - c_{l,0} \right\} \\
\times \sin \frac{(2l+1)\pi z'}{2h} = 0, \\
\sum \omega \left\{ 2\omega\lambda c_l + \lambda g_l - \frac{a_l}{r} + \frac{8h(-1)^l}{\pi^2(2l+1)^2}(2w_0 - 2(\lambda+\omega)w) - g_{l,0} \right\} \\
\times \sin \frac{(2l+1)\pi z'}{2h} = 0, \\
\lambda b_l - \frac{(2l+1)\pi}{2h} a_l - b_{l,0} = 0.
\end{gathered} \tag{7.19}$$

By equating to zero the coefficients at $\sin\left[\dfrac{(2l+1)\pi z}{2h}\right]$ and solving the obtained system with respect to c_l and g_l, we have

$$[\lambda^2 - 4\omega^2] c_l - \lambda a_l' + 2\omega \frac{a_l}{r} - \frac{16h(-1)^l}{\pi^2(2l+1)^2}(\lambda - 2\omega)$$

$$\times [(\lambda+\omega)w - w_0] - \lambda c_{l,0} + 2\omega g_{l,0} = 0,$$

$$[\lambda^2 - 4\omega^2]g_l + 2\omega a_l' - \lambda\frac{a_l}{r} - \frac{16h(-1)^l}{\pi^2(2l+1)^2}(\lambda - 2\omega) \tag{7.20}$$
$$\times[(\lambda+\omega)w - w_0] + 2\omega c_{l,0} - \lambda g_{l,0} = 0,$$
$$\lambda b_l - \frac{(2l+1)\pi}{2h}a_l - b_{l,0} = 0.$$

Using the equation of continuity, we obtain

$$-\lambda(ra_l')' + \frac{\lambda a_l}{r} + \frac{(2l+1)^2\pi^2}{4h^2}\,\frac{\lambda^2 - 4\omega^2}{\lambda}ra_l - \lambda(rc_{l,0})' - 2\omega c_{l,0}$$
$$+2\omega(r, g_{l,0})' + \lambda g_{l,0} + \frac{\lambda^2 - 4\omega^2}{\lambda}\,\frac{(2l+1)\pi}{2h}b_{l,0} = 0 \tag{7.21}$$

or

$$(ra_l')' + \left\{\frac{4\omega^2 - \lambda^2}{\lambda^2}\,\frac{(2l+1)\pi^2}{4h^2}r - \frac{1}{r}\right\}a_l = f_l, \tag{7.22}$$

where

$$f_l = -(rc_{l,0})' - \frac{2\omega}{\lambda}c_{l,0} + \frac{2\omega}{\lambda}(rg_{l,0})' + g_{l,0} - \frac{4\omega^2 - \lambda^2}{\lambda^2}\,\frac{(2l+1)\pi}{2h}b_{l,0}. \tag{7.23}$$

Equation (7.22) allows determining a_l, and the first of equations (7.20) provides the boundary conditions for this function.

Indeed, $c_{l,0} = 0$, $c_l = 0$ for $r = b$. Hence,

$$r\lambda a_l'|_{r=b} - 2\omega a_l|_{r=b} = -\frac{16h(-1)^l}{\pi^2(2l+1)^2}(\lambda - 2\omega)r[(\lambda+\omega)w - w_0]. \tag{7.24}$$

Assuming

$$k^2 = \frac{4\omega^2 - \lambda^2}{\lambda^2}\,\frac{(2l+1)^2\pi^2}{4h^2},$$

for a_l we obtain the expression

$$a_l(r) = -\frac{16hb(-1)^l(\lambda - 2\omega)}{\pi^2(2l+1)^2}\,\frac{J_1(kr)}{\lambda kbJ_1'(kb) - 2\omega J_1(kb)}$$
$$\times[(\lambda+\omega)w - w_0] + \int_0^b K(r, r_1)f_l(r_1)\,dr_1. \tag{7.25}$$

Here

$$K(r, r_1) = \begin{cases} \dfrac{\pi}{2}J_1(kr)\left[Y_1(kr_1) - \dfrac{\lambda kbY_1'(kb) - 2\omega Y_1(kb)}{\lambda kbJ_1'(kb) - 2\omega J_1(kb)}J_1(kr_1)\right], & r \le r_1, \\ \dfrac{\pi}{2}J_1(kr_1)\left[Y_1(kr) - \dfrac{\lambda kbY_1'(kb) - 2\omega Y_1(kb)}{\lambda kbJ_1'(kb) - 2\omega J_1(kb)}J_1(kr)\right], & r \ge r_1. \end{cases} \tag{7.26}$$

The expression obtained for a_l allows us to compute

$$N^{**}(p) = \beta w + N_1, \tag{7.27}$$

where N_1 depends only on w_0, $\mathbf{v}_0$, but does not depend on w.

For the goals stated it suffices to know the value of β. Taking into account that

$$\int\limits_{-h}^{h} z' \sin \frac{(2l+1)\pi z'}{2h}\, dz' = \frac{8h^2(-1)^l}{\pi^2(2l+1)^2},$$

we obtain

$$\frac{1}{\omega} N^{**}(p) = i\varrho[\lambda w - w_0] \left\{ -2\pi h \int\limits_0^b r^3\, dr + \pi b^2 \int\limits_{l_2-h}^{l_2+h} z^2\, dz \right\}$$

$$-2\pi i \varrho[(\lambda+\omega)w - w_0] \int\limits_{l_2-h}^{l_2+h} b^2 l_2 z\, dz$$

$$+ i\varrho \sum_{l=0}^{\infty} 2\pi(-1)^l \left\{ \int\limits_0^b a_l(r) r^2\, dr - \frac{4h^2 b a_l(b)}{\pi^2(2l+1)^2} \right\}$$

$$= iw \left\{ -(C_2 - A_2^{(0)})\lambda + 2\pi\varrho \frac{16bh(\lambda - 2\omega)(\lambda+\omega)}{\pi^2(2l+1)^2} \right.$$

$$\left. \times \sum_{l=0}^{\infty} \left[\frac{4h^2 b J_1(kb)}{\pi^2(2l+1)^2} - \int\limits_0^b J_1(kr) r^2\, dr \right] - l_2^2 M_2(\lambda + 2\omega) 2\omega \right\} + \frac{1}{\omega} N_1,$$

where $A_2^{(0)}$ and C_2 denote the principal moments of inertia of the fluid

$$A_2^{(0)} = \frac{2}{3}\pi h^3 b^2 \varrho + \frac{1}{2}\pi h b^4 \varrho, \quad C_2 = \pi h b^4 \varrho, \quad A_2 = l_2^2 M_2 + A_2^{(0)}.$$

Let us calculate the quantity

$$\psi = \frac{4h^2 b J_1(kb)}{\pi^2(2l+1)^2} - \int\limits_0^b J_1(kr) r^2\, dr.$$

We have

$$\int\limits_0^b J_1(kr) r^2\, dr = \frac{1}{k^3} \int\limits_0^{kb} J_1(x) x^2\, dx = -\frac{1}{k^3} \int\limits_0^{kb} J_0'(x) x^2\, dx,$$

$$x^2 J_0'(x) = x^2 J_0'(x) + 2xJ_0(x) + 2xJ_0''(x) + 2J_0'(x)$$
$$= \frac{d}{dx}[x^2 J_0(x) + 2xJ_0'(x)]$$

and, hence,

$$\int_0^b J_1(kr) r^2\, dr = -\frac{1}{k^3}[k^2 b^2 J_0(kb) - 2kbJ_1(kb)],$$

and

$$\psi = \frac{b^2}{k} J_0(kb) - \frac{2b}{k^2} J_1(kb)\left[1 - \frac{4\omega^2 - \lambda^2}{2\lambda^2}\right]$$
$$= \frac{4h^2 b}{(2l+1)^2 \pi^2} \frac{\lambda^2 kbJ_0(kb) + (4\omega^2 - 3\lambda^2) J_1(kb)}{4\omega^2 - \lambda^2}.$$

Collecting together all the above calculations, we obtain

$$\frac{1}{\omega} N^{**}(p) = iw\Bigg\{ -\lambda(C_2 - A_2^{(0)}) + \frac{128h^3 b^2 \varrho}{\pi^3} \frac{(\lambda - 2\omega)(\lambda + \omega)}{4\omega^2 - \lambda^2}$$
$$\times \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} \frac{\lambda^2 kbJ_0(kb) + (4\omega^2 - 3\lambda^2) J_1(kb)}{\lambda kbJ_0(kb) - (\lambda + 2\omega) J_1(kb)}$$
$$-l_2^2 M_2(\lambda + 2\omega)\Bigg\} + \frac{1}{\omega} N_1. \tag{7.28}$$

By substituting this expression into equations (7.11), we obtain the system

$$\Bigg[(A_1 + l_2^2 M_2 - A_2^{(0)} + C_2)\lambda - (B_1 - 2l_2^2 M_2)\omega + \frac{128h^3 b^2 \varrho}{\pi^3} \frac{(\lambda - 2\omega)(\lambda + \omega)}{4\omega^2 - \lambda^2}$$
$$\times \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} \frac{\lambda^2 kbJ_0(kb) + (4\omega^2 - 3\lambda^2) J_1(kb)}{\lambda kbJ_0(kb) - (\lambda + 2\omega) J_1(kb)}\Bigg] w - L\omega Z = Aw_0 - i\frac{N_1}{\omega},$$
$$\lambda Z - \omega w = Z_0. \tag{7.29}$$

The determinant of this system has the form

$$\varDelta(\lambda) = (A^* - A_2^{(0)} + C_2)\lambda^2 - (B\omega\lambda - L\omega^2)$$
$$-\frac{128h^3 b^2 \varrho}{\pi^3} \frac{\lambda(\lambda + \omega)}{\lambda + 2\omega} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} \frac{\lambda^2 kbJ_0(kb) + (4\omega^2 - 3\lambda^2) J_1(kb)}{\lambda kbJ_0(kb) - (\lambda + 2\omega) J_1(kb)}. \tag{7.30}$$

Here

$$A^* = A_1 + l_2^2 M_2, \qquad B = B_1 - 2l_2^2 M_2.$$

For such values λ that $\varDelta(\lambda) = 0$, there exists a solution of the homogeneous system corresponding to (7.29). If we can find from this system the value w, then (7.25) determines p, and, hence, all other unknowns.

8 The Investigation of Results

1^0. The goal of the present section is to investigate the function $\Delta(\lambda)$ and the conditions in which it has complex roots. Denoting $\lambda = 2\omega q$, and dividing both parts by ω^2, we have

$$\Delta(\lambda) = (A^* - A_2^{(0)} + C_2)4q^2 - 2Bq - L - \frac{256}{\pi^3} h^3 b^2 \varrho \frac{q(2q+1)}{q+1} D_1(q), \quad (8.1)$$

where

$$D_1(q) = \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} \frac{q^2 kbJ_0(kb) - (3q^2-4)J_1(kb)}{qkbJ_0(kb) - (q+1)J_1(kb)}. \quad (8.2)$$

Let us rewrite equations (8.1) in the form

$$A^* 4q^2 - 2Bq - L$$
$$= \left((A_2^{(0)} - C_2)4q^2 + \frac{256}{\pi^3} h^3 b^2 \varrho \frac{q(2q+1)}{q+1} D_1(q) \right) = \varphi_1(q).$$

The quadratic trinomial on the left side sometimes is convenient to express in another form[6]

$$4A^* q^2 - 2Bq - L = 4A^* q^2 - (C_1 - 2A^*)2q + (K/\omega^2 + A^* - C_1)$$
$$+ (A_2^{(0)} - C_2) = A^*(2q+1)^2 - C_1(2q+1) + K/\omega^2 + (A_2^{(0)} - C_2). \quad (8.3)$$

Then equation (8.1) has the form

$$A^*(2q+1)^2 - C_1(2q+1) + \frac{K}{\omega^2}$$
$$= (A_2^{(0)} - C_2)(4q^2 - 1) + \frac{256}{\pi^3} h^3 b^2 \varrho \frac{q(2q+1)}{q+1} D_1(q). \quad (8.4)$$

In such notation we see that the interval, cut off by the parabola $y = A^*(2q+1)^2 - C_1(2q+1) + \frac{K}{\omega^2}$ from the line $q = -\frac{1}{2}$, is equal to the value of the tilting moment K divided by the square of the angular velocity of the top.

Let us first study the right side of equation (8.4). Let

$$\xi = \sqrt{\frac{1}{q^2} - 1}. \quad (8.5)$$

Let us make a cut in the plane q along the interval $-1 \leq q \leq 1$ of the real axis. In the remaining part of the plane, the quantity

[6] S. L. Sobolev uses formulas $L = C_1 + C_2 - A_1 - A_2 - \frac{K}{\omega^2}$, $A_1 = A^* - l_2^2 M_2$, $A_2 = A_2^0 + l_2^2 M_2$. – *Ed.*

$$kb = \frac{(2l+1)\pi b}{2h}\xi \tag{8.6}$$

is a regular function. Let us agree to chose always the values of the root such that k has a positive imaginary part for $q > 1$. In this case the entire plane q gets mapped into the twice repeated upper half-plane of k with a critical point on the imaginary axis. The imaginary axis of q and the parts of the real axis, where $|q| > 1$, correspond to the positive part of the imaginary axis. The upper part of the interval $-1 < q < 0$ maps to $0 < k < \infty$, and the upper part of the interval $0 < q < 1$ maps to $-\infty < k < 0$, and also on the upper lip k is monotonically increasing on these intervals. On the lower lip k has a sign opposite to the sign on the upper one. Near the point $q = \infty$ we have the expansion

$$\xi = i\left(1 - \frac{1}{2}\frac{1}{q^2} + \cdots\right), \tag{8.7}$$

and near $q = 0$ for the values from the upper half-plane

$$\xi = -\frac{1}{q} + \frac{q}{2} + \cdots . \tag{8.8}$$

It is obvious that for q from the upper half-plane

$$q = -\frac{1}{\xi}\left(1 - \frac{1}{2}\frac{1}{\xi^2} + \cdots\right) = -\frac{1}{\sqrt{1+\xi^2}}. \tag{8.9}$$

2^0. Let us establish convergence of the series $D_1(q)$ for all q outside the cut $|q| < 1$. To do so, we consider the terms of this series as functions of ξ. For any q outside the cut, the quantity ξ has a complex value from the upper half-plane. In this case the quantity $kb = \left[\frac{(2l+1)\pi b}{2h}\right]\xi$, as l increases, runs over a sequence of discrete values located on the same ray passing through the origin.

For a large in absolute value kb for integer n, the following formula [3] is valid[7]

$$J_n(kb) = \left(\frac{2}{\pi kb}\right)^{1/2}\left(\cos(kb - \frac{1}{2}n\pi - \frac{1}{4}\pi)U_n(kb)\right.$$
$$\left. + \sin(kb - \frac{1}{4}n\pi - \frac{1}{4}\pi)V_n(kb)\right). \tag{8.10}$$

Here

$$U_n(z) = 1 + \sum_{r=1}^{\infty}\frac{(-1)^r(4n^2-1^2)(4n^2-3^2)\dots(4n^2-(4r-1)^2)}{2r!2^{6r}z^{2r}},$$
$$V_n(z) = \sum_{r=1}^{\infty}\frac{(-1)^r(4n^2-1^2)(4n^2-3^2)\dots(4n^2-(4r-3)^2)}{(2r-1)!2^{6r-3}z^{2r-1}}. \tag{8.11}$$

[7] The Hankel asymptotical expansion. – *Ed.*

Taking into account that for large y,

$$\cos(x+iy) \sim \frac{1}{2}e^{y-ix}, \qquad \sin(x+iy) \sim \frac{i}{2}e^{y-ix}, \tag{8.12}$$

and assuming $kb = x+iy$, we have for sufficiently large y,

$$U_l(q) \equiv \frac{q^2\frac{(2l+1)\pi b}{2h}\xi J_0\left(\frac{(2l+1)\pi b}{2h}\xi\right) - (3q^2-4)J_1\left(\frac{(2l+1)\pi b}{2h}\xi\right)}{q\frac{(2l+1)\pi b}{2h}\xi J_0\left(\frac{(2l+1)\pi b}{2h}\xi\right) - (q+1)J_1\left(\frac{(2l+1)\pi b}{2h}\xi\right)} \approx q. \tag{8.13}$$

From (8.13) the uniform convergence of $D_1(q)$ immediately follows everywhere in the plane q, except for the cut mentioned.

Let us point out one important corollary. Let

$$\sum_{l=0}^{N} \frac{1}{(2l+1)^4} U_l(q) = D_1^{(N)}(q). \tag{8.14}$$

For $\dfrac{3b^2}{4} < h^2$ we consider the approximate equation

$$\frac{1}{A_2^{(0)} - C_2}\varphi(q) = (4q^2-1) + \frac{384}{\pi^4}\frac{1}{1-\frac{3b^2}{4h^2}}\frac{q(2q+1)}{q+1}D_1^{(N)}(q). \tag{8.15}$$

The complex roots of equation (8.4), if they exist, are arbitrarily close to the complex roots of the approximate equation. Therefore, instead of studying complex roots of (8.4), we are going to consider the same roots of equation (8.15).

3^0. Let us consider one auxiliary question. Let the equation

$$\frac{a_{-1}}{z} + a_0 + a_1 z = h(z) \tag{8.16}$$

be given, where a_{-1}, a_0, a_1 are real numbers, $h(z)$ is an analytic function of z, small for small values and real on the real axis.

Let us try to calculate its roots. Throwing the term $h(z)$ off, we obtain

$$a_{-1} + a_0 z + a_1 z^2 = 0. \tag{8.17}$$

Hence,

$$z_{1,2}^{(1)} = -\frac{a_0}{2a_1} \pm \sqrt{\frac{a_0^2}{4a_1^2} - \frac{a_{-1}}{a_1}}. \tag{8.18}$$

Obviously, the roots are complex if $a_0^2 < 4a_1 a_{-1}$ or $|a_0| < 2\sqrt{a_1 a_{-1}}$.

The maximum value of the imaginary part of the root, as a function of the parameter a_0, is equal to $\sqrt{\dfrac{a_{-1}}{a_1}}$.

To find the exact solution, we apply the method of successive approximations. Let

$$a_0^{(n)} = a_0 - \frac{z_2^{(n-1)} h(z_1^{(n-1)}) - z_1^{(n-1)} h(z_2^{(n-1)})}{z_2^{(n-1)} - z_1^{(n-1)}},$$
$$a_1^{(n)} = a_1 - \frac{h(z_2^{(n-1)}) - h(z_1^{(n-1)})}{z_2^{(n-1)} - z_1^{(n-1)}}. \tag{8.19}$$

In this case

$$[a_0^{(n)} + a_1^{(n)} z]_{z=z_1^{(n-1)}} = [a_0 + a_1 z - h(z)]_{z=z_1^{(n-1)}},$$
$$[a_0^{(n)} + a_1^{(n)} z]_{z=z_2^{(n-1)}} = [a_0 + a_1 z - h(z)]_{z=z_2^{(n-1)}}, \tag{8.20}$$

and let

$$\frac{a_{-1}}{z^{(n)}} + a_0^{(n)} + a_1^{(n)} z^{(n)} = 0. \tag{8.21}$$

The convergence of successive approximations for sufficiently small h follows from usual estimates.

4^0. Equation (8.15) is such an equation whose right side contains a meromorphic function with simple poles at the points, where

$$qkbJ_0(kb) - (q+1)J_1(kb) \tag{8.22}$$

vanishes for different l.

Near each such pole we can investigate its complex roots by using the arguments of 3^0. Let us set up a goal for ourselves to solve the following problem.

To find for which values of the angular velocity a top with the given weight and shape looses stability. As shown above, the solution of this problem reduces to finding complex roots of (8.15), having a large imaginary part.

It is convenient to define these roots graphically. For this, for a given value of $\dfrac{b}{h}$ we construct the curve

$$y = \varphi_N(q) = \left(1 - \frac{C_2}{A_2^{(0)} - \frac{1}{2C_2}}\right)(4q^2 - 1) + \frac{384}{\pi^4}\frac{q(2q+1)}{q+1} D_1^{(N)}(q). \tag{8.23}$$

Roots of (8.15) are found at the intersection points of the parabola

$$y = \frac{1}{\gamma}\left[A^*(2q+1)^2 - C_1(2q+1) + \frac{K}{\omega^2}\right] \tag{8.24}$$

with a curve defined by (8.23).

The equation in question has imaginary roots when the parabola passes between two branches of function (8.23). If we draw curve (8.23) on paper, and parabola (8.24) on a transparency, then shifting it so that the axis of the curves coincide, we can find all risky intervals of the change of frequency by length of the interval cut off by the parabola on the line $q = -\frac{1}{2}$.

It is not difficult to see that the poles $D_1^{(N)}(q)$, are values of q corresponding to fluid vibrations for the stationary cylinder. The equation

$$A^*(2q+1)^2 - C_1(2q+1) + \frac{K}{\omega^2} = 0,$$

roughly speaking, gives frequencies of fundamental vibrations of the top shell with some corrections due to the moments of inertia of the fluid. Therefore, the entire phenomenon bears the nature of peculiar resonance.

The domains of the risky values $\frac{K}{\omega^2}$ lie approximately, where the fundamental frequency of the shell vibrations is close to one of the fundamental frequencies of the fluid vibrations in the top with a stationary axis.

5^0. For the more convenient application of this method, let us indicate how to determine the poles of functions (8.22) and values of residues of the right side of (8.4) corresponding to these roots. From (8.22) we have

$$q = \frac{J_1(kb)}{kbJ_0(kb) - J_1(kb)}. \tag{8.25}$$

In this case, we have

$$kb = \frac{(2l+1)\pi b}{2h} \frac{\sqrt{k^2b^2J_0^2(kb) - 2kbJ_1(kb)J_0(kb)}}{J_1(kb)} \tag{8.26}$$

or

$$\frac{h}{(2l+1)b} = \pi\sqrt{\frac{(kbJ_0(kb) - 2J_1(kb))J_0(kb)}{4[J_1(kb)]^2kb}}, \tag{8.27}$$

i.e.,

$$h^* = \frac{h}{(2l+1)b} = \frac{\pi}{2}\sqrt{\frac{-J_0(kb)J_2(kb)}{[J_1(kb)]^2}}. \tag{8.28}$$

We consider kb as the independent parameter, then (8.27) and (8.25) give us a parametric equation for determining pairs of the values h^* and those q, for which equation (8.22) has a root. The reality of the values needed requires that $J_0(kb)J_2(kb)$ be negative. This circumstance occurs for distinct kb in different intervals between two roots of these functions. Thus, we obtain many branches of h^* as functions of q, and vice versa.

6^0. To determine the values of residues for each root, let us use the theory of the Riccati equations. Let us consider the equation

$$y' + py^2 + qy + r = 0 \tag{8.29}$$

with analytic coefficients in a certain interval of the domain of the independent variable $a < x < b$. Suppose that in a neighborhood of a certain point x_0 from this interval,

$$\begin{aligned} p &= p_0 + p_0'(x - x_0) + \frac{p_0''}{2}(x - x_0)^2 + \cdots , \\ q &= q_0 + q_0'(x - x_0) + \cdots , \quad r = r_0 + r_0'(x - x_0) + \cdots . \end{aligned} \tag{8.30}$$

Assume that $p \neq 0$. If equation (8.29) has a solution with a pole at the point x_0, then this pole can be only the simple pole. In the neighborhood of x_0 the representation

$$y = \frac{a_{-1}}{x - x_0} + a_0 + a_1(x - x_0) + \cdots . \tag{8.31}$$

is valid in this case.

Substituting (8.31) into (8.29), we find

$$\begin{aligned} a_{-1} &= \frac{1}{p_0}, \qquad a_0 = -\frac{1}{2p_0}\left(\frac{p_0'}{p_0} + q_0\right), \\ a_1 &= -\frac{1}{3p_0^2}p_0'' + \frac{1}{12p_0}\left(\frac{p_0'}{p_0} + q_0\right)\left(\frac{3p_0'}{p_0} - 2q_0\right) - \frac{1}{3}\left(\frac{q_0'}{p_0} + r_0\right) . \end{aligned} \tag{8.32}$$

Each function

$$y = \frac{\alpha x J_0(x) + \beta_1 J_1(x)}{\gamma x J_0(x) + \delta_1 J_1(x)} \tag{8.33}$$

is a solution of some Riccati equation (8.29). The coefficients of this equation are

$$\begin{aligned} p &= \frac{1}{\Delta}\left[\delta\left(\gamma' + \frac{\gamma}{x} + \frac{\delta}{x}\right) - \gamma\left(\delta' - \gamma x - \frac{\delta}{x}\right)\right], \\ r &= \frac{1}{\Delta}\left[\beta\left(\alpha' + \frac{\alpha}{x} + \frac{\beta}{x}\right) - \alpha\left(\beta' - \alpha x - \frac{\beta}{x}\right)\right], \\ q &= \frac{1}{\Delta}\left[\alpha\left(\delta' - \gamma x - \frac{\delta}{x}\right) - \delta\left(\alpha' + \frac{\alpha}{x} + \frac{\beta}{x}\right)\right. \\ &\quad \left. - \beta\left(\gamma' + \frac{\gamma}{x} + \frac{\delta}{x}\right) + \gamma\left(\beta' - \alpha x - \frac{\beta}{x}\right)\right], \end{aligned} \tag{8.34}$$

where $\Delta = \alpha\delta - \beta\gamma$.

If we use (8.34) and (8.32), then, after simple computations, we obtain that at every pole the function

$$\frac{q(2q+1)}{q+1}\,\frac{384}{\pi^4}\,\frac{1}{(2l+1)^4}\,\frac{q^2 kbJ_0(kb) - (3q^2 - 1)J_1(kb)}{qkbJ_0(kb) - (q+1)J_1(kb)} \tag{8.35}$$

must have the residue

$$\operatorname{res}_l(q_0) = \frac{384}{\pi^4}\frac{1}{(2l+1)^4}\frac{q_0^3(2q_0+1)^2(1-q_0)}{(q_0+1)[q_0+1+(2l+1)^2\pi^2b^2/4h^2]}.$$

7^0. Comparing the tops with the ellipsoidal and cylindrical cavities, we see that the behavior of such tops depends significantly on the shape of the cavity.

The important distinction of the motion of the top with the ellipsoidal cavity is the following: for arbitrary values of the fundamental angular velocity, the vibrations of the shell are related only with the principal frequency of the fluid vibrations in the rotating top with a stationary axis.

The generalized resonance discussed by us above, is possible here only in one bounded interval of ν, and the instability, besides the values of ν from this interval, can occur only for very small values of the angular velocity.

In the case of the cylinder the shell vibrations are connected with the entire infinite set of the forms of the fluid vibrations, and therefore there are infinitely many domains of resonance.

Hence the top with the ellipsoidal cavity should have calmer rotations. The engaged top with the ellipsoidal cavity, as its angular velocity is decreasing, either immediately leaves the stable state for good, or only once before that passes the unstable state returning then to calm motion.

The behavior of a top with the cylindrical cavity is agitated. Engaged with a certain angular velocity, it will, as the velocity decreases, loose and again regain the stability for many times.

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[8] Paper [9] of Part I of this book. – *Ed.*

11. On a Class of Problems of Mathematical Physics*

S. L. Sobolev

1. In statements of problems for partial differential equations of mathematical physics, as is known, one should assume that unknown functions are sufficiently smooth. These equations describe physical phenomena in domains large enough to be able to interpret in terms of differentiable functions such statistical notions as density of a medium, pressure, velocity of a fluid, values of mechanical, electrical, and magnetic stresses in a body, etc.

For all these quantities we have to consider partial derivatives with respect to the spatial and the time variables instead of certain integral relations.

Obviously, the description of physical laws in terms of partial derivatives assumes that the described functions and their derivatives do not vary much in the domains where we apply these arguments.

Thus, it is quite natural under the study of these equations to seek solutions which remain smooth for all moments t. The use of discontinuous solutions, solutions with discontinuous derivatives and even weak solutions such as, for example, distributions, is accepted in two cases.

1. A solution obtained with discontinuous derivatives or a distribution is a good approximation of a smooth solution. This occurs, for instance, for the vibrating string equation.
2. A discontinuous solution is used only as an intermediate auxiliary element needed for construction of a smooth solution.

For the classical equations of vibrations of continuous media (string, membrane, elastic body, gas, etc.) the existence of partial derivatives of a solution can be guaranteed for an infinite interval of time, if initial conditions are sufficiently smooth.

It is known that this circumstance, generally speaking, does not hold for nonlinear equations, for example, for the equations of gas motion, where the

* Simpos. Internaz. Appl. Anal. Fis. Mat. (Cagliari-Sassari, 1964), Edizioni Cremonese, Rome (1965), pp. 192–208.

classical solution exists only on a bounded interval t. Then strong discontinuities appear, and the mechanical process can be studied by using a different theory.

In my report one class of linear problems of mathematical physics will be discussed, where there is no hope to obtain a physically sound solution. However, the reasons are somewhat less transparent than the appearance of discontinuities. Solutions of these problems have noncompact trajectory in a corresponding abstract space.

Let us illustrate the above.

Consider the classical equation of vibrations corresponding to several physical problems

$$\Delta u - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

in a domain Ω with boundary conditions

$$Bu|_\Gamma = 0. \tag{2}$$

It is convenient to consider as an element of a solution the pair $w = \left(u, \dfrac{\partial u}{\partial t}\right)$. In this case the natural metric can be defined as the energy metric

$$\|w\|^2_X = \|u\|^2_{W_2^1} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2}, \tag{3}$$

$$\|u\|^2_{W_2^1} = \int_\Omega |\mathrm{grad}\, u|^2 dx, \quad \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2} = \int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx.$$

In such a metric the compactness of a trajectory is related, generally speaking, with the compactness of the domain Ω. The trajectory is compact for finite domains and noncompact for infinite domains such as a half-space, a semibounded or unbounded cylinder, etc.

For all domains with corresponding conditions (2) the law of energy preservation holds,

$$\frac{d}{dt}\|w\|^2_X = 0. \tag{4}$$

This law means that the set of all points of the trajectory is bounded.

For a set $\mathfrak{M}$ of functions $w(x)$ defined in a bounded domain to be compact in the space X, it is necessary and sufficient that:

1) this set is bounded

$$\|w(x)\|_X < M; \tag{5}$$

2) it is equicontinuous in the natural metric, i.e.,

$$\|w(x + \Delta x) - w(x)\|_X < \eta(|\Delta x|), \tag{6}$$

where neither M, nor $\eta(|\Delta x|)$ depend on the choice of w, and $\eta(\xi)$ tends to zero as ξ tends to zero.

The necessary condition of applicability of partial differential equations for the study of a physical phenomenon is the condition that functions $w(x,t)$, expressing different physical fields in the problem, always have restrictions $w_{\Omega}(x,t)$ equicontinuous for all moments of t in each prescribed part $\Omega \subset E_n$. More exactly, the estimates of the norm must hold,

$$\|w_{\Omega}(x+\Delta x,t) - w_{\Omega}(x,t)\|_X \leq \eta(|\Delta x|,\Omega), \tag{7}$$

where η does not depend on t. If inequality (7) fails, it is easy to see that the partial differential equations mentioned above make no sense.

Indeed, deriving these equations, we assume that we can describe fields of velocities, for example, of a fluid by functions $u(x,t)$ such that their values present, in some sense, the average for a large number of molecules. However, if (7) fails, then there exists a constant η_0 such that for the given $|\Delta x|$, even when this value is much less than the intermolecular distance, we can point out a moment t, after which the shift by $|\Delta x|$ gives a difference exceeding η_0, which is nonsense.

In my report I would like to indicate some quite simple problems of mathematical physics, whose solutions do not satisfy condition (6) uniformly in t, therefore, they lose their physical meaning.

We should note one more circumstance playing a significant part in this question. The noncompactness of trajectories means that it is impossible in principle to construct an approximation of solutions of such problems for all moments of time by a finite number of elements from a bounded set in a finite-dimensional space.

Therefore, the usual methods of stability study, such as the Ritz method or the Bubnov–Galerkin method of moments, are unacceptable in principle.

The stated problems are closely related to very important problems of mathematical analysis.

2. We now move on to examples. Let Ω be filled with an ideal uncompressible fluid, rotating with a constant angular velocity as a solid body.

The velocity and the pressure could be taken in the form

$$\overrightarrow{v} = \omega(\overrightarrow{k} \times \overrightarrow{r}), \quad p = \frac{\omega^2(r^2 - z^2)}{2} - gz + p_0, \tag{8}$$

where ω is the angular velocity, $\overrightarrow{r}$ is the coordinate vector[1]. Here $\overrightarrow{k}$ is the unit vector parallel to the z-axis. Functions (8) satisfy the Euler equations.

In the rotating system of coordinates the fluid will be at rest.

Let us consider now small vibrations of the fluid about this equilibrium in the rotating system.

If we keep only the linear terms on the left sides of the Euler equations, then we have[2]

[1] The density of the fluid $\rho = 1$. – *Ed.*

[2] For details, see the paper [10] of Part I of this book. – *Ed.*

$$\frac{\partial \vec{v}}{\partial t} + 2\omega(\vec{k} \times \vec{v}) + \operatorname{grad} p = 0, \tag{9}$$

$$\operatorname{div} \vec{v} = 0. \tag{10}$$

Without loss of generality, for convenience we replace equation (10) by

$$\operatorname{div} \frac{\partial \vec{v}}{\partial t} = 0. \tag{11}$$

We consider two boundary value problems for this system.

Problem 1. Find a solution of system (9), (11) with conditions

$$p|_{\Gamma} = \text{const}, \tag{12}$$

$$\vec{v}|_{t=0} = \vec{v}_0(x). \tag{13}$$

Problem 2. Find a solution of system (9), (11) with condition (13) and

$$\vec{v}_n|_{\Gamma} = 0. \tag{14}$$

These two problems will be studied either in a bounded domain, or in a cylinder with generators parallel to the y-axis.

The problem of motion of a top with a cavity filled with an ideal uncompressible fluid has been studied as well.

The equations of motion of this top and the boundary conditions for the fluid can be easily written.

Solutions of Problems 1 and 2 can be obtained in the form of series in powers of t. It is more convenient to consider the real Hilbert space H with the inner product

$$(\vec{u}_1, \vec{u}_2) = \int (u_x^1 u_x^2 + u_y^1 u_y^2 + u_z^1 u_z^2)\, dx\, dy\, dz. \tag{15}$$

System (9), (11) can be expressed in the form

$$\frac{d\vec{v}}{dt} = \mathcal{A}\vec{v}. \tag{16}$$

Here the operator $\mathcal{A}$ can be defined in the following way[3]. Consider the subspace $\mathcal{J}_1$ of the space H generated by vector fields orthogonal to gradients of all φ_0, i.e.,

$$\int_{\Omega} \langle \vec{v}, \operatorname{grad} \varphi_0 \rangle\, dx = 0 \tag{17}$$

for all φ_0 compactly-supported in Ω with continuous first-order derivatives. This space is the closure of the space of all $\operatorname{rot}\vec{\psi}$ in H. Another subspace $\mathcal{J}_0$

[3] For details, see the paper [9] of Part I of this book. – *Ed.*

is the space of fields satisfying (17) for all φ_0. $\mathcal{J}_0$ is the closure of the space of all $\operatorname{rot}\overrightarrow{\psi}$ satisfying the condition

$$\operatorname{rot}_n \overrightarrow{\psi}|_\Gamma = 0. \tag{18}$$

For Problem 1 the operator $\mathcal{A}$ is the projector of the vector $2\omega(\overrightarrow{k} \times \overrightarrow{v})$ on $\mathcal{J}_1$, and for Problem 2 it is the projector of the same vector on $\mathcal{J}_0$. Since the vector product $2\omega(\overrightarrow{k} \times \overrightarrow{v})$ and the projectors are bounded operators, we can see that the operator $\mathcal{A}$ is bounded

$$\|\mathcal{A}\| \leq 2\omega. \tag{19}$$

Therefore, the solution of system (16) with condition (13) can be expressed in the form of the series

$$\overrightarrow{v}(x,t) = e^{t\mathcal{A}}\overrightarrow{v}_0(x) = \overrightarrow{v}_0(x) + \frac{t}{1!}\mathcal{A}\overrightarrow{v}_0(x) + \frac{t^2}{2!}\mathcal{A}^2\overrightarrow{v}_0(x) + \cdots, \tag{20}$$

convergent for all t.

In the same way the Cauchy problem (16), (13) is studied in unbounded domains.

From these formulas one easily observes that the three classical Hadamard conditions of well-posedness hold: the existence of solution, uniqueness and continuous dependence on initial conditions.

3. S. Bochner proved a remarkable theorem about the behavior of all solutions of problems of mathematical physics with energy integral independent of t. He proved that for trajectories in a corresponding space to be compact it is necessary that $u(t,x)$ be almost periodic in t, i.e., admit the Fourier series expansion

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x) e^{i\lambda_k t}. \tag{21}$$

The converse is also valid. Every almost periodic function, i.e., the function of the form (21), generates also a compact trajectory in X.

In the classical problems of mathematical physics without relaxation, there always exists an integral of energy, and the compactness of a trajectory is simply related to the compactness of Ω, occupied by the vibrating medium. In the case of one independent variable C. F. Muckenhoupt, and in the case of several variables the author, proved this fact in the simplest cases by using construction of positive integrals of higher order[4].

[4] Almost periodicity of solutions of the first and the second boundary value problems for the wave equation in cylindrical domains was proved in the works: Muckenhoupt, C. F.: J. Math. Phys. Massachusetts Inst. of Technology, **8**, 163–199 (1929); Sobolev, S. L.: Dokl. Akad. Nauk SSSR, **48**, 570–573 (1945); **48**, 646–648 (1945); **49**, 12–15 (1945). – *Ed.*

It is impossible to build these positive integrals of higher order in our problems. R. A. Aleksandryan directly proved in certain special cases the noncompactness of trajectories in Hilbert space. In other cases, on the contrary, the compactness of trajectories was directly proved (R. A. Aleksandryan, R. T. Denchev)[5].

Therefore R. A. Aleksandryan proved that, generally speaking, the behavior of the solution depends on the domain Ω; however, this behavior is more complicated than in the classical case. We will return later to the work of R. A. Aleksandryan.

The alternative to the Fourier series is the Fourier integral, and the alternative to the discrete spectrum is the continuous spectrum. Let us refine these two points.

The classical Fourier method assumes that it is known how to find eigenfunctions of a problem of mathematical physics, i.e., solutions of the equation

$$\mathcal{A}w + \lambda w = 0 \tag{22}$$

in our Hilbert space X. Strictly speaking, a solution of the problem is obtained by the superposition of solutions of the form $e^{i\lambda_k t}w_k(x)$.

Already in Fourier's times, in unbounded domains, mathematicians used solutions of (22), which do not belong to X, for example,

$$e^{i[(a,x)+\lambda_k t]}. \tag{23}$$

It is also known that instead of the Fourier series in this case we deal with the integral

$$w(x,t) = \int\limits_{-\infty}^{+\infty} e^{i\lambda t} d(\varepsilon_\lambda w), \tag{24}$$

where ε_λ is the unit decomposition, and

$$w_0(x) = \int\limits_{-\infty}^{+\infty} d(\varepsilon_\lambda w), \quad \mathcal{A}w_0(x) = \int\limits_{-\infty}^{+\infty} \lambda d(\varepsilon_\lambda w). \tag{25}$$

λ is called a point of the spectrum of $\mathcal{A}$, if the operator $\mathcal{A} + \lambda E$ does not have a bounded inverse operator.

The study of problems included in my report has led in the present day to a different notion of a point of the spectrum, which better suits our goal.

This notion, first introduced by R. A. Aleksandryan without rigorous theory, was developed in the works of I. M. Gelfand, A. G. Kostyuchenko, and somewhat later by Yu. M. Berezanskii[6].

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[5] See, for instance, the works: Aleksandryan, R. A.: Tr. Moskov. Mat. Obshch., **9**, 455–505 (1960); Denchev, R. T.: Dokl. Akad. Nauk SSSR, **126**, 259–262 (1959). – *Ed.*

[6] See references in: Berezanskii, Yu. M.: Expansions in Eigenfunctions of Selfadjoint Operators. Naukova Dumka, Kiev (1965). – *Ed.*

The set $\mathcal{S}(\lambda)$ of values λ is called *spectral* if for all $\lambda \in \mathcal{S}(\lambda)$ we can find a functional $w(\lambda)$ defined on a dense set $X_\lambda \subset X$ such that $\mathcal{A}w + \lambda w$ is identically zero, in other words[7],

$$(w, \mathcal{A}w_1 + \lambda w_1) = 0 \tag{26}$$

for all w_1 from a set everywhere dense in X.

The functional w is called the *eigenfunctional*[8]. For differential equations, eigenfunctionals are functionals continuous in a metric slightly stronger than the metric of X. By eigenfunctionals, the Fourier expansion can be written as

$$v(x,t) = \int e^{i\lambda t} g(\lambda) d\varrho(\lambda), \tag{27}$$

where

$$g(\lambda) = (w(\lambda), v(x,0)). \tag{28}$$

A solution $v(x,t)$ of a problem of mathematical physics generates a compact trajectory in X if and only if the spectrum of the problem is a denumerable set.

The functionals $e^{i(a,x)}$, which were eigenfunctionals in the classical problems of mathematical physics, are equivalent to certain functions not square integrable.

These investigations allowed discovery of new properties of the class of the problems in question.

To find eigenvalues of the operator from equation (22), we need to construct weak solutions of the system $\mathcal{A}w + \lambda w = 0$, which can be reduced in our special case to the problem

$$\mathcal{A}_1 w + \lambda \mathcal{A}_2 w = 0, \tag{29}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are differential operators of the second order with corresponding boundary conditions.

Omitting calculations, let us just note that finding solutions of the system of equations (9) and (11) of the rotating fluid for $\omega = \dfrac{1}{2}$ is reduced to finding solutions of the equation[9]

$$\frac{\partial^2}{\partial t^2} \Delta p + \frac{\partial^2 p}{\partial z^2} = 0. \tag{30}$$

For Problem 1 the boundary condition is still (12). The problem of eigenvalues is to find such a λ that the equation

[7] Here the symbol of the inner product is taken not in the sense of the Hilbert space.

[8] The notion of the eigenfunctional or the generalized eigenfunction was introduced by R. A. Aleksandryan in his Ph. D. Thesis (Moskovsk. Gosudarstv. Univ., Moscow (1949)). – *Ed.*

[9] See appropriate arguments in the paper [9] of Part I of this book. – *Ed.*

$$-\lambda^2 \Delta p + \frac{\partial^2 p}{\partial z^2} = 0 \tag{31}$$

with condition (12) has a nontrivial weak solution. For $-1 < \lambda < 1$, equation (31) is hyperbolic. For all other remaining values of λ (meaning complex values) we can prove that equation (31) has no solutions except for $p = 0$.

Thus, generalized eigenfunctions are weak solutions of the boundary value problem for the hyperbolic equation with the boundary condition $p|_\Gamma = 0$. This problem was solved only in some special cases, which I point out somewhat later.

First, R. A. Aleksandryan[10] studied Problem 1 in the case when the domain Ω is a cylinder with generators parallel to the y-axis, and the solution does not depend on y. In this case the problem is reduced to the search for solutions of the equation

$$\left(-\lambda^2 \frac{\partial^2}{\partial x^2} + (1 - \lambda^2)\frac{\partial^2}{\partial z^2}\right) p = 0 \tag{32}$$

in a domain Ω' with two variables x and z, i.e., to the search of the set of values of λ, for which there exist weak solutions of this equation, which is the vibrating string equation vanishing on the contour.

The problem was studied in detail. Weak solutions were explicitly constructed. They are the step functions accepting at most three values: -1, 0, and 1. Their structure is clear from Fig. 1, where the subdomains are marked for which the solution takes its values.

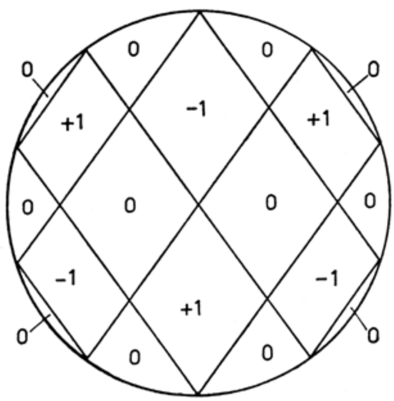

Fig. 1.

[10] The main results of R. A. Aleksandryan's thesis (Moskovsk. Gosudarstv. Univ., 1949) are published in the works: Dokl. Akad. Nauk SSSR, **73**, 631–634 (1950); **73**, 869–872 (1950); Izv. Akad. Nauk Arm. SSR, Ser. Fiz.-Mat. Nauk, **10**, 69–83 (1957); Tr. Moskov. Mat. Obshch., **9**, 455–505 (1960). – *Ed.*

It was shown that the necessary and sufficient condition of the existence of such solutions is the existence of a closed cycle compounded from segments of straight lines

$$\lambda x \pm \sqrt{1-\lambda^2}\, z = C \tag{33}$$

crossing in the contour.

R. A. Aleksandryan also proved that for some special domains, for example, for a disk, an ellipse or a rectangle with sides parallel to the coordinate axis, closed cycles exist for a denumerable set of values of λ everywhere dense in the interval $-1 < \lambda < 1$. Thus, he proved that for these domains all weak solutions are almost periodic and form compact sets in the corresponding Hilbert space.

He also presented an example of a domain with the analytic contour

$$\varrho \leq 1 + \varepsilon \sin^4 \theta \tag{34}$$

arbitrarily close to the disk (see Fig. 2), where weak solutions exist for every value of λ of a certain interval.

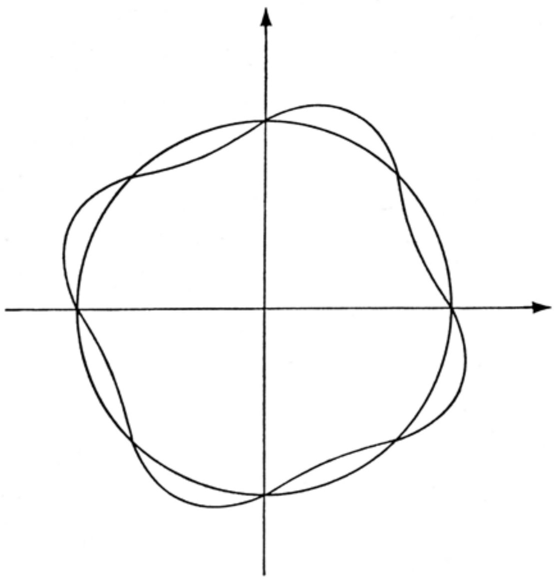

Fig. 2.

The noncompactness of trajectories of solutions for all ε follows from the Bochner theorem.

Thus, the existence of compact trajectories does not depend continuously on the shape of the domain.

An interesting result was established by T. I. Zelenyak[11]. For one class of problems, somewhat more general than the class studied by R. A. Aleksandryan, he presented an explicit form of the asymptotic behavior of solutions. The velocity is expressed as

$$v(x,z,t) = w(x,z)e^{i\mu(x,z)t}. \tag{35}$$

This formula shows that for large enough values of t the difference in phases between two arbitrary points (x_1, z_1) and (x_2, z_2), where the function $\mu(x,z)$ has distinct values, is arbitrarily large, i.e., there are an arbitrary number of maximums and minimums between these two points . The difference between two extreme points is asymptotically equal to $\dfrac{2\pi}{t|\operatorname{grad}\mu|}$ for sufficiently large t, and it can be significantly smaller than the distance between molecules of the fluid. Obviously, this solution does not present any physical sense for large t. This fact, known for nonlinear equations, holds also for linear equations in the absence of viscosity.

4. The subject of my report is one more problem related to these studies.

Let us consider a finite interval $0 \le t \le T$. Does the solution of the problem depend continuously on the shape of the domain?

The answer is positive. I would like to present here a sketch of the proof[12].

We can restrict ourselves to the case when $\Omega_\varepsilon \subset \Omega$. The general problem can be reduced to this special case. Consider the Hilbert space H generated by vector-functions v in Ω and given by (15)

$$(\vec{u}_1, \vec{u}_2) = \int (u_x^1 u_x^2 + u_y^1 u_y^2 + u_z^1 u_z^2)\, dx\, dy\, dz.$$

Let $\vec{k}$ be the unit vector parallel to the z-axis, and $\omega = \dfrac{1}{2}$.

The operator $\mathcal{A}$ in (16) is the projection operator $(\vec{k} \times \vec{v})$ on the space $\mathcal{J}_1$ of all solenoidal vectors. For $\vec{k} \times \vec{v}$ we have the representation

$$\vec{k} \times \vec{v} = \vec{v}_1 + \operatorname{grad} p, \tag{36}$$

where $\int\limits_\Omega \langle \vec{v}_1, \operatorname{grad}\varphi\rangle\, dx = 0$ for each φ satisfying the condition

$$\varphi|_\Gamma = 0,$$

and

$$p|_\Gamma = 0.$$

[11] Zelenyak, T. I.: Differ. Uravn., **2**, 47–64 (1966) – *Ed.*

[12] The continuous dependence of solutions of mixed problems for the Sobolev equation for $n = 2, 3$ were studied also in: Zelenyak, T. I.: Dokl. Akad. Nauk SSSR, **164**, 1225–1228 (1965). – *Ed.*

Then $\mathcal{A}\overrightarrow{v} = \overrightarrow{v}_1$.

Let $\mathcal{A}_\varepsilon^*$ be the operator defined as follows.

Consider $\overrightarrow{k} \times \overrightarrow{v}$ in the domain Ω_ε. In the domain Ω_ε the operator $\mathcal{A}_\varepsilon$ is defined, as well as the operator $\mathcal{A}$ in Ω, by the projection $\overrightarrow{k} \times \overrightarrow{v}$ in the corresponding space.

Assume that

$$\mathcal{A}_\varepsilon^* \overrightarrow{v} = \begin{cases} \mathcal{A}_\varepsilon \overrightarrow{v}, & x \in \Omega_\varepsilon, \\ \mathcal{A}\overrightarrow{v}, & x \notin \Omega_\varepsilon. \end{cases} \tag{37}$$

We show that for all $\overrightarrow{v} \in H$,

$$\|(\mathcal{A} - \mathcal{A}_\varepsilon^*)\overrightarrow{v}\|_{L_2(\Omega)} \to 0, \qquad \varepsilon \to 0.$$

Obviously,

$$(\mathcal{A} - \mathcal{A}_\varepsilon^*)\overrightarrow{v} = \begin{cases} \operatorname{grad} p_\varepsilon - \operatorname{grad} p, & x \in \Omega_\varepsilon, \\ 0, & x \notin \Omega_\varepsilon. \end{cases}$$

The difference $p - p_\varepsilon$ is a function harmonic in Ω_ε.

Indeed,

$$\int \langle (\overrightarrow{k} \times \overrightarrow{v}) - \operatorname{grad} p, \operatorname{grad} \varphi \rangle \, dx = 0$$

and

$$\int \langle (\overrightarrow{k} \times \overrightarrow{v}) - \operatorname{grad} p_\varepsilon, \operatorname{grad} \varphi \rangle \, dx = 0$$

for any function φ compactly-supported in Ω. Hence,

$$\int \langle \operatorname{grad}(p - p_\varepsilon), \operatorname{grad} \varphi \rangle \, dx = 0, \tag{38}$$

which is equivalent to the existence of all derivatives of the function $p - p_\varepsilon$ such that

$$\Delta(p - p_\varepsilon) = 0. \tag{39}$$

Boundary values of this function coincide with values of the function p, since p_ε is equal to zero on Γ_ε.

The norm of $p - p_\varepsilon$ in the space $W_2^1(\Omega_\varepsilon)$, i.e., the norm of $\operatorname{grad}(p - p_\varepsilon)$ in the space $L_2(\Omega_\varepsilon)$, satisfies the condition

$$\|\operatorname{grad}(p - p_\varepsilon)\|_{L_2(\Omega_\varepsilon)} \le c\|(p - p_\varepsilon)|_{\Gamma_\varepsilon}\|_{W_2^{1/2}(\Gamma_\varepsilon)} = c\|p|_{\Gamma_\varepsilon}\|_{W_2^{1/2}(\Gamma_\varepsilon)} \tag{40}$$

(see E. Gagliardo, N. Aronszajn, V. M. Babich, M. Schechter, etc.).

Next, by the embedding theorem, the values of the function p on Γ_ε are arbitrarily close to the values of p on Γ in the norm of $W_2^{1/2}$.

More exactly, if we have sufficiently regular one-to-one correspondence between points of Γ and Γ_ε, then the following inequality holds (E. Gagliardo[13]),

[13] Gagliardo, E.: Rend. Sem. Mat. Univ. Padova, **27**, 284–305 (1957) – *Ed.*

$$\|p|_\Gamma - p_\varepsilon|_{\Gamma_\varepsilon}\|_{W_2^{1/2}(\Gamma)} \le \delta(\varepsilon, p), \tag{41}$$

where $\delta(\varepsilon, p)$ tends to zero for any $p \in W_1^2(\Omega)$. From inequalities (40) and (41) it follows that

$$\|\mathrm{grad}(p - p_\varepsilon)\|_{L_2(\Omega_\varepsilon)} \le c\|p|_{\Gamma_\varepsilon}\|_{W_2^{1/2}(\Gamma_\varepsilon)} \le \delta(\varepsilon, p). \tag{42}$$

By the inequality

$$\|(\mathcal{A} - \mathcal{A}_\varepsilon^*)\overrightarrow{v}\|_{L_2(\Omega_\varepsilon)} \le \|\mathrm{grad}(p - p_\varepsilon)\|_{L_2(\Omega_\varepsilon)},$$

we conclude that for any $\overrightarrow{v}$,

$$\|(\mathcal{A} - \mathcal{A}_\varepsilon^*)\overrightarrow{v}\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We can now prove that

$$\|(e^{t\mathcal{A}_\varepsilon} - e^{t\mathcal{A}})\overrightarrow{v}_0(x)\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for all $\overrightarrow{v}_0(x)$ and $0 \le t \le T$. For all T and ε we can indicate a number $N(\varepsilon, T)$ such that

$$\left\|\left(E + \frac{t}{1!}\mathcal{A} + \frac{t^2}{2!}\mathcal{A}^2 + \ldots + \frac{t^N}{N!}\mathcal{A}^N\right) - e^{t\mathcal{A}}\right\| < \varepsilon$$

for $0 \le t \le T$ and $N > N(\varepsilon, T)$. We have

$$\mathcal{A}_\varepsilon^k - \mathcal{A}^k = \sum_{s=0}^{k-1} \mathcal{A}_\varepsilon^s(\mathcal{A}_\varepsilon - \mathcal{A})\mathcal{A}^{k-s-1}.$$

Hence,

$$\|(e^{t\mathcal{A}_\varepsilon} - e^{t\mathcal{A}})\overrightarrow{v}_0(x)\| \le 2\varepsilon\|\overrightarrow{v}_0(x)\| + \left\|\sum_{k=0}^N \frac{t^k}{k!}(\mathcal{A}_\varepsilon^k - \mathcal{A}^k)\overrightarrow{v}_0(x)\right\|$$

$$= 2\varepsilon\|\overrightarrow{v}_0(x)\| + \left\|\sum_{k=0}^N \frac{t^k}{k!}\sum_{s=0}^{k-1}\left(\mathcal{A}_\varepsilon^s(\mathcal{A}_\varepsilon - \mathcal{A})\mathcal{A}^{k-s-1}\overrightarrow{v}_0(x)\right)\right\|.$$

If we take here ε so small that

$$\|(\mathcal{A}_\varepsilon - \mathcal{A})\mathcal{A}^{k-s-1}\overrightarrow{v}_0(x)\|_{L_2(\Omega)} < \frac{\varepsilon}{N},$$

we obtain

$$\|(e^{t\mathcal{A}_\varepsilon} - e^{t\mathcal{A}})\overrightarrow{v}_0(x)\|_{L_2(\Omega)} \le K\varepsilon,$$

which is required.

5. I would like to say a few words about some connections existing between these problems and other problems of the theory of partial differential equations despite the fact that this is not our main goal.

I would like to speak on three main directions of research.

1. The principal problems of the theory of partial differential equations.

I have already mentioned the works of I. M. Gelfand, A. G. Kostyuchenko and Yu. M. Berezanskii in the theory of generalized eigenfunctions.

In these works, as we have already noted, they established that it is always possible to expand a function in a series (or integral) in “eigenfunctionals”. However, several questions still remain unexplained. There are no methods of constructing of the complete system of eigenfunctionals. The classical theory, dealing with integral equations with compact operators, does not present any solution. Eigenfunctionals are weak solutions of the equations

$$(\mathcal{A}_1 + \lambda \mathcal{A}_2)u = 0 \tag{43}$$

with the boundary conditions

$$Bu|_{\Gamma} = 0. \tag{44}$$

We know what kind of relation exists between a nonhomogeneous problem for partial differential equations and a corresponding homogeneous problem. The study of eigenvalues of (43) is quite close to the study of boundary value problems for these nonhomogeneous equations, which are ill-posed in the sense of Hadamard.

In the works of several authors (F. John, N. N. Vakhaniya, R. T. Denchev, V. I. Arnold, A. Finzi et al.) there were discovered certain remarkable and very delicate properties of these problems[14].

We do not have enough space here to consider it in detail.

2. Another direction, first introduced by R. A. Aleksandryan and his disciples[15], consists of the indication of a new sufficiently broad class of problems for partial differential equations, similar to the problem on vibrations of a rotating fluid. Classes of equations and systems of such type were considered.

If we abstract ourselves from spectral properties and study only the Cauchy problem and mixed problems for partial differential equations not solvable with respect to the highest derivative in t, we should point out some very general results of M. I. Vishik and for certain special cases results of O. A. Ladyzhenskaya[16].

[14] See the following: John, F.: Amer. J. Math., **63**, 141–154 (1941); Vakhaniya, N. N.: Dokl. Akad. Nauk SSSR, **116**, 906–909 (1957); Arnold, V. I.: Izv. Akad. Nauk SSSR, Ser. Mat., **25**, 21–86 (1961); Finzi, A.: Ann. Sci. Ecole Norm. Sup., **69**, 371–430 (1952). – *Ed.*

[15] For the review and references see the paper by Aleksandryan R. A., Berezanskii Yu. M., Il’in V. A., Kostyuchenko A. G. in the book: Partial Differential Equations (Proceedings of the Symposium Dedicated to the 60th Anniversary of Academician S. L. Sobolev) Nauka, Moscow (1970), pp. 3–35. – *Ed.*

[16] Vishik, M. I.: Mat. Sb., **39**, 51–148 (1956); Ladyzhenskaya, O. A.: The Mathematical Problems of Dynamics of Viscous Incompressible Fluid. Nauka, Moscow (1961); English edition: Gordon and Breach Science Publishers, New York – London (1963). – *Ed.*

3. The asymptotical behavior of solutions of one quite broad class of systems not solvable with respect to the highest derivative in t (systems not of Kovalevskaya type) was the subject of the work of T. I. Zelenyak[17]. In several cases he established that trajectories of solutions are noncompact in every finite part of the space, i.e., unstable in some sense.

We have given an impression of one class of problems of mathematical physics, and we see that it is still in its initial stages of study[18].

[17] The review and references can be found in the book: Zelenyak, T. I.: Selected Questions of Qualitative Theory of Partial Differential Equations. Novosibirsk. Gosudarstv. Univ., Novosibirsk (1970). – *Ed.*

[18] S. L. Sobolev's investigations of the problem on small oscillations of a rotating fluid originated the most intense interest in equations not solvable with respect to the highest-order derivative

$$\mathcal{A}_0 D_t^l u + \sum_{k=0}^{l-1} \mathcal{A}_{l-k} D_t^k u = f,$$

where $\mathcal{A}_0$, $\mathcal{A}_1, \dots, \mathcal{A}_l$ are linear differential operators in $x = (x_1, \dots, x_n)$. At present, equations of such form are often called *equations of Sobolev type* in the literature. Bibliographical comments and extensive references devoted to the theory of boundary value problems for equations of Sobolev type can be found in the book: Demidenko, G. V., Uspenskii, S. V.: Partial Differential Equations and Systems not Solvable with Respect to the Highest-Order Derivative. Marcel Dekker, New York, Basel (2003). – *Ed.*

Part II

Computational Mathematics and Cubature Formulas

1. Schwarz's Algorithm in Elasticity Theory*

S. L. Sobolev

The conventional Schwarz algorithm in potential theory allows us to construct a solution of the Dirichlet problem in the domain $D_{12}^{'}$, which is the set-theoretic sum of two domains D_1 and D_2, which are partially overlapping, and we do it by solving the Dirichlet problems in each of these subdomains successively. A similar algorithm was proposed by S. G. Mikhlin[1] for solving the problem of elasticity theory in the multiply-connected domain D_{12}, which is the set-theoretic product of D_1 and D_2. S. G. Mikhlin established the convergence of this algorithm only in the case when individual boundary contours of the multiply-connected domain under consideration are located sufficiently far from each other.

Using established existence theorems for the problems of elasticity theory, we show convergence of both the Schwarz algorithm, and the Mikhlin algorithm in the problem of elastic displacements when D_1 and D_2 are domains from a wider class.

Let the boundary of D_1 be S_1 and let the boundary of D_2 be S_2. Denote by $S_1^{'}$ the part of S_1 lying inside D_2 and by $S_1^{''}$ the part of S_1 outside D_2. Similarly, let $S_2^{'}$ be the part of S_2 inside D_1, and let $S_2^{''}$ be the part of S_2 outside D_1. Also, let $\mathcal{L}$ stand for the intersection line of S_1 and S_2, separating S_1' from $S_1^{''}$ and S_2' from $S_2^{''}$. We assume that all listed surfaces are piece-wise smooth.

The first problem under study consists of solving the equations of elasticity theory

$$L^{\alpha\beta}u_\beta = (\lambda+\mu)\nabla^\alpha\nabla^\beta u_\beta + \mu\nabla^2 u^\alpha = (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mu\Delta\mathbf{u} = 0 \tag{1}$$

in the domain $D_{12}^{'}$ under two conditions:

$$u_\beta\big|_{S_1^{''}} = \psi_\beta \quad \text{and} \quad u_\beta\big|_{S_2^{''}} = \chi_\beta. \tag{2}$$

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* Dokl. Akad. Nauk SSSR, **4**, 235–238 (1936)

[1] Mikhlin, S. G.: The method of successive approximations in application to biharmonic problem. Tr. Seism. Inst., **39** (1934), 14 p. – *Ed.*

The second problem consists of solving the same equations in the domain D_{12} under two conditions:

$$u_\beta\big|_{S_1'} = \psi_\beta \quad \text{and} \quad u_\beta\big|_{S_2'} = \chi_\beta. \tag{3}$$

Here u_β stands for the unknown displacement vector, whose values are given on the boundary of the corresponding domain.

To each vector v_γ considered as an elastic displacement and given in the domain D_{12}, there corresponds the *strain energy*, which, as is known, may be written as follows:

$$E(v_\gamma) = \int\limits_{D_{12}'} \left[\lambda(\nabla^\alpha v_\alpha)^2 + \frac{\mu}{2}(\nabla^\alpha v_\beta + \nabla_\beta v^\alpha)^2\right] dxdydz. \tag{4}$$

The problem for solving (1) with boundary conditions on the boundary of a given domain is equivalent to the problem of variational calculus that consists of the search for the minimum of an integral of type (4), over the given domain on the set of all vectors satisfying the given boundary conditions.

The sequence of functions $\{u_\beta^{(i)} \mid i = 0, 1, 2, \dots\}$, converging to the solution of the first problem under study, is constructed according to the following rule.

All vector functions $u_\beta^{(2k)}$ are regular in D_2 and in $D_1' = D_1 \setminus D_2$, and satisfy the system of equations (1) in these subdomains. In D_{12}' these vector functions are continuous. Moreover, they satisfy the following conditions:

$$u_\beta^{(2k)}\big|_{S_1''} = \psi_\beta, \quad u_\beta^{(2k)}\big|_{S_2''} = \chi_\beta; \quad u_\beta^{(2k)}\big|_{S_2'} = \begin{cases} \omega_\beta & \text{for} \quad k = 0, \\ u_\beta^{(2k-1)}\big|_{S_2'} & \text{for} \quad k > 0. \end{cases} \tag{5}$$

Here ω_β is an arbitrary sufficiently smooth vector function admitting on $\mathcal{L}$ the same values as ψ_β and χ_β.

All vector functions $u_\beta^{(2k+1)}$ are continuous in D_{12}', regular in D_1 and in $D_2' = D_2 \setminus D_1$, satisfy the system of equations (1) in D_1 and D_2'. Moreover, they satisfy the following conditions:

$$u_\beta^{(2k+1)}\big|_{S_1''} = \psi_\beta, \; u_\beta^{(2k+1)}\big|_{S_2''} = \chi_\beta, \; u_\beta^{(2k+1)}\big|_{S_1'} = u_\beta^{(2k)}\big|_{S_1'} \text{ for } k \geq 0. \tag{6}$$

For $k > 0$ the vector $u_\beta^{(2k)}$ gives the minimum of $E(\mathbf{u}_\gamma)$ on the set of all $\mathbf{u}_\gamma$ satisfying (5). In particular, $u_\gamma^{(2k-1)}$ can be chosen as $\mathbf{u}_\gamma$; hence,

$$E\left(u_\gamma^{(2k)}\right) \leq E\left(u_\gamma^{(2k-1)}\right). \tag{7}$$

Similarly, we establish that

$$E\left(u_\gamma^{(2k+1)}\right) \leq E\left(u_\gamma^{(2k)}\right). \tag{8}$$

The monotone decreasing sequence of the positive numbers $E\left(u_\gamma^{(n)}\right)$ is convergent. If c stands for its limit, then for sufficiently large k the following inequalities hold:

$$E\left(u_\gamma^{(2k-1)}\right) \leq c+\varepsilon \quad \text{and} \quad E\left(u_\gamma^{(2k)}\right) \leq c+\varepsilon. \tag{9}$$

Using the fact that the function $(u_\beta^{(2k)}+u_\beta^{(2k-1)})/2$ satisfies conditions (5), we obtain

$$E\left(\frac{u_\beta^{(2k)}+u_\beta^{(2k-1)}}{2}\right) \geq c. \tag{10}$$

Whence, and from (9), it follows that

$$E\left(u_\beta^{(2k)}-u_\beta^{(2k-1)}\right) = 2E\left(u_\beta^{(2k)}\right)+2E\left(u_\beta^{(2k-1)}\right)$$
$$-4E\left(\frac{u_\beta^{(2k)}+u_\beta^{(2k-1)}}{2}\right) \leq 4\varepsilon, \tag{11}$$

and, similarly,

$$E\left(u_\beta^{(2k+1)}-u_\beta^{(2k)}\right) \leq 4\varepsilon. \tag{12}$$

By the means of usual estimates from these inequalities, we establish that the sequence of the functions $w_\beta^{(n)}=u_\beta^{(n)}-u_\beta^{(n-1)}$ is uniformly convergent to zero in the domain $D_{12}^{'}$ together with the sequence of the derivatives.

Let us now construct the Green function for the domain $D_{12}^{'}$, i.e., the function $G^{\alpha\beta}$ of two points P and P_1 such that, as a function of P_1, it has a singularity at the point P, and is expressed as

$$G^{\alpha\beta}(P,P_1)=g^{\alpha\beta}+\frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\nabla^\alpha\nabla^\beta r-\frac{1}{4\pi\mu}\frac{1}{r}\delta^{\alpha\beta}. \tag{13}$$

Here r stands for the distance between P_1 and P, and, for a fixed α, $g^{\alpha\beta}$ is such a solution of the equations of elasticity theory that the function $G^{\alpha\beta}(P,P_1)$ vanishes on the boundary of the domain $D_{12}^{'}$.

Let n be an even number. Then $u_\beta^{(n)}$ has a discontinuity of derivatives on $S_2^{'}$, and $u_\beta^{(n-1)}$ has a discontinuity of derivatives on $S_1^{'}$. In this case, the function $u_\alpha^{(n)}$ can be represented in D_2 as

$$u_\alpha^{(n)}=\int\limits_{S_2^{''}}\left(w_\beta^{(n)}\Gamma_{\alpha\gamma}^{\beta}-G_{\alpha\beta}T_\gamma^{(n)\beta}\right)dS^\gamma+\int\limits_{S_1^{'}}\psi_\beta\Gamma_{\alpha\gamma}^{\beta}\,dS^\gamma+\int\limits_{S_2^{'}}\chi_\beta\Gamma_{\alpha\gamma}^{\beta}\,dS^\gamma, \tag{14}$$

and the function $u_\alpha^{(n-1)}$ can be represented in D_1 as

$$u_\alpha^{(n-1)} = \int\limits_{S_1''} \left(G_{\alpha\beta} T_\gamma^{(n)\beta} - w_\beta^{(n)} \Gamma_{\alpha\gamma}^\beta \right) dS^\gamma + \int\limits_{S_1'} \psi_\beta \Gamma_{\alpha\gamma}^\beta \, dS^\gamma + \int\limits_{S_2'} \chi_\beta \Gamma_{\alpha\gamma}^\beta \, dS^\gamma. \quad (15)$$

Here $T^{(n)\alpha\beta}$ is a stress tensor corresponding to the displacements $w_\beta^{(n)}$, i.e., the tensor expressed through $w_\beta^{(n)}$ by the formula

$$T^{(n)\alpha\beta} = \lambda \nabla^\gamma w_\gamma^{(n)} \delta^{\alpha\beta} + \mu(\nabla^\alpha w^{(n)\beta} + \nabla^\beta w^{(n)\alpha}), \quad (16)$$

and, for a fixed α, $\Gamma^{\alpha\beta\gamma}$ is a stress tensor corresponding to the displacements $G^{\alpha\beta}$, i.e.,

$$\Gamma^{\alpha\beta\gamma} = \lambda \nabla_\zeta G^{\alpha\zeta} \delta^{\beta\gamma} + \mu(\nabla^\beta G^{\alpha\gamma} + \nabla^\gamma G^{\alpha\beta}). \quad (17)$$

Indeed, the functions defined by the right sides of (14) and (15), obviously, satisfy conditions in (2). Their difference coincides with $w_\beta^{(n)}$ in D_{12}, and each of them is regular in the corresponding domain. Since the expansion of $w_\beta^{(n)}$ in such a sum is unique, for this expansion is determined uniquely to within an additive function vanishing on the boundary of D'_{12} and regular inside, then the validity of (14) and (15) is established.

Using (14) and (15), it is already not difficult to show that both $u_\alpha^{(2k)}$, and $u_\alpha^{(2k-1)}$ converge to the same limit:

$$\lim_{n\to\infty} u_\alpha^{(n)} = \int\limits_{S_1'} \psi_\beta \Gamma_{\alpha\gamma}^\beta \, dS^\gamma + \int\limits_{S_2'} \chi_\beta \Gamma_{\alpha\gamma}^\beta \, dS^\gamma. \quad (18)$$

As is known, this limit is the solution of the first problem under study. The algorithm for the solution of the second problem is similar. We construct the sequence of $v_\beta^{(n)}$ according to the following rule.

All functions $v_\beta^{(2k)}$ are regular in D_2 and in D'_1, and they satisfy the system of equations (1) in these domains. In D'_{12} they are continuous. Moreover, they satisfy the following conditions:

$$v_\beta^{(0)}\big|_{S_2'} = \chi_\beta, \quad v_\beta^{(0)}\big|_{S_2''} = \omega_\beta, \quad v_\beta^{(0)}\big|_{S_1''} = \lambda_\beta, \quad (19)$$

where ω_β and λ_β are two sufficiently smooth vector functions admitting on $\mathcal{L}$ the same values as ψ_β and χ_β;

$$v_\beta^{(2k)}\big|_{S_2'} = v_\beta^{(2k-1)}\big|_{S_2'}, \quad v_\beta^{(2k)}\big|_{S_2''} = 0, \quad v_\beta^{(2k)}\big|_{S_1''} = 0 \quad \text{for} \quad k \geq 1. \quad (20)$$

All functions $v_\beta^{(2k+1)}$ are continuous in D'_{12}, regular in D'_1 and in D_2, satisfy the system of equations (1) in D'_1 and D_2. Moreover, on the boundaries under consideration, they satisfy the following conditions:

$$v_\beta^{(1)}\big|_{S_1'} = -\psi_\beta + v_\beta^{(0)}\big|_{S_1'}, \quad v_\beta^{(1)}\big|_{S_1''} = 0, \quad v_\beta^{(1)}\big|_{S_2''} = 0, \quad (21)$$

$$v_\beta^{(2k+1)}\big|_{S_1'} = v_\beta^{(2k)}\big|_{S_1'}, \quad v_\beta^{(2k+1)}\big|_{S_1''} = 0, \quad v_\beta^{(2k+1)}\big|_{S_2''} = 0 \quad \text{for} \quad k \geq 1. \quad (22)$$

By the same way as in the case of the first problem, we establish that $v_\beta^{(n)}$ converges to zero in $D_{12}^{'}$. Hence, the series

$$v_\beta = v_\beta^{(0)} - v_\beta^{(1)} + v_\beta^{(2)} - \dots + (-1)^n v_\beta^{(n)} + \dots \quad (23)$$

is regularly convergent on the boundary. Thus, it is also convergent in the domain D_{12}, where its sum is the required solution of the second problem under study.

The same arguments can be applied in the infinite domain. In this case, it is useful to note that the Green function $G^{\alpha\beta}$ must satisfy the corresponding conditions at infinity. In particular, if $D_{12}^{'}$ is the whole space, then the function $g^{\alpha\beta}$ is equal to 0.

Based on the same principles it is possible to prove the convergence of the method in many other cases: for example, in solving the Dirichlet problem for the self-adjoint elliptic equation with variable coefficients, in solving the fundamental problem for the biharmonic equation, etc. Moreover, the number of overlapping or added domains is unessential; we can consider the case with more than two such domains.

2. On Solution Uniqueness of Difference Equations of Elliptic Type*

S. L. Sobolev

Let us consider the equation

$$Lu_{m,n} = \frac{1}{4}\{u_{m+1,n+1} + u_{m-1,n-1} + u_{m+1,n-1} + u_{m-1,n+1} - 4u_{m,n}\} = 0, \quad (1)$$

where

$$-\infty < m < +\infty, \quad -\infty < n < +\infty,$$

for the function $u_{m,n}$ defined at nodes of the net $m + n = 2k + 1$. We prove the following theorem.

Theorem. *The solution of equation* (1), *increasing at infinity slower than* $\sqrt{m^2 + n^2}$, *can be reduced to a constant.*

Proof. Let us consider differences of values of the function $u_{m,n}$ in two adjacent points: $u_{m'+1,n'} - u_{m'-1,n'}$ and $u_{m',n'+1} - u_{m',n'-1}$. We show that by the assumptions of the theorem these differences are equal to zero. Let us choose the coordinate origin at the point (m', n'). Then it suffices to establish that the differences

$$\delta_1 = u_{1,0} - u_{-1,0} \quad \text{and} \quad \delta_2 = u_{0,1} - u_{0,-1} \quad (2)$$

are equal to zero.

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* Dokl. Akad. Nauk SSSR, **87**, 179–182 (1952)
In the same journal (**88**, p. 740) the following letter of the author is published in relation to the article in question: "Yu. G. Reshetnyak pointed my attention on the fact that all main results of my notes "On solution uniqueness of difference equations of elliptic type" and "On one difference equation", published in Vol. 87 of the journal "Doklady Akademii Nauk SSSR", are contained in the paper by A. Stöhr'a "Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten", published in 1950 in the journal "Mathematische Nachrichten" (Bd. 3, H. 4, 5, 6),which I have not noticed before. I express my sincere gratitude to Yu. G. Reshetnyak for this indication. S. Sobolev. January 12, 1953."

Let us consider the square

$$|m| = 2p+1, \qquad |n| = 2p+1. \tag{3}$$

For equation (1) the maximum principle holds, and, hence, we have the uniqueness theorem for the solution of the Dirichlet problem. Thus, the function $u_{m,n}$ must coincide with the sum of four functions:

$$u_{m,n} = u^I_{m,n} + u^{II}_{m,n} + u^{III}_{m,n} + u^{IV}_{m,n},$$

satisfying, in turn, equation (1). Each of these functions is zero on three sides of the square, and coincides with $u_{m,n}$ on the fourth side. For example,

$$u^I_{m,n}\big|_{n=2p+1} = u_{m,n}\big|_{n=2p+1}, \qquad u^I_{m,n}\big|_{n=-2p-1} = u^I_{m,n}\big|_{|m|=2p+1} = 0. \tag{4}$$

In the same way, the remaining functions u^{II}, u^{III}, and u^{IV} are constructed. If we establish that the differences δ_1 and δ_2 vanish on each $u^{(j)}$ separately, then the theorem is proved.

By symmetry it suffices to establish the theorem only for one of the functions $u^{(j)}$, for example, for $u^I_{m,n}$. For the function $u^I_{m,n}$ we can give the explicit expression

$$u^I_{m,n} = \sum_{k=1}^{2p+1} a_k U^{(k)}_{m,n}, \tag{5}$$

where

$$U^{(k)}_{m,n} = \sqrt{\frac{2}{2p+1}}\,\frac{\sin{(m+2p+1)\alpha_k}\ \sinh{(n+2p+1)\beta_k}}{\sinh{(4p+2)\beta_k}}, \tag{6}$$

$$\text{for} \quad k = 1, 2, \ldots, 2p, \quad \alpha_k = \frac{k\pi}{4p+2}, \quad e^{\beta_k} = \cot\left(\frac{\alpha_k}{2} + \frac{\pi}{4}\right), \tag{7}$$

and

$$U^{(2p+1)}_{m,n} = \begin{cases} (-1)^{m/2}\sqrt{\dfrac{2}{2p+1}}\sin{(m+2p+1)\alpha_{2p+1}} & \text{for} \quad n = 2p+1, \\ 0 & \text{for} \quad n < 2p+1. \end{cases} \tag{8}$$

The system of functions (6) and (8), as is not difficult to see, satisfies conditions (4). Moreover, they are linearly independent, since the orthogonal conditions hold for them:

$$\sum_{m=-2p}^{2p} U^{(k)}_{m,2p+1} U^{(l)}_{m,2p+1} = \begin{cases} 1 & \text{for} \quad k = l, \\ 0 & \text{for} \quad k \neq l. \end{cases} \tag{9}$$

The total number of these functions equals $(2p+1)$, and thus the system of functions (6) and (8) is complete.

It is easy to see that

$$\sum_{k=1}^{2p+1} a_k^2 = \sum_{m=-2p}^{2p} u_{m,2p+1}^2 = \left(A^I\right)^2(p). \tag{10}$$

In other words, the sum of squares of the Fourier coefficients is equal to the sum of squares of values of the function u on the side $n = 2p+1$.

The differences δ_1 and δ_2 can be estimated by the quantity $A^I(p)$. Indeed,

$$\delta_1 = \sum_{k=1}^{2p+1} a_k (U_{1,0}^{(k)} - U_{-1,0}^{(k)}) \quad \text{and} \quad \delta_2 = \sum_{k=1}^{2p+1} a_k (U_{0,1}^{(k)} - U_{0,-1}^{(k)}).$$

Therefore, by the Cauchy inequality for sums, we have

$$\delta_1^2 + \delta_2^2 \le \left(\sum_{k=1}^{2p+1} a_k^2\right) \sum_{k=1}^{2p+1} \left[\left(U_{1,0}^{(k)} - U_{-1,0}^{(k)}\right)^2 + \left(U_{0,1}^{(k)} - U_{0,-1}^{(k)}\right)^2\right]. \tag{11}$$

The computation shows that

$$\sum_{k=1}^{2p+1} \left[(U_{1,0}^{(k)} - U_{-1,0}^{(k)})^2 + (U_{0,1}^{(k)} - U_{0,-1}^{(k)})^2\right]$$
$$\le \frac{2}{2p+1} \sum_{k=1}^{2p+1} \left(\frac{\tan\gamma_k - \cot\gamma_k}{\tan^{2p+1}\gamma_k - \cot^{2p+1}\gamma_k}\right)^2, \tag{12}$$

where $\gamma_k = \alpha_k/2 + \pi/4 = k\pi/(8p+4) + \pi/4$. Assuming

$$\frac{2}{2p+1} \sum_{k=1}^{2p+1} \left(\frac{\tan\gamma_k - \cot\gamma_k}{\tan^{2p+1}\gamma_k - \cot^{2p+1}\gamma_k}\right)^2 = S(p), \tag{13}$$

from (11) and (12) we obtain

$$\delta_1^2 + \delta_2^2 \le \left(A^I\right)^2(p) S(p). \tag{14}$$

Let us estimate the sum $S(p)$. To do this, it is convenient to compare it with the integral

$$I(p) = \frac{8}{\pi} \int_{\pi/4}^{\pi/2} \left(\frac{\tan\gamma - \cot\gamma}{\tan^{2p+1}\gamma - \cot^{2p+1}\gamma}\right)^2 d\gamma. \tag{15}$$

The integrand in (15) is a decreasing function of the variable γ. Therefore the sum $S(p)$ is the value of the integral $I(p)$, calculated using the Darboux sum with deficiency, and, hence,

$$S(p) \le I(p). \tag{16}$$

Estimating $I(p)$ and taking $\cot\gamma = y$ as the new variable, we have

$$I(p) = \frac{8}{\pi}\int_0^1 \frac{(1-y^2)^2}{(1-y^{4p+2})^2}\,\frac{y^{4p}}{1+y^2}\,dy = I^{(1)}(p) + I^{(2)}(p), \tag{17}$$

where
$$\begin{cases} I^{(1)}(p) = \dfrac{8}{\pi}\displaystyle\int_0^{1-\frac{1}{4p+2}} \frac{(1-y^2)^2}{(1-y^{4p+2})^2}\,\frac{y^{4p}}{1+y^2}\,dy, \\ I^{(2)}(p) = \dfrac{8}{\pi}\displaystyle\int_{1-\frac{1}{4p+2}}^{1} \frac{(1-y^2)^2}{(1-y^{4p+2})^2}\,\frac{y^{4p}}{1+y^2}\,dy. \end{cases} \tag{18}$$

Let us introduce the new independent variable

$$z = (1-y)(4p+2) \tag{19}$$

in the integral $I^{(2)}(p)$. This enables us to express $I^{(2)}(p)$ as

$$\frac{8}{\pi(4p+2)^3}\int_0^1 \frac{\left(2-\dfrac{z}{4p+2}\right)^2}{\left[1+\left(1-\dfrac{z}{4p+2}\right)^2\right]\left(1-\dfrac{z}{4p+2}\right)^2}$$

$$\times\frac{\left(1-\dfrac{z}{4p+2}\right)^{4p+2} z^2}{\left[1+\left(1-\dfrac{z}{4p+2}\right)^{4p+2}\right]^2}\,dz.$$

The first factor under the integral sign is bounded, while the second factor increases and tends to $\dfrac{e^{-z}z^2}{(2-e^{-z})^2}$. Hence,

$$I^{(2)}(p) \le \frac{B}{(4p+2)^3}\int_0^1 \frac{e^{-z}z^2}{(1-e^{-z})^2}\,dz \le \frac{C}{2p^3}, \tag{20}$$

where B and C are certain constants. In the case of $I^{(1)}(p)$, we have:

$$\left(1-y^{4p+2}\right)^2 \ge \left[1-\left(1-\frac{1}{4p+2}\right)^{4p+2}\right]^2 \ge \left(1-\frac{1}{e}\right)^2.$$

Therefore

$$I^{(1)}(p) \le B\int_0^{1-\frac{1}{4p+2}} (1-y)^2 y^{4p}dy$$

$$< B\int\limits_0^1 (1-y)^2 y^{4p}\,dy = B\frac{\Gamma(4p+1)\Gamma(3)}{\Gamma(4p+4)} \le \frac{C}{2p^3}. \tag{21}$$

Whence and from (16), (17), and (20) it follows that

$$S(p) \le \frac{C}{2p^3}. \tag{22}$$

If now

$$\big(A^I\big)^2(p) = o\big((2p+1)^3\big), \tag{23}$$

then $\delta_1^2 + \delta_2^2 = o(1)$, and, hence,

$$\delta_1^2 + \delta_2^2 = 0. \tag{24}$$

Obviously, estimate (23) holds in the assumptions of the theorem, since from these assumptions it follows that

$$\frac{1}{4(2p+1)}\Big[\big(A^I\big)^2(p) + \big(A^{II}\big)^2(p) + \big(A^{III}\big)^2(p) + \big(A^{IV}\big)^2(p)\Big] = o(p^2), \tag{25}$$

from which (23) follows. $\square$

3. On One Difference Equation*

S. L. Sobolev

In the present note we consider the equation

$$\frac{1}{4}\bigl(w_{m+1,n+1}+w_{m-1,n-1}+w_{m+1,n-1}+w_{m-1,n+1}-4w_{m,n}\bigr)$$
$$=\begin{cases}1, & m^2+n^2=0,\\ 0, & m^2+n^2>0\end{cases} \tag{1}$$

for the values of m and n in the whole plane:

$$-\infty<m<+\infty,\quad -\infty<n<+\infty. \tag{2}$$

We construct a solution of this equation which increases at infinity as $\ln\sqrt{m^2+n^2}$ and is equal to zero for $m=n=0$:

$$w_{00}=0. \tag{3}$$

Such solution, as follows from [1], is unique.

The unknown solution has the form:

$$w_{m,n}=-\frac{1}{\pi^2}\oint\limits_{|u|=1}\Bigl(\oint\limits_{|v|=1}\frac{u^m v^n-1}{u^2v^2+u^2+v^2+1-4uv}\,dv\Bigr)\,du. \tag{4}$$

Let us indicate how to verify the validity of this formula, and also let us explicitly calculate the solution and estimate its asymptotical behavior as m and $n\to\infty$.

Below it is shown that the integral in the right side of (4) has meaning for all values of m and n. Therefore we can substitute (4) into equation (1), and after this substitution, we have

$$Lw_{m,n}=-\frac{1}{\pi^2}\oint\limits_{|u|=1}\Bigl(\oint\limits_{|v|=1}\frac{L(u^m v^n-1)}{G(u,v)}\,dv\Bigr)\,du$$

* Dokl. Akad. Nauk SSSR, **87**, 341–343 (1952)

$$= -\frac{1}{4\pi^2} \oint\limits_{|u|=1} \Big(\oint\limits_{|v|=1} u^{m-1} v^{n-1}\, dv \Big)\, du = -\frac{1}{4\pi^2} \oint\limits_{|u|=1} u^{m-1}\, du \oint\limits_{|v|=1} v^{n-1}\, dv.$$

Hence, the validity of equation (1) follows.

The denominator of the integrand in formula (4) may be written as

$$G(u,v) = u^2v^2 + u^2 + v^2 + 1 - 4uv = (u^2+1)(v - v_1)(v - v_2), \tag{5}$$

where

$$v_1 = i\frac{u-i}{u+i}, \quad v_2 = \frac{1}{v_1} = -i\frac{u+i}{u-i}. \tag{6}$$

Function $G(u,v)$ has two roots for fixed u. These roots are both equal to 1 in module, if u is real; for u from the upper half-plane, $|v_1| < 1$, $|v_2| > 1$, and for u from the lower half-plane, conversely, $|v_1| > 1$, $|v_2| < 1$.

Applying to (4) the residue theorem, we obtain:

$$w_{m,n} = \frac{1}{\pi} \int\limits_{\substack{-\pi \le \arg\, u \le 0 \\ |u|=1}} \frac{u^m \Big(-i\dfrac{u+i}{u-i}\Big)^n - 1}{u^2 - 1}\, du$$

$$-\frac{1}{\pi} \int\limits_{\substack{0 \le \arg\, u \le \pi \\ |u|=1}} \frac{u^m \Big(i\dfrac{u-i}{u+i}\Big)^n - 1}{u^2 - 1}\, du. \tag{7}$$

The integrand in the first term is regular in the complex plane, except for the point $u = i$, where it has a pole, since the roots of the denominator at the points $u = \pm 1$ cancel the roots in the numerator. In the second integral, the integrand is regular in the complex plane, except for the point $u = -i$, where it has a pole. The singularities at the points $u = \pm 1$ cancel each other. From formula (4) it follows that

$$w_{m,n} = w_{-m,n} = w_{m,-n} = w_{-m,-n}$$

$$= w_{n,m} = w_{-n,m} = w_{n,-m} = w_{-m,-n}. \tag{8}$$

Thus, the lines $m = 0$, $n = 0$, $m = n$, and $m = -n$ are the symmetry axis for the system of the values of $w_{m,n}$.

Formula (7) allows us to compute the values of $w_{m,n}$. After elementary transformations we have:

$$w_{2k,0} = w_{0,2k} = w_{-2k,0} = w_{0,-2k} = \frac{4}{\pi}\Big(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1}\Big). \tag{9}$$

If we know the values of $w_{m,n}$ on the coordinate axis, using the symmetry of this function, we can easily compute its values at the other points, using directly equation (1). Obviously,

$$w_{1,1} = w_{-1,1} = w_{1,-1} = w_{-1,-1} = 1. \tag{10}$$

At any point (m, n) the function $w_{m,n}$ is the sum of two terms:

$$w_{m,n} = w^{I}_{m,n} + w^{II}_{m,n}. \tag{11}$$

Here $w^{I}_{m,n}$ is an integer number; $w^{II}_{m,n}$ is a transcendental number of the form r/π, where r is rational. Both functions $w^{I}_{m,n}$ and $w^{II}_{m,n}$ for $\sqrt{m^2+n^2} \to \infty$ increase significantly faster than logarithmically. The function $w^{I}_{m,n}$ is a certain particular solution of equation (1), and $w^{II}_{m,n}$ is a particular solution of the corresponding homogeneous equation.

The estimate of $w_{m,n}$ for a large value of $\sqrt{m^2+n^2}$ follows from formula (7). First, let us note that $w_{m,n}$ for $m > n$ can be written as

$$w_{m,n} = \frac{4}{\pi}\mathrm{Re}\left\{ \int\limits_0^1 \frac{u^m\left(i\dfrac{u-i}{u+i}\right)^n - 1}{u^2-1}\, du \right\}. \tag{12}$$

Estimating formula (12), instead of u, we introduce the new variable

$$\psi = u^{\cos\alpha}\left(i\frac{u-i}{u+i}\right)^{\sin\alpha}, \tag{13}$$

where

$$\cos\alpha = \frac{m}{\sqrt{m^2+n^2}} \quad \text{and} \quad \sin\alpha = \frac{n}{\sqrt{m^2+n^2}}. \tag{14}$$

Formula (12) in the new variables has the form

$$w_{m,n} = \frac{4}{\pi}\mathrm{Re}\left\{ \int\limits_0^1 \frac{\psi^{\sqrt{m^2+n^2}} - 1}{\psi^2-1}\, d\psi + \int\limits_0^1 \psi^{\sqrt{m^2+n^2}}\left(\frac{du}{u^2-1} - \frac{d\psi}{\psi^2-1}\right) \right.$$
$$\left. + \int\limits_0^1 \left(\frac{d\psi}{\psi^2-1} - \frac{du}{u^2-1}\right) \right\}. \tag{15}$$

Choosing the integration path so that the function ψ is real, and using the boundedness of the function $\left(\dfrac{du/d\psi}{u^2-1} - \dfrac{1}{\psi^2-1}\right)$, we see that the second term in parenthesis tends to zero as $1/\sqrt{m^2+n^2}$. The third term is, as computations show, pure imaginary, and is equal to $i\alpha$. Estimating the first term, we have:

$$\left| w_{m,n} - \frac{2}{\pi}\left(C + \ln 2 + \ln\sqrt{m^2+n^2}\right)\right| \le \frac{A}{\sqrt{m^2+n^2}},$$

where C is the Euler constant. Thus, the order of growth of $w_{m,n}$ is $\ln\sqrt{m^2+n^2}$, as required.

References

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4. Certain Comments on the Numeric Solutions of Integral Equations*

S. L. Sobolev

Summary. In this work we consider the algorithm of solving integral equations of the second kind and of Fredholm type with a continuous kernel for the functions of one independent variable that is based on replacement of the integral by a sum. The possibility of this replacement is established using the theorem on a regular approximation of completely continuous operators (the strong convergence for uniform complete continuity of the approximating operators). We introduce a definition of closure of the computational algorithm, and indicate the possibility of the loss of significant digits in computations in the case when the algorithm is irregularly closed. We also give other applications of the closure of computational algorithms (see [1, 2]).

1 The Closure of Computational Algorithms

The solution of many problems of mathematical physics and of analysis in general, when the unknowns are the functions of one or many independent variables, or more generally, members of a certain functional space, is often rather complicated and consists of a large number of arithmetic and logic operations. Ultimately, these actions lead to an approximate solution that is expressed through a finite set of elements and the values of the approximate solution are written down as real numbers containing only finitely many digits.

Mostly, the approximate solution depends on one or several certain parameters determining the quality of this approximation. For example, such parameters could be the mesh-sizes of the lattice in the method of finite differences, or the number of successive approximations in iterations. When the parameters tend to the limits (including, for example, infinity), the approximate solution converges to the true solution in the sense of convergence of the space in question.

The typical technique for solving the equation

* Izv. Akad. Nauk SSSR, Ser. Mat., **20**, 413–436 (1956)

$$Lu = f \tag{1}$$

in the functional space U with the right side from the space F is replacement of the operator L by an approximate operator L_k acting over the finite approximation U_k of U and such that its inverse could be found in a finite number of steps. To solve the equation

$$L_k u_k = f_k, \tag{2}$$

where $u_k \in U_k$ and $f_k \in F_k$, we transform it into a sequence of the equations

$$L_k^{(m)} u_k = D_k^{(m)} f_k, \quad m = 1, 2, \dots, N. \tag{3}$$

For $m = N$ it is assumed that $L_k^{(N)} \equiv E$, where E is the identity operator, i.e., $u_k = D_k^{(N)} f_k$. In general, the approximation depends on certain parameters h_1, h_2, …, h_r, and as $h_s \to 0$ for $s = 1, 2, \dots, r$ the closure of the discrete set U_k is the space U, the closure of the operator L_k is the operator L, and the limit of f_k is the function f.

Sequence (3) together with the methods of approximation of U, L, and f by U_k, L_k, and f_k constitutes the *computational algorithm T of solving* (1).

Sometimes, equalities (3) are written as

$$L_k^{(m)} u_k = \varphi_k^{(m)}, \quad m = 1, 2, \dots, N, \tag{3*}$$

where $\varphi_k^{(m)} = D_k^{(m)} f_k$ is calculated in fact.

It could happen that not only u_k, L_k, and f_k tend to u, L, and f, respectively, as $h_s \to 0$ for $s = 1, 2, \dots, r$, but also all of equations (3) (or (3*)), or more precisely, all of $L_k^{(m)}$ and $D_k^{(m)}$ or $\varphi_k^{(m)}$, make sense in the limit.

Suppose that there is an ordered set of the functions $m(h_1, \dots, h_r, z)$ of the real parameter z such that they are given on the set $\mathcal{E}_z$ and

$$m(h_1, h_2, \dots, h_r, z_1) \ge m(h_1, h_2, \dots, h_r, z_2) \quad \text{for} \quad z_1 > z_2. \tag{4}$$

Assume that with the corresponding definition of convergence (usually it does not cause any difficulties) in this way and with $h_s \to 0$ for $s = 1, 2, \dots, q$ the space U_k passes into a certain space $U_k^{(z, h_{q+1}, \dots, h_r)}$, F_k passes into $F_k^{(z, h_{q+1}, \dots, h_r)}$, while the operators $L_k^{(m)}$ and $D_k^{(m)}$ tend to the limiting operators

$$L_k^{(z, h_{q+1}, \dots, h_r)} \quad \text{and} \quad D_k^{(z, h_{q+1}, \dots, h_r)}$$

with the range in the space $\varPhi_k^{(z, h_{q+1}, \dots, h_r)}$, and finally, $\varphi_k^{(m)}$ and f_k also tend to the limit. Thus, as a limit case of (3) or (3*) we obtain the equalities

$$L_k^{(z, h_{q+1}, \dots, h_r)} u_k^{(z, h_{q+1}, \dots, h_r)} = D_k^{(z, h_{q+1}, \dots, h_r)} f_k^{(z, h_{q+1}, \dots, h_r)} \tag{5}$$

or

$$L_k^{(z, h_{q+1}, \dots, h_r)} u_k^{(z, h_{q+1}, \dots, h_r)} = \varphi_k^{(z, h_{q+1}, \dots, h_r)}. \tag{5*}$$

Definition. *Let formulas* (5) (*or* (5*)) *make sense. Then we call* (5) (*or* (5*)) *the closure of the computational algorithm* T.

It is easy to see that in general the closure of the computational algorithm depends on the method of the passage to the limit.

Let us present an example of the closure of the computational algorithm. Consider the Poisson equation in two variables:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{6}$$

in the square

$$0 \le x \le 1, \quad 0 \le y \le 1. \tag{7}$$

Suppose that we are looking for the solution of this equation satisfying the Dirichlet conditions:

$$u\Big|_{Q \in S} = \omega(Q). \tag{8}$$

In this case Lu is the operator sending a twice continuously differentiable function $u(x, y)$ to a pair of the functions: $u\Big|_S$, defined on the boundary S of square (7), and Δu, defined in the interior of this square. Thus, U is the space of twice continuously differentiable functions in the square, and F is a space of pairs of continuous functions, one of which is defined on the boundary of the square, and the other of which is defined in the interior of it.

Let us study the computational *algorithm of iterations* T_{it} for this problem consisting in the following. We replace the space of functions defined in the square by the spaces of discrete functions given with accuracy η on the net

$$x = jh_1, \quad y = lh_2, \quad \text{where} \quad h_1 = 1/M_1, \quad h_2 = 1/M_2,$$

M_1 and M_2 are integer numbers. We define $L_k u$ by the following simplest way. It simply assumes the values of u at the corresponding points of the boundary of the square, and

$$L_k u = -\frac{1}{h_1^2}\big(u(x + h_1, y) + u(x - h_1, y) - 2u(x, y)\big)$$

$$-\frac{1}{h_2^2}\big(u(x, y + h_2) + u(x, y - h_2) - 2u(x, y)\big) \tag{9}$$

at the points in the interior of the square. We solve the equation $L_k u = f_k$ for $h_1 = h_2$ using the method of iterations. For the initial approximation we take an arbitrary continuous function $u_0(x, y)$ satisfying the condition

$$u_0\Big|_{Q \in S} = \omega(Q). \tag{10}$$

Performing the computation with accuracy η, we find the mth approximation of u at all points in the interior of the square by means of the equality

$$u_m(x,y) = \frac{h^2}{4} f(x,y) + \frac{1}{4}\Big(u_{m-1}(x+h,y)$$

$$+u_{m-1}(x-h,y) + u_{m-1}(x,y+h) + u_{m-1}(x,y-h)\Big). \qquad (11)$$

At the points of the boundary, u_m and u_0 assume the values of $\omega(Q)$. At sufficiently large m the function u_m provides the solution of the problem with an arbitrary accuracy.

Already G. O'Brien, S. Kaplan, and M. Hymen [5] focused their attention on the analogue between the described algorithm and solving the heat equation

$$\frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - f(x,y) \qquad (12)$$

by the method of nets. Let us trace this connection in more detail.

If we replace the domain $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z < \infty$ by the net $x = jh$, $y = lh$, $z = mh^2/4$, and the operator $\partial v/\partial z - \Delta v$ by

$$\frac{4}{h^2}\Big[v(x,y,z+\tau)$$

$$-\frac{v(x+h,y,z) + v(x-h,y,z) + v(x,y+h,z) + v(x,y-h,z)}{4}\Big], \qquad (13)$$

then the formulas for $v(jh, lh, mh^2/4)$ are the same as formulas (11) for $u_m(jh, lh)$.

Hence, for the function $u_m(x,y)$ given by recurrent formulas (11) the equality holds,

$$u_m(jh, lh) = v\left(jh, lh, \frac{mh^2}{4}\right). \qquad (14)$$

Equality (14), where the values of $v(jh, lh, mh^2/4)$ are calculated in fact, is the individual realization of sequence (3^*) for the problem under study. In this event, $v(jh, lh, mh^2/4)$ coincides with $\varphi_k^{(m)}$.

It is easy to see that the algorithm T_{it} has a closure as $h \to 0$. Let

$$m(h,\eta,z) = \left[\frac{4z}{h^2}\right], \qquad (15)$$

where the brackets stands for the integer part of a number. If $h \to 0$ and $\eta \to 0$, then instead of (14) we obtain the equality

$$u(x,y,z) = v(x,y,z). \qquad (16)$$

Equality (16), where $v(x,y,z)$ is the solution of (12) under the conditions

$$v(x,y,0) = u_0(x,y), \quad v\Big|_{Q\in S} = \omega(Q),$$

is the closure of the algorithm T_{it}.

Similarly, we can also construct other examples of closures of algorithms. For example, passing in (9) to the limit as $h_2 \to 0$, we obtain the formulas of the so-called "*method of lines*" for equation (6).

2 The Regular and Irregular Closures of Computational Algorithms

Let $L^z u = D^z f$ be the closure of the computational algorithm T. If L^z and D^z are operators acting in a metric space, then they could be both bounded and unbounded in the corresponding metric. We say that the algorithm T has a *regular closure* provided that L^z and D^z are uniformly bounded. If L^z and D^z are unbounded, we say that the algorithm T has an *irregular closure.*

It is not difficult to verify that if the algorithm T has an irregular closure, one could experience serious difficulties using it, when we improve the accuracy of computations.

In the following sections we demonstrate different circumstances, which could occur in this case, on the example of integral equations of the second kind and of Fredholm type with a continuous kernel for the functions of one independent variable.

3 The Approximation of the Solution of Integral Equations of Fredholm Type

In the present work we consider a linear integral equation of the second kind and of Fredholm type for the function $\varphi(x)$ of one independent variable x, defined in the bounded interval $0 \le x \le 1$:

$$\varphi(x) = \int\limits_0^1 K(x,y)\varphi(y)\,dy + f(x), \tag{17}$$

$$\text{or} \quad (E-A)\varphi = f, \quad A\varphi \equiv \int\limits_0^1 K(x,y)\varphi(y)\,dy. \tag{18}$$

We assume that the kernel $K(x,y)$ of the operator A is a continuous function in both variables.

To solve equation (17), one often replaces it by a system of algebraic linear equations

$$\varphi_i = \sum_{j=1}^{N} K_{ij}\varphi_j + f_i, \quad i = 1, 2, \ldots, N, \tag{19}$$

with N unknowns φ_1, φ_2, $\ldots$, φ_N, where

$$\varphi_i = \varphi(t_i), \quad 0 \le t_1 < t_2 < \cdots < t_N \le 1, \tag{20}$$

and K_{ij} are certain quadrature coefficients of the approximate expression for the operator A.

There are many different techniques of solving (19). Mainly, they are reduced to the method of iterations or the method of successive elimination of unknowns. As is known, the iterations do not always converge, while the method of successive eliminations allows, in principle, finding a solution of the problem, though sometimes we have to perform a very large number of steps in this case.

However, in the method of successive eliminations one could also encounter some difficulties of a fundamental nature, as we show further. We show that the algorithm of solving equation (17), using its replacement by a finite system and successive eliminations, could have an irregular closure. As we will see later, this is also related to a number of further troubles: solving the equation with the aid of such an algorithm, we lose significant digits in our calculations. Therefore, to obtain solutions with a required accuracy, we would carry out the computation with an excessive number of digits.

4 Approximations of the Linear Operator

The linear operator $A\varphi \equiv \int\limits_0^1 K(x,y)\varphi(y)\,dy$ with the continuous kernel $K(x,y)$ is a completely continuous operator in the space C of continuous functions defined in the interval $(0,1)$. If equation (17) has a solution for any right side, then the operator $E-A$ has the inverse $(E-A)^{-1} = E+\Gamma$, where $\Gamma f(x) = \int\limits_0^1 \Gamma(x,y)f(y)\,dy$. The function $\Gamma(x,y)$ is called the *resolvent of the kernel* $K(x,y)$. The solution of equation (18) can be written as

$$\varphi = \big(E+\Gamma\big)f. \tag{21}$$

Let us recall that $E+\Gamma$ is the left and right inverse of $E-A$ simultaneously, and, hence, it is the unique inverse of $E-A$.

From the formulas $(E+\Gamma)(E-A) = E$ and $(E-A)(E+\Gamma) = E$, after opening the parenthesis, it follows that

$$\Gamma - A - \Gamma A = 0, \tag{22}$$

$$\Gamma - A - A\Gamma = 0. \tag{23}$$

These equations are the *integral equations on the resolvent.*

Let us consider a certain sequence of the linear operators A_N. It is customary to say that this sequence *strongly converges* to the operator A, if for each function φ from C the sequence

$$A_N\varphi \tag{24}$$

converges to $A\varphi$ uniformly, i.e., it converges in the metric of the space C. Of course, from strong convergence of the operators A_N to A, convergence in the norm (or so-called uniform convergence) of the operators A_N to A does not follow. In other words, the sequence

$$\|A_N - A\| \tag{25}$$

does not have to converge to zero in this case (recall that $\|A\|$ is defined by the equality $\|A\| = \sup\limits_{\|\varphi\|_C \leq 1} \|A\varphi\|$). However, the following theorem holds.

Theorem (Banach). *If the sequence A_N strongly converges to A, then $\|A_N\|$ is bounded,*

$$\|A_N\| \leq M. \tag{26}$$

This theorem is valid for all complete spaces (see, e.g., [6]).

We say that the set of linear completely continuous operators A_N is *uniformly completely continuous*, if it transforms a bounded set Φ from C into a compact set; in other words, if the union of sets $A_N\Phi$ is compact.

Remark. The set of all uniformly completely continuous operators is bounded.

This remark is obvious, since if there exist elements ψ_N such that

$$\|\psi_N\| = 1 \quad \text{and} \quad A_N\psi_N \to \infty, \tag{27}$$

then we would have the noncompact set $\{A_N\psi_N\}$ for the bounded sequence $\{\psi_N\}$, which contradicts the condition of the uniform complete continuity.

If the operators A_N strongly converges to A, and also they are uniformly completely continuous, then we say that the operators A_N *approximate A properly.*

Lemma. *If there exists a sequence $\{\psi_N\}$ such that*

$$\|\psi_N\| \geq h > 0 \quad \textit{and} \quad \lim_{N\to\infty} (E - A_N)\psi_N = 0, \tag{28}$$

and the operators A_N approximate A properly, then the operator $E - A$ has no inverse.

Proof. Without loss of generality, we can assume that $\|\psi_N\| = 1$. Otherwise we would consider the new sequence $\{\frac{1}{\|\psi_N\|}\psi_N\}$, that also satisfies the conditions of the lemma.

The sequence $A_N\psi_N$ is compact, and we can take it strongly convergent by keeping a convergent subsequence and discarding everything else. Let

$$\lim_{N\to\infty} A_N\psi_N = \xi_0. \tag{29}$$

We prove that ψ_N strongly converges to ξ_0 in C. Indeed,

$$\lim_{N\to\infty} \psi_N = \lim_{N\to\infty} (E - A_N)\psi_N + \lim_{N\to\infty} A_N\psi_N = \xi_0.$$

Obviously, $\|\xi_0\| = 1$. Let us show that ξ_0 is an eigenelement for A. Indeed,

$$(E - A)\xi_0 = (A_N - A)\xi_0 + (E - A_N)(\xi_0 - \psi_N) + (E - A_N)\psi_N.$$

The first term on the right side of this equality tends to zero in view of strong convergence of A_N to A, and the second tends to zero by convergence of ψ_N to ξ_0 and the boundedness of $E - A_N$. Finally, the last term tends to zero by the condition of the lemma.

Hence, the norm of $(E-A)\xi_0$ is less than any given number, and therefore it is equal to zero, i.e., ξ_0 is indeed the eigenelement of the operator A.

In view of the Fredholm theorems, the inverse of $E - A$ does not exist, as required. □

Let us denote by Γ the operator of the resolvent for A, i.e., let

$$(E - A)^{-1} = E + \Gamma, \tag{30}$$

and let Γ_N be the resolvent for the operator A_N.

Theorem. *If the operators A_N approximate the operator A properly, and $E - A$ has an inverse, then, from some N on, the operators $E - A_N$ also have inverse operators. In this case the resolvents Γ_N for A_N approximate the resolvent Γ for A properly.*

Proof. Indeed, from some N on, the operators $E - A_N$ must have inverses. Otherwise, by the Fredholm theorem, we could find a subsequence of $E - A_N$ such that each its terms would have the normed eigenelement ψ_N:

$$(E - A_N)\psi_N = 0, \quad \|\psi_N\| = 1. \tag{31}$$

By the previous lemma it would immediately imply that the operator $E - A$ has no inverse, which contradicts the condition of the theorem.

Let us establish that the operators $E + \Gamma_N$ strongly converge to $E + \Gamma$. Assume the contrary. Then we can find such element ξ_0 and such subsequence Γ_{N_k} of Γ_N that the norm of the element

$$\psi_k = (E + \Gamma_{N_k})\xi_0 - (E + \Gamma)\xi_0 \tag{32}$$

is positive; $\|\psi_k\| > 0$. By applying to both parts of (32) the operator $E - A_{N_k}$, we obtain

$$\begin{aligned}(E - A_{N_k})\psi_k &= \xi_0 - (E - A_{N_k})(E + \Gamma)\xi_0 \\ &= \xi_0 - (E - A)(E + \Gamma)\xi_0 + (A_{N_k} - A)(E + \Gamma)\xi_0 \\ &= (A_{N_k} - A)(E + \Gamma)\xi_0. \end{aligned} \tag{33}$$

By strong convergence of A_{N_k} to A, the right side of (33) tends to zero. Hence, the left side also tends to zero, i.e.,

$$\lim_{k\to\infty} (E - A_{N_k})\psi_k = 0. \tag{34}$$

By applying the previous lemma, we obtain that $(E - A)^{-1}$ does not exist, which contradicts the condition of the theorem. Hence, $E + \Gamma_N$ strongly converges to $E + \Gamma$, i.e., Γ_N strongly converges to Γ.

It remains to prove that Γ_N is uniformly completely continuous. In view of (23) we have

$$\Gamma_N = A_N(E + \Gamma_N). \tag{35}$$

By the mentioned Banach theorem, the operators $E + \Gamma_N$ are uniformly bounded. Moreover, by condition, A_N is uniformly completely continuous. Hence, $A_N(E + \Gamma_N)$ send a given bounded set to a compact set, and $\Gamma_N = A_N(E + \Gamma_N)$ is uniformly completely continuous, as required. □

5 Approximations by Means of Sums

The completely continuous operator $A\varphi \equiv \int\limits_0^1 K(x,y)\varphi(y)\,dy$ can be approximated by the operators of the form

$$A_N\varphi = \sum_{n=1}^{N} K_n(x)\varphi(t_n), \tag{36}$$

where the points $t_1, t_2, \ldots, t_N$ lie in the interval $0 \le t \le 1$. An example of such approximations is the approximation using the replacement of the integral $A\varphi$ by the sum as in the rectangle rule, or using some other quadrature formula.

Dividing the interval $0 \le y \le 1$ into N equal parts of the same length h, where $Nh = 1$, and taking $t_n = h/2 + (n-1)h$ and $K_n(x) = hK(x, t_n)$, we obtain the approximate formula

$$\int\limits_0^1 K(x,y)\varphi(y)\,dy \cong \sum_{n=1}^{N} K_n(x)\varphi(t_n) = A_N^*\varphi. \tag{37}$$

For a given continuous function φ at sufficiently large N the right side of (37) is arbitrarily close to the left side uniformly. This follows from the fact that by the Weierstrass theorem, the oscillation

$$\operatorname*{osc}_y K(x,y)\varphi(y)$$

of the function $K(x,y)\varphi(y)$ with respect to y converges to zero independent of x in the interval $(n-1)h \le y \le nh$, and the difference

$$\int\limits_0^1 K(x,y)\varphi(y)\,dy - \sum_{n=1}^{N} K_n(x)\varphi(t_n)$$

does not exceed the value of

$$\sup_n \operatorname*{osc}_{(n-1)h\leq y\leq nh} K(x,y)\varphi(y).$$

The operators A_N defined by (36) can approximate the operator A properly. In particular, this is valid for the approximations A_N^* constructed above by using integration by the rectangle rule. Indeed, if

$$\|\varphi\| = \max_{0\leq y\leq 1} |\varphi(y)| = M, \tag{38}$$

then $|A_N^*\varphi| \leq M \sum\limits_{n=1}^{N} |K_n(x)| \leq MA$, where $A \geq \sup\limits_{x,y} |K(x,y)|$, and, hence,

$$\|A_N^*\| \leq MA.$$

Next,

$$\Big|A_N^*\varphi(x'') - A_N^*\varphi(x')\Big| = \Big|\sum_{n=1}^{N}\big(K_n(x'') - K_n(x')\big)\varphi(y_n)\Big|$$

$$\leq M \max_n \Big|K(x'',t_n) - K(x',t_n)\Big| \sum_{n=1}^{N} h. \tag{39}$$

For sufficiently small $|x'' - x'|$ the right side of (39) is arbitrarily small by the Weierstrass theorem on the uniform continuity of the function $K(x,y)$. Thus, for uniformly bounded functions φ_m the family $A_N^*(\varphi_m)$ is also uniformly bounded and equicontinuous.

In what follows we are going to consider not only the operators A_N^*, but also arbitrary operators approximating A properly.

Let a sequence of the operators $A_N\varphi$ given by (36) approximate the operator $A\varphi$ properly. Then, in view of the established lemmas, for the solutions of equations

$$\varphi = A_N\varphi + f, \tag{40}$$

$$\varphi = A\varphi + f, \tag{41}$$

we obtain the formulas

$$\varphi = f + \Gamma_N f, \tag{42}$$

$$\varphi = f + \Gamma f, \tag{42*}$$

where the operators Γ_N approximate the operator Γ properly.

We refer the computational algorithm leading to the replacement of (41) by (40) and solving (40) by the successive elimination of unknowns as the *algorithm of nets* T_c. Let us consider this algorithm in detail.

The solution of (40) is essentially reduced to a solution of the system of linear algebraic equations (19). Indeed, in order to calculate $A_N\varphi$, and, hence, φ, satisfying (40), it suffices to know the values of φ at the points t_1, t_2, …, t_N.

Let $B_1\varphi$ be the operator that sends continuous function φ to the vector $(\varphi_1, \varphi_2, \dots, \varphi_N)$ from the space R^N, where $\varphi_s = \varphi(t_s)$. The operator A_N acting on the function φ is, obviously, the product of two operators:

$$A_N\varphi = A_N^* B_1\varphi, \tag{43}$$

where A_N^* sends the vector $(\varphi_1, \varphi_2, \dots, \varphi_N)$ to the continuous function by the rule

$$A_N^*\varphi = \sum_{n=1}^{N} K_n(x)\varphi_n. \tag{44}$$

In these settings, equation (40) has the form

$$\varphi = A_N^* B_1\varphi + f. \tag{45}$$

By applying to both parts of (45) the operator B_1, we have

$$B_1\varphi = B_1 A_N^* B_1\varphi + B_1 f. \tag{46}$$

Let $B_1 A_N^* \equiv \widetilde{A}_N$, $B_1\varphi \equiv \widetilde{\varphi}$, and $B_1 f \equiv \widetilde{f}$. Then (46) is equivalent to

$$\widetilde{\varphi} = \widetilde{A}_N\widetilde{\varphi} + \widetilde{f}. \tag{47}$$

Denoting by $\mathcal{E}$ the identity operator in R^N, we write the solution of (47) as

$$\widetilde{\varphi} = (\mathcal{E} + \widetilde{\Gamma}_N)\widetilde{f}, \tag{48}$$

where

$$(\mathcal{E} + \widetilde{\Gamma}_N)(\mathcal{E} - \widetilde{A}_N) = (\mathcal{E} - \widetilde{A}_N)(\mathcal{E} + \widetilde{\Gamma}_N) = \mathcal{E}, \tag{49}$$

or

$$\widetilde{\Gamma}_N - \widetilde{A}_N - \widetilde{\Gamma}_N\widetilde{A}_N = \widetilde{\Gamma}_N - \widetilde{A}_N - \widetilde{A}_N\widetilde{\Gamma}_N = 0.$$

We can verify that the resolvent of the operator $E - A_N^* B_1$ is given by the formula $(E - A_N^* B_1)^{-1} = E + A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)B_1$, or, if we set

$$\Gamma_N^* = A_N^*(\mathcal{E} + \widetilde{\Gamma}_N), \tag{50}$$

by the formula

$$(E - A_N^* B_1)^{-1} = E + \Gamma_N^* B_1. \tag{51}$$

Indeed, in view of (49) we have:

$$\begin{aligned} &(E - A_N^* B_1)\big(E + A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)B_1\big) \\ &= E - A_N^*\mathcal{E}B_1 + A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)B_1 - A_N^*\widetilde{A}_N(\mathcal{E} + \widetilde{\Gamma}_N)B_1 \\ &= E + A_N^*(\widetilde{\Gamma}_N - \widetilde{A}_N - \widetilde{A}_N\widetilde{\Gamma}_N)B_1 = E, \end{aligned} \tag{52}$$

and also

$$\begin{aligned} &\big(E + A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)B_1\big)(E - A_N^* B_1) \\ &= E - A_N^*\mathcal{E}B_1 + A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)B_1 - A_N^*(\mathcal{E} + \widetilde{\Gamma}_N)\widetilde{A}_N B_1 \\ &= E + A_N^*(\widetilde{\Gamma}_N - \widetilde{A}_N - \widetilde{\Gamma}_N\widetilde{A}_N)B_1 = E, \end{aligned} \tag{53}$$

as required.

6 Another Method of Reducing the Problem to an Algebraic System

We can indicate one more method of reducing equation (41) to a system of linear algebraic equations. Let us approximate the kernel $K(x,y)$ by the piece-wise constant function defined by

$$K_N^{(1)}(x,y) = K(t_k,t_l) \quad \text{for} \quad |x-t_k| \leq \frac{h}{2} \quad \text{and} \quad |y-t_l| \leq \frac{h}{2}, \tag{54}$$

where t_k and t_l are given before. In this case the operator

$$A_N^{(1)}\varphi \equiv \int\limits_0^1 K_N^{(1)}(x,y)\varphi(y)\,dy$$

already maps C not in C, but in its isometric expansion $\mathcal{M}$, the space of bounded measurable functions. In the space $\mathcal{M}$ the operators $A_N^{(1)}$ converges to A uniformly. Indeed,

$$\|(A_N^{(1)}-A)\varphi\| \leq \max_x \int\limits_0^1 \left|K_N^{(1)}(x,y)-K(x,y)\right| dy \cdot \sup_y |\varphi(y)| \leq \varepsilon\|\varphi\|, \tag{55}$$

as required.

It is well-known that in the case of uniform convergence of $A_N^{(1)}$ to A the existence of the inverse of $E-A$ leads to the existence of the inverses of all operators $E-A_N^{(1)}$, from some N on. Moreover, the operators $\Gamma_N^{(1)} = (E-A_N^{(1)})^{-1}-E$ approximate the operator $\Gamma = (E-A)^{-1}-E$ uniformly. The following equation

$$(E-A_N^{(1)})\varphi = f \tag{56}$$

can be solved by a method similar to the one used in solving (40).

To solve (56), we introduce the operator B_2 mapping C (or $\mathcal{M}$) in the N-dimensional Euclidean space of the vectors $\boldsymbol{\varphi} = (\varphi_1,\varphi_2,\ldots,\varphi_N)$, where

$$\varphi_k = \frac{1}{h}\int\limits_{(k-1)h+h/2}^{kh+h/2} \varphi(y)\,dy \quad \text{for} \quad k=1,2,\ldots,N.$$

Further, let $A_N^{(1)*}$ be the operator sending the vector $\boldsymbol{\varphi} = (\varphi_1,\varphi_2,\ldots,\varphi_N)$ to the function

$$A_N^{(1)*}\boldsymbol{\varphi} = \sum_{k=1}^N K_N^{(1)}\Big(x,(k-1)h+\frac{h}{2}\Big)\varphi_k. \tag{57}$$

Then, by the definition of $A_N^{(1)}\varphi$, we have

$$A_N^{(1)}\varphi = A_N^{(1)*}B_2\varphi. \tag{58}$$

Writing (56) as $\varphi - A_N^{(1)*}B_2\varphi = f$, and acting on both parts of this equality by B_2, we obtain:

$$B_2\varphi - B_2 A_N^{(1)*}B_2\varphi = B_2 f. \tag{59}$$

Assuming $B_2\varphi = \approx\!\!\!\!\varphi$, $B_2A_N^{(1)*} = \overset{\approx}{A}_N$, and $B_2 f = \overset{\approx}{f}$, we rewrite (59) as follows

$$(\mathcal{E} - \overset{\approx}{A}_N)\,\overset{\approx}{\varphi} = \overset{\approx}{f}\ . \tag{60}$$

Obviously, this vector equation can be solved, if we find the inverse

$$(\mathcal{E} - \overset{\approx}{A}_N)^{-1} = \mathcal{E} + \overset{\approx}{\varGamma}_N\ . \tag{61}$$

Hence, similarly to the case of equation (48), we conclude:

$$(E - A_N^{(1)*}B_2)^{-1} = E + \varGamma_N^{(1)*}B_2, \tag{62}$$

where

$$\varGamma_N^{(1)*} = A_N^{(1)*}(\mathcal{E} + \overset{\approx}{\varGamma}_N). \tag{63}$$

7 Partially Solved Equations

Let us return to the study of the integral equation

$$\varphi(x) = \int_0^1 K(x,y)\varphi(y)\,dy + f(x). \tag{64}$$

Let us consider the equation

$$\varphi(x) - \int_z^1 \varGamma(x,y,z)\varphi(y)\,dy = f(x) + \int_0^z \varGamma(x,y,z)f(y)\,dy, \tag{65}$$

assuming that for $z = 0$ it becomes equation (64). For $z = 1$ we have another particular case of (65),

$$\varphi(x) = f(x) + \int_0^1 \varGamma(x,y)f(y)\,dy. \tag{66}$$

Thus, the solution of (64) is expressed through the Fredholm resolvent. We establish that, in general, for a given equation (64) we can find an equivalent system of form (65), where z ranges over the interval $0 \le z \le 1$.

Let $z_1 < z_2$. For $z = z_1$ equation (65) may be written as

$$\varphi(x) - \int_{z_1}^{z_2} \Gamma(x, y, z_1)\varphi(y)\,dy = f(x)$$

$$+ \int_0^{z_1} \Gamma(x, y, z_1) f(y)\,dy + \int_{z_2}^{1} \Gamma(x, y, z_1)\varphi(y)\,dy. \tag{67}$$

For $z_1 \le x \le z_2$ equation (67) may be considered as an integral equation with respect to the function $\varphi(x)$ in this interval.

For the sake of convenience, instead of the function $\varphi(x)$ defined on the interval $0 \le x \le 1$, we temporarily introduce three functions:

$$\begin{array}{lll} \varphi_1(x) & \text{for} & 0 \le x \le z_1, \\ \varphi_2(x) & \text{for} & z_1 \le x \le z_2, \\ \varphi_3(x) & \text{for} & z_2 \le x \le 1, \end{array} \tag{68}$$

and rewrite (67) as

$$\begin{aligned} \varphi_1 &= A_{11}\varphi_1 + A_{12}\varphi_2 + A_{13}\varphi_3 + f_1, \\ \varphi_2 &= A_{21}\varphi_1 + A_{22}\varphi_2 + A_{23}\varphi_3 + f_2, \\ \varphi_3 &= A_{31}\varphi_1 + A_{32}\varphi_2 + A_{33}\varphi_3 + f_3. \end{aligned} \tag{69}$$

Suppose that

$$(E - A_{22})^{-1} = E + \Gamma_{22}. \tag{70}$$

Then

$$(E - A_{22})(E + \Gamma_{22}) = (E + \Gamma_{22})(E - A_{22}) = E,$$

$$\text{or} \quad A_{22}\Gamma_{22} = \Gamma_{22}A_{22} = \Gamma_{22} - A_{22}. \tag{71}$$

By excluding φ_2 from the right side of all equations (69) after calculations, we obtain:

$$\begin{aligned} \varphi_1 &= B_{11}\varphi_1 + B_{12}f_2 + B_{13}\varphi_3 + f_1, \\ \varphi_2 &= B_{21}\varphi_1 + B_{22}f_2 + B_{23}\varphi_3 + f_2, \\ \varphi_3 &= B_{31}\varphi_1 + B_{32}f_2 + B_{33}\varphi_3 + f_3, \end{aligned} \tag{72}$$

$$\text{where} \quad B_{ij} = A_{ij} + A_{i2}A_{2j} + A_{i2}\Gamma_{22}A_{2j}. \tag{73}$$

For $i = 2$ or $j = 2$ the equality in (73) is simplified. Using (71), we obtain

$$\begin{aligned} B_{i2} &= A_{i2} + A_{i2}\Gamma_{22}, \\ B_{2j} &= A_{2j} + \Gamma_{22}A_{2j}, \\ B_{22} &= \Gamma_{22}. \end{aligned} \tag{74}$$

Obviously, formulas (72) are mutual with (69), which we can easily verify by noticing that $E - A_{22} = (E + \Gamma_{22})^{-1}$. Equations (73) and (72) may be rewritten in integral form as

$$\Gamma(x,y,z_2) = \Gamma(x,y,z_1) + \int\limits_{z_1}^{z_2} \Gamma(x,t,z_1)\Gamma(t,y,z_1)\,dt$$

$$+ \int\limits_{z_1}^{z_2}\int\limits_{z_1}^{z_2} \Gamma(x,t,z_1)\Gamma(t,u,z_2)\Gamma(u,y,z_1)\,dtdu \tag{75}$$

and

$$\varphi(x) = f(x) + \int\limits_0^{z_2} \Gamma(x,y,z_2)f(y)\,dy + \int\limits_{z_2}^{1} \Gamma(x,y,z_2)\varphi(y)\,dy, \tag{76}$$

respectively. Similar equations have been studied by M. G. Krein [3] and N. P. Sergeev [4].

From (65), holding for all z, it is possible to obtain one more important equation of Volterra type. Setting in (65) $z = x$, we have

$$\varphi(x) - \int\limits_x^1 \Gamma(x,y,x)\varphi(y)\,dy = f(x) + \int\limits_0^x \Gamma(x,y,x)f(y)\,dy.$$

Denoting $\Gamma(x,y,x)$ by $V(x,y)$, we obtain

$$\varphi(x) - \int\limits_x^1 V(x,y)\varphi(y)\,dy = f(x) + \int\limits_0^x V(x,y)f(y)\,dy. \tag{77}$$

Equation (77) is called the *resolving Volterra equation* for (64). The solution of (77) can be obtained by the method of successive approximations, or by some other method. By the construction itself we see that the solution of (77) coincides with the solution of (64) provided that $V(x,y)$ is a continuous function.

Simultaneously with the partially solved integral equation for (41) we are going to consider the partially solved equations for the sequences of equations (40) and (47).

It is not difficult to calculate the corresponding operators in fact. Let us consider the vector $(\varphi_1, \varphi_2, \dots, \varphi_N)$ as a union of two vectors $\varphi^{(1)} = (\varphi_1, \dots, \varphi_{k(z)})$ and $\varphi^{(2)} = (\varphi_{k(z)+1}, \varphi_{k(z)+2}, \dots, \varphi_N)$. In this case the operator B_1 introduced above is a pair of the operators $B_{11}^{(z)}$ and $B_{12}^{(z)}$. The first of them $B_{11}^{(z)}$ sends φ to the k-dimensional vector $\varphi^{(1)}$, and the second operator $B_{12}^{(z)}$ sends φ to the $(N-k)$-dimensional vector $\varphi^{(2)}$. Respectively, the operator A_N^* is the sum of two operators $A_{N,1}^*$ and $A_{N,2}^*$:

$$A_N^*\varphi = A_{N,1}^*\varphi^{(1)} + A_{N,2}^*\varphi^{(2)},$$

while the operator $\widetilde{A}_N$ has a form of the square matrix:

$$\widetilde{A}_N = \begin{pmatrix} \widetilde{A}_{N,11} & \widetilde{A}_{N,12} \\ \widetilde{A}_{N,21} & \widetilde{A}_{N,22} \end{pmatrix} = \begin{pmatrix} B_{11}^{(z)} A_{N,1}^* & B_{12}^{(z)} A_{N,1}^* \\ B_{11}^{(z)} A_{N,2}^* & B_{12}^{(z)} A_{N,2}^* \end{pmatrix}.$$

Assuming $(\mathcal{E} - \widetilde{A}_{N,11})^{-1} = \mathcal{E} + \widetilde{\varGamma}_{N,11}$, we can consider instead of (47) the following partially solved system

$$\begin{aligned} \varphi^{(1)} &= \widetilde{B}_{N,11}\varphi^{(1)} + \widetilde{B}_{N,12} f^{(2)} + f^{(1)}, \\ \varphi^{(2)} &= \widetilde{B}_{N,21}\varphi^{(1)} + \widetilde{B}_{N,22}\widetilde{f}^{(2)} + \widetilde{f}^{(1)}, \end{aligned} \tag{78}$$

where

$$\begin{aligned} \widetilde{B}_{N,22} &= \widetilde{A}_{N,22} + \widetilde{A}_{N,21}\widetilde{A}_{N,12} + \widetilde{A}_{N,21}\widetilde{\varGamma}_{N,11}\widetilde{A}_{N,12}, \\ \widetilde{B}_{N,21} &= \widetilde{A}_{N,21} + \widetilde{A}_{N,21}\widetilde{\varGamma}_{N,11}, \\ \widetilde{B}_{N,12} &= \widetilde{A}_{N,12} + \widetilde{\varGamma}_{N,11}\widetilde{A}_{N,12}, \\ \widetilde{B}_{N,11} &= \widetilde{\varGamma}_{N,11}. \end{aligned} \tag{79}$$

Multiplying the first of equalities (78) by $A_{N,1}^*$ from the left, and the second of them by $A_{N,2}^*$, we obtain

$$\varphi = (A_{N,1}^*\widetilde{B}_{N,11} + A_{N,2}^*\widetilde{B}_{N,21})\varphi + (A_{N,1}^*\widetilde{B}_{N,12} + A_{N,2}^*\widetilde{B}_{N,22}) f + f. \tag{80}$$

This equality is the partially solved equation corresponding to (47), since, obviously, its right side depends only on the values of φ for $x < z$ and on the values of f for $x \geq z$.

Let $A_N^{(z)}$ stand for the operator $A_{N,1}^*\widetilde{B}_{N,11} + A_{N,2}^*\widetilde{B}_{N,21}$. Then, by the theorem proved in Sect. 4, we can conclude that *if $E - A^{(z)}$ has an inverse and $(E - A^{(z)})^{-1} = E + \varGamma^{(z)}$, then, from some N on, there exist the operators $(E - A_N^{(z)})^{-1} = E + \varGamma_N^{(z)}$, and the operators $\varGamma_N^{(z)}$ approximate $\varGamma^{(z)}$ properly.*

Thus, we can formulate an important corollary about the algorithm T_c. More precisely, *the algorithm T_c of solving integral equation* (17) *has a closure. This closure is given by partially solved integral equations* (65). Also, we have the two theorems.

Theorem 1. *Let all of the integral equations*

$$\varphi(x) - \int_0^z K(x,y)\varphi(y)\,dy = \psi(x), \quad z \leq 1, \tag{81}$$

obtained by shortening the integration domain in (64)*, be nonsingular, which means that for neither of these equations does the second case of the Fredholm alternative hold. Then the algorithm of solving the integral equation under study has a regular closure.*

For sufficiently good replacement of (64) by a system of linear equations we can solve the obtained algebraic system by the method of successive elimination of unknowns.

Theorem 2. *Assume that equation* (81) *is unsolvable for some z and some function $\psi(x)$ on the right side of it. Then the successive elimination of unknowns in the corresponding system of algebraic equations leads to the algorithm T_c for solving the equation with the irregular closure.*

Under the conditions of Theorem 2, in the process of elimination of unknowns, we pass through the system that as the value of h varies, becomes however close to the unsolvable system. The better we approximate the operator, the worse the system is in intermediate computations. In Sect. 8 and Sect. 9 there is an example establishing the importance of Theorem 2.

8 The Example of Construction of a Partially Solved Equation

To illustrate certain applications, we present one example. Let

$$\varphi(x) = \int_0^1 K(x,y)\varphi(y)\,dy + f(x), \quad K(x,y) = a + b\cos 2\pi x \cos 2\pi y. \tag{82}$$

Let us construct a partially solved equation. From (82) it obviously follows that

$$\begin{aligned} &\varphi(x) - \int_0^z (a + b\cos 2\pi x \cos 2\pi y)\varphi(y)\,dy = \omega(x,z), \\ &\omega(x,z) = f(x) + \int_z^1 (a + b\cos 2\pi x \cos 2\pi y)\varphi(y)\,dy. \end{aligned} \tag{83}$$

From (83) we obtain

$$\varphi(x) = \omega(x,z) + aC_1(z) + bC_2(z)\cos 2\pi x, \tag{84}$$

$$C_1(z) = \int_0^z \varphi(y)\,dy, \quad C_2(z) = \int_0^z \varphi(y)\cos 2\pi y\,dy. \tag{85}$$

Inserting (84) in (85), we have:

$$C_1(z) = C_1(z)\,a\int_0^z dy + C_2(z)\,b\int_0^z \cos 2\pi y\,dy + \int_0^z \omega(y,z)\,dy,$$

$$C_2(z) = C_1(z)\,a\int_0^z \cos 2\pi y\,dy + C_2(z)\,b\int_0^z \cos^2 2\pi y\,dy + \int_0^z \omega(y,z)\cos 2\pi y\,dy.$$

Hence,

$$C_1(z)(1-az) - C_2(z)\frac{b\sin 2\pi z}{2\pi} = \int\limits_0^z \omega(y,z)\,dy,$$

$$-C_1(z)\frac{a\sin 2\pi z}{2\pi} + C_2(z)\left(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\right) = \int\limits_0^z \omega(y,z)\cos 2\pi y\,dy.$$

Solving these equations for $C_1(z)$ and $C_2(z)$, we obtain

$$C_1(z) = \frac{\int\limits_0^z \omega(y,z)\left(\left(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\right) + \frac{b}{2\pi}\sin 2\pi z\cos 2\pi y\right)dy}{(1-az)\Big(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\Big) - \frac{ab}{4\pi^2}\sin^2 2\pi z},$$

$$C_2(z) = \frac{\int\limits_0^z \omega(y,z)\left((1-az)\cos 2\pi y + \frac{a}{2\pi}\sin 2\pi z\right)dy}{(1-az)\Big(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\Big) - \frac{ab}{4\pi^2}\sin^2 2\pi z}.$$

Finally, substituting the values of $C_1(z)$ and $C_2(z)$ in (84), we obtain

$$\varphi(x) = \omega(x,z) + \int\limits_0^z \omega(y,z)\left[\frac{a\Big(1 - b\big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\big)\Big) + \frac{ab\sin 2\pi z\cos 2\pi y}{2\pi}}{(1-az)\Big(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\Big) - \frac{ab\sin^2 2\pi z}{4\pi^2}}\right.$$
$$\left. + \frac{b\cos 2\pi x\big((1-az)\cos 2\pi y + \frac{a\sin 2\pi z}{2\pi}\big)}{(1-az)\Big(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\Big) - \frac{ab\sin^2 2\pi z}{4\pi^2}}\right] dy. \tag{86}$$

From (86) it follows that for $x \le z$ and $y \le z$ the function $\Gamma(x,y,z)$ is expressed as fraction in brackets. We also see that the resolvent $\Gamma(x,y,z)$ is an analytic function of the variables (x,y,z), and, hence, it has to coincide with $K(x,y,z)$ for all values of (x,y,z). Finally, we obtain

$$\varphi(x) - \int\limits_z^1 K(x,y,z)\varphi(y)\,dy = f(x) + \int\limits_0^z K(x,y,z)f(y)\,dy, \tag{87}$$

where

$$K(x,y,z) = \frac{a\left(1 - b\big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\big)\right)}{(1-az)\Big(1 - b\Big(\frac{z}{2} + \frac{\sin 4\pi z}{8\pi}\Big)\Big) - \frac{ab\sin^2 2\pi z}{4\pi^2}}$$

$$+\frac{(\cos 2\pi x+\cos 2\pi y)\dfrac{ab\sin 2\pi z}{2\pi}+(1-az)b\cos 2\pi x\cos 2\pi y}{(1-az)\Big(1-b\Big(\dfrac{z}{2}+\dfrac{\sin 4\pi z}{8\pi}\Big)\Big)-\dfrac{ab\sin^2 2\pi z}{4\pi^2}}. \tag{88}$$

Formulas (87) and (88) explicitly express the closure of the algorithm T_c for solving the equation in (82).

Let us study how the kernel $K(x, y, z)$ behaves when z varies. The denominator in (88) can vanish for certain values of z. Namely, it vanishes for those z, for which the equation

$$\varphi(x)=\int\limits_0^z K(x,y,z)\varphi(y)\,dy+\omega(x,z) \tag{89}$$

has 1 as an eigenvalue of it. Depending on whether the roots of the denominator of the fraction in (88) lie in the interior of the interval $(0, 1)$ or not, the closure of the algorithm T_c will be either irregular or regular.

For example, let $a = 2$ and $b \neq 4$. Then $z = 1/2$ is the root of the denominator, since in this case both $(1 - az)$ and $\sin^2 2\pi z$ vanish, and the closure of T_c is irregular. In this case let us consider the behavior of the kernel $K(x, y, z)$ as z tends to $1/2$ in more detail.

Expanding the numerator and denominator of (88) in powers of $(z - 1/2)$ for $a = 2$ and $b \neq 4$, we obtain

$$K(x,y,z)=\frac{\dfrac{1}{2}-2b}{\Big(2b-\dfrac{1}{2}\Big)\Big(z-\dfrac{1}{2}\Big)+o\Big(\big(z-\dfrac{1}{2}\big)^4\Big)}$$

$$-\frac{\big(z-\dfrac{1}{2}\big)b\big[1+\cos 2\pi x+\cos 2\pi y+\cos 2\pi x\cos 2\pi y\big]+o\Big(\big(z-\dfrac{1}{2}\big)^3\Big)}{\Big(2b-\dfrac{1}{2}\Big)\Big(z-\dfrac{1}{2}\Big)+o\Big(\big(z-\dfrac{1}{2}\big)^4\Big)}. \tag{90}$$

In a neighborhood about the point $z = 1/2$ the function $K(x, y, z)$ may be written as

$$K(x,y,z)=\frac{2}{1-2z}+\frac{2b}{1-4b}(1+\cos 2\pi x)(1+\cos 2\pi y)+o\left(\big(z-\frac{1}{2}\big)^2\right). \tag{91}$$

Let us study how the system of algebraic equations corresponding to (82) is solved.

In the case of $a = 2$ and $b \neq 4$ solving the integral equation by the method of reducing it to the system of algebraic equations, we replace the integral using the quadrature formulas, i.e., we compose the equation $(E - A_N)\varphi = f$. For the values of $z = 1/2 - \eta$, as η tends to zero, the operator $\mathit{\Gamma}_N$ becomes however close to $\mathit{\Gamma}$, and, hence, the matrix $\dfrac{1}{h}\mathit{\Gamma}_{N,n,m}$ becomes however large.

In this case its singular part of the form $2/(2z-1)$ dominates, and at a certain step of the method of elimination the other terms of the matrix become relatively insignificant in magnitude.

If we conduct calculations keeping the same relative error (for example, the same number of digits), then the smaller we choose h, i.e., the better approximate A using A_N, the worse the accuracy of the final result is. This remark emphasizes the significance of Theorem 2 from the previous section.

9 Differential Equation on the Resolvent

It is not difficult to obtain from equation (75) one interesting corollary. Moving $\Gamma(x,y,z_1)$ to the left side of (75), and dividing both sides of the result by z_2-z_1, we have

$$\frac{\Gamma(x,y,z_2)-\Gamma(x,y,z_1)}{z_2-z_1} = \frac{1}{z_2-z_1}\int\limits_{z_1}^{z_2} \Gamma(x,t,z_1)\Gamma(t,y,z_1)\,dt$$

$$+\frac{1}{z_2-z_1}\int\limits_{z_1}^{z_2}\int\limits_{z_1}^{z_2} \Gamma(x,t,z_1)\Gamma(t,u,z_2)\Gamma(u,y,z_1)\,dudt. \tag{92}$$

Obviously, the right side of (92) tends to

$$\Gamma(x,z_1,z_1)\Gamma(z_1,y,z_1)$$

as $z_2 \to z_1$. Hence, the left side of (92) also has the limit, and we obtain

$$\frac{d\Gamma(x,y,z)}{dz} = \Gamma(x,z,z)\Gamma(z,y,z). \tag{93}$$

Equation (93) is a generalized ordinary differential equation in the variable z with a functional right side.

Let us denote, as above, by $\omega(x,z)$ the function

$$\omega(x,z) = f(x) + \int\limits_0^z \Gamma(x,y,z)f(y)\,dy. \tag{94}$$

Obviously, $\omega(x,1)=\varphi(x)$. Differentiating both parts of (94) with respect to z and applying (93), we obtain

$$\frac{d\omega(x,z)}{dz} = \Gamma(x,z,z)f(z) + \int\limits_0^z \frac{d\Gamma(x,y,z)}{dz}f(y)\,dy$$

$$= \Gamma(x,z,z)\Big(f(z) + \int_0^z \Gamma(z,y,z) f(y)\,dy\Big) = \Gamma(x,z,z)\omega(z,z). \tag{95}$$

From (95) it follows that for the given function $\Gamma(x,z,z)$ the problem of determining the function $\omega(x,z)$ reduces to the simple quadrature. From (95) it also follows that

$$\omega(z,z) = \omega(0,0) \exp \int_0^z \Gamma(z_1,z_1,z_1)\,dz_1,$$

$$\omega(x,z) = \omega(x,0) + \int_0^z \Gamma(x,z_1,z_1)\omega(z_1,z_1)\,dz_1.$$

Hence, if we would try to find the solution of the integral equation under study using the resolvent of it, it would be sufficient to know the function $\Gamma(x,y,z)$ only for $y \geq z$.

Let us note that for (93) the existence and uniqueness theorems established in the theory of ordinary differential equations keep their meaning. Let us prove one of them.

Theorem. *In a given interval $\alpha \leq z \leq \beta$ the bounded solution $K(x,y,z)$ of* (93) *such that*

$$K(x,y,\alpha) = \varphi(x,y), \tag{96}$$

is unique.

Proof. Let $K_1(x,y,z)$ and $K_2(x,y,z)$ be two solutions of the problem in question such that $|K_1(x,y,z)| \leq M$ and $|K_2(x,y,z)| \leq M$ in the considered interval. Then

$$\frac{\partial(K_2-K_1)}{\partial z} = K_2(x,z,z)K_2(z,y,z) - K_1(x,z,z)K_1(z,y,z)$$

$$= \big[K_2(x,z,z) - K_1(x,z,z)\big]K_2(z,y,z) + K_1(x,z,z)\big[K_2(z,y,z) - K_1(z,y,z)\big].$$

Hence,

$$\Big|\frac{\partial(K_2-K_1)}{\partial z}\Big| \leq M\left(|K_2(x,z,z) - K_1(x,z,z)| + |K_2(z,y,z) - K_1(z,y,z)|\right).$$

However, $(K_2-K_1)(x,y,z) = \int_\alpha^z \frac{\partial(K_2-K_1)}{\partial u}(x,y,u)\,du$ and, hence,

$$|(K_2-K_1)(x,y,z)| \leq M \int_\alpha^z \big(|K_2(x,u,u) - K_1(x,u,u)|$$

$$+|K_2(u,y,u) - K_1(u,y,u)|)\, du. \tag{96*}$$

Hence, by induction, we obtain the estimate

$$|(K_2 - K_1)(x,y,z)| \le \frac{(2Mz)^n}{n!}. \tag{97}$$

From the assumption that (97) holds for certain n, and from (96*), as is easy to see, the validity of (97) follows also for $n+1$. □

If in the example of Sect. 8 $b = 0$, then (82) becomes the equation

$$\varphi(x) = a \int_0^1 \varphi(y)\, dy + f(x). \tag{98}$$

In this case it is easy to find the resolvent by integrating equation (93) with the initial condition $\Gamma(x,y,0) = a$. It is natural to search the solution of the problem as a function independent of x and y. We have

$$\frac{d\Gamma(z)}{dz} = \Gamma^2(z), \quad d\frac{1}{\Gamma(z)} = -dz, \tag{99}$$

and, hence,

$$\Gamma(z) = \frac{a}{1 - az}, \tag{100}$$

which coincides with the result obtained earlier. In this case the singularity of the solution is movable, and it appears at $z = 1/a$.

It is interesting to trace the relation between the algorithm of successive eliminations of unknowns in the algebraic system of equations and solving the corresponding differential equation (93).

We establish that the method of successive eliminations of unknowns is quite close to the polygonal Euler method applied to differential equation (93).

Let us point out one important circumstance. In the example we just considered all unknowns appear in the problem symmetrically, therefore the elimination order makes no difference. Thus, generally speaking, we cannot avoid the mentioned loss of significant digits applying the Gauss elimination method. The origin of these losses depends on reasons deeper than unsuccessful order of the elimination of unknowns.

Let us study in more detail the process of solving the system of the equations

$$\varphi_i = \sum_{j=1}^{N} K_{ij}\varphi_j + f_i, \quad i = 1, 2, \dots, N, \tag{101}$$

by means of the successive elimination of unknowns. Let

$$\varphi_i - \sum_{j=s+1}^{N} K_{ij}^{(s)} \varphi_j = f_i + \sum_{j=1}^{s} K_{ij}^{(s)} f_j, \quad s = 0, 1, \ldots, N, \tag{102}$$

be the partially solved systems of equations. From the condition that all of equations (102) must be mutually equivalent, it is not difficult to find the dependence between the numbers $K_{ij}^{(s)}$. This dependence is well known. Solving the equation of system (102) with the number $s+1$ with respect to φ_{s+1}, we substitute the result in all other equations of the system. Then after some simple calculations we obtain

$$\begin{aligned} &\varphi_i - \sum_{j=s+2}^{N} \left(K_{ij}^{(s)} + \frac{K_{i,s+1}^{(s)} K_{s+1,j}^{(s)}}{1 - K_{s+1,s+1}^{(s)}} \right) \varphi_j \\ &= f_i + \sum_{j=1}^{s} \left(K_{ij}^{(s)} + \frac{K_{i,s+1}^{(s)} K_{s+1,j}^{(s)}}{1 - K_{s+1,s+1}^{(s)}} \right) f_j, \end{aligned} \tag{103}$$

which immediately entails the known final formulas

$$K_{ij}^{(s+1)} = K_{ij}^{(s)} + \frac{K_{i,s+1}^{(s)} K_{s+1,j}^{(s)}}{1 - K_{s+1,s+1}^{(s)}}, \tag{104}$$

$$K_{ij}^{(0)} = K_{ij}. \tag{105}$$

As is easy to see, (105) follows from (101) and (102) for $s = 0$.

The similarity between formulas (104) and (93) strikes. Introducing the notation $\Gamma_{ij}^{(s)} = h\Gamma(t_i, t_j, t_s)$, where t_1, t_2, …, t_N are the same points in the interval $[0, 1]$ as before, from (93) we approximately obtain:

$$\frac{\Gamma(t_i, t_j, t_{s+1}) - \Gamma(t_i, t_j, t_s)}{h} = \Gamma_{i,s+1}^{(s)} \Gamma_{s+1,j}^{(s)}, \tag{106}$$

which coincides with (104) with the accuracy to within small terms of higher order.

Thus, the principal term in (104) is the same as one would obtain by integrating (93) by the polygonal Euler method.

The remark makes it possible to expect that for solving the integral equation there are algorithms based on the approximate solution of (93), and they lead to the solution faster than both algorithms indicated above. As is known, the polygonal method is not the best in solving differential equations. It is possible that we could apply to equation (93) the techniques based on more exact formulas, for example, the Runge–Kutta method or the Adams method.

The singular points in the closure of the algorithm, as we have observed, prevent the normal course of calculations. We indicate here one theoretical method how to avoid these singular points.

Let us look for the solution $\varphi(x)$ of equation (18) in the parametric form

$$\varphi = \psi + \Gamma_N^{(1)*} B_2 \psi = (E + \Gamma_N^{(1)})\psi, \tag{107}$$

where $\Gamma_N^{(1)*}$ is defined by formula (63). In this case we obtain

$$(E + \Gamma_N^{(1)})\psi = A(E + \Gamma_N^{(1)})\psi + f. \tag{108}$$

From (108) it follows that

$$\Big(E + (\Gamma_N^{(1)} - A_N^{(1)} + A_N^{(1)}\Gamma_N^{(1)}) + (A_N^{(1)} - A)(E + \Gamma_N^{(1)})\Big)\psi$$

$$\equiv \Big(E + (A_N^{(1)} - A)(E + \Gamma_N^{(1)})\Big)\psi = f.$$

Let

$$(A - A_N^{(1)})(E + \Gamma_N^{(1)}) \equiv D. \tag{109}$$

Then we obtain the new equation

$$(E - D)\psi = f. \tag{110}$$

If $A_N^{(1)}$ is sufficiently close to A, then the norm of the operator D, and even the kernel of this integral operator, can be made less than 1, and then equation (110) turns out to be solvable using both the algorithms of nets T_c, and the algorithm of successive approximations T_{it}, having a regular closure in this case. It is not difficult to compute the operator D by solving a finite number of equations on $\stackrel{\approx}{\Gamma}_N$.

If $(E - A)^{-1}$ exists, then for some N the operator $\stackrel{\approx}{\Gamma}_N$ satisfies the conditions stated. After that, by (107) it is not difficult to compute φ.

It seems to us that the closure of the computational algorithm could be also useful in other cases, in particular, in solving different boundary value problems for partial differential equations. The study of the closure of operators is also useful from one more side. It is possible to assume that the systems of equations with many unknowns are closer in their properties to their closures than to their two or three dimensional analogues. Then we should think on extension for solution of such systems of certain methods of mathematical analysis. If the present work attracts the attention of researchers to these questions, then our goal will be achieved.

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5. Certain Modern Questions of Computational Mathematics*

S. L. Sobolev

1. The subject of numerical mathematics from the modern point of view. The sets of functions and functional spaces. Tables, graphs, approximate formulas, certain numeric values as finite-dimensional approximations in the functional space. How to study sets that cannot be reduced to finite-dimensional? The bounded ε-net in finite-dimensional spaces. Compactness as an important property of all objects of numerical mathematics.

Numerical mathematics as one of the branches of functional analysis. New methods, directly introduced by functional analysis into computational applications.

2. Numerical mathematics and the functions of a discrete argument. Binary representations of numbers. The two-valued functions of many variables admitting only two values: 0 and 1.

Connection between numeric mathematics and mathematical logic. Data and information. The problems of information theory related to the large amount of information. The estimate of an algorithm by its complexity (by a number of operations).

3. Mathematical machines. Universal high speed electronic computers. Programming, its theory and practice. The influence of machine technology on the problems of mathematics as a whole.

Mathematical logic and its applications.

Extension of the classes of solvable problems. Appearance of a need for solution of complicated mathematical problems simultaneously with enlargement of the possibilities of solutions.

Space problems and nonlinear problems.

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* Tr. 3 Vsesoyuz. Mat. S'ezda, vol. 2 (1956), p. 77.
Resume of a lecture given at the Third All-Union Mathematical Congress in Moscow (1956). – *Ed.*

4. Approximation theory. New problems in the theory of approximation of functions related to the use of functions in calculations.

The problems of constructing algorithms for the best approximations.

The interpolation of functions of many variables.

5. Special questions of approximation of operators. Quadrature formulas and representations of derivatives by differences in the case of functions of many variables.

Inverses for approximate operators and the approximate operators for the inverses. The explicit forms of certain inverses.

6. The Cauchy problem for differential and difference equations. Problems solved step by step, their stability, the stability of computations in different schemes. Purely computational effects related to rounding in calculations.

7. Systems with a large number of algebraic equations.

The border problems between algebra and analysis. Systems with a large number of equations, corresponding to the given integral equation.

Elliptic equations and corresponding difference systems. The methods of analysis in algebraic equations. Algorithmization of classical analysis as a result of enlarged computational possibilities.

8. Conclusion.

6. Functional Analysis and Computational Mathematics*

L. V. Kantorovich, L. A. Lyusternik, and S. L. Sobolev

1. **Historical review.** Computational mathematics as one of the sources of ideas in functional analysis.

2. **Computational mathematics as the science about finite approximations of general compacts** (not necessarily metric).

3. **The main branches of computational mathematics in its historical order.** The approximations of numbers, functions, and operators.

4. **Approximations in spaces with different topologies.** Approximations in C_n and in $C_{\rho(\infty)}$ (integral transformations on the axis in L_2). Weak approximations. Integral as the limit of sums, convergence of quadrature formulas. Partially ordered spaces.

5. **The types of approximation of operators.** Uniform approximations. Strong approximation. Regular approximation. Approximation by n-dimensional manifolds. Preservation of the qualitative properties of an operator in replacing it by its approximations. (The invertibility of the operator, the maximum principle, integral estimates.)

6. **Approximation of the functions of operators.** Symbolic calculus for functions of one and several variables. Application of these methods to the quadrature and cubature formulas. Approximation of the resolvent by operator polynomials. (The Chebyshev polynomials, continuous fractions, orthogonalization of the sequence A_x^n.)

7. **Difference approximations.** The question on solutions of difference equations. Stability of difference computations.

8. **Computational algorithms** and their direct study. General properties of computational algorithms. Closures of computational algorithms.

* Tr. 3 Vsesoyuz. Mat. S'ezda, vol. 2 (1956), p. 43.
Resume of a lecture given at the Third All-Union Mathematical Congress in Moscow (1956). – *Ed.*

9. Transferring the computational ideas of algebra and elementary analysis to functional spaces. The method of successive approximations. Linearization. The Newton method and its different modifications. Chaplygin's estimates. Generalization of the principle of root separation. The Schauder theorem on rotation of a vector field. The deepest descent method.

10. New problems of computational nature in functional analysis. Equations in variational derivatives. Integration in a functional space.

7. Formulas of Mechanical Cubatures in n-Dimensional Space*

S. L. Sobolev

For different classes of functions, formulas of mechanical cubature

$$(l, \varphi) \equiv \int\limits_{\Omega} \varphi(x)\, dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}) \cong 0 \tag{1}$$

give various degrees of approximation. In (1) x is a point in a bounded n-dimensional domain Ω, the c_k are coefficients, and the $x^{(k)}$ are nodes of the formula. In what follows we assume that the error (l, φ) equals zero for polynomials of a certain degree m_1, and the domain Ω has a piece-wise smooth boundary.

Of particular theoretical interest is a special case of cubature formulas, when the function φ is periodic with periods $H\beta$, where H is the basic matrix of periods

$$H = (\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_n}), \tag{2}$$

each period $\mathbf{h_k}$ is a column vector

$$\mathbf{h_k} = \begin{pmatrix} h_{1k} \\ \vdots \\ h_{nk} \end{pmatrix}; \tag{3}$$

β is a column vector of integers

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad -\infty < \beta_k < +\infty. \tag{4}$$

In this case, the domain of integration Ω_0 is a fundamental paralleloid such that the system of all paralleloids Ω_β, obtained by translations of Ω_0 by the vectors $H\beta$, covers the whole space R^n without intersection. In this case, we put $m_1 = 0$. Hence, formula (1) is exact for $\varphi = 1$, and therefore

* Dokl. Akad. Nauk SSSR, **137**, 527–530 (1961)

$$\sum_{k=1}^{N} c_k = |\Omega_0|. \tag{5}$$

The main problem of the theory of mechanical cubature is to find

$$\min_{c_k, x^{(k)}} \left[\max_{\varphi \in X} |(l, \varphi)|\right] = d(X, N) \tag{6}$$

for a given class X and a given number of nodes N. The values c_k and $x^{(k)}$, for which this minimax is attained, give the optimal formulas of mechanical cubature. For $n = 1$, this problem was studied by S. M. Nikol'skii and his students. A summary of the results obtained is given in [3] along with a major bibliography on the subject. Many authors [7–13] have recently studied cubature formulas for $n > 1$.

It is convenient to consider as X the unit sphere in a certain Banach space B, where (l, φ) is a linear functional. In the space C of continuous functions on Ω or Ω_0, there are no functionals of the form (l, φ) such that they are small on the unit sphere of C, even though they are all linear. In this event, we have

$$\sup_{\|\varphi\|_C = 1} |(l, \varphi)| = |\Omega_0| + \sum_{k=1}^{N} |c_k|. \tag{7}$$

Therefore, the main problem has no meaning in C.

In the present note, we study the first half of the main problem. More precisely, we look for

$$\max_{\varphi \in X} |(l, \varphi)| = d(c_k, x^{(k)}), \tag{8}$$

where X is the unit sphere in the space of functions and the mth-order[1] derivatives of functions under consideration are square integrable.

The second half of the main problem consists in finding $\min d(c_k, x^{(k)})$. This is the problem about the minimum of functions of $(n+1)N$ variables, and we are not going to touch on it here.

Further we consider simultaneously spaces $\widetilde{W}_2^{(m)}$ and $\widetilde{L}_2^{(m)}$ of functions periodic in R^n with periods $H\beta$ for any integer vector β. In this event, as the domain of integration Ω we take the fundamental paralleloid Ω_0.

The $W_2^{(m)}$- and $\widetilde{W}_2^{(m)}$-norms are given by formulas [2]

$$\|\varphi\|^2_{W_2^{(m)}} = \|\Pi\varphi\|^2_{\mathbf{P}_{m-1}} + \|\varphi\|^2_{L_2^{(m)}} = \|\Pi\varphi\|^2_{\mathbf{P}_{m-1}} + D(\varphi), \tag{9}$$

$$\|\varphi\|^2_{\widetilde{W}_2^{(m)}} = \left(\int_{\Omega_0} \varphi(x)\, dx\right)^2 + D(\varphi), \tag{10}$$

where Π is a projection operator from $W_2^{(m)}$ into the space $\mathbf{P}_{m-1}$ of polynomials of degree $m-1$, and $L_2^{(m)}$ is the quotient space $W_2^{(m)}/\mathbf{P}_{m-1}$. Also in (9) and (10) we use the notation

[1] Here and in what follows $m > n/2$. – *Ed.*

$$\|\varphi\|^2_{L_2^{(m)}} = D(\varphi) = \int\limits_{\Omega} \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx. \tag{11}$$

In the nonperiodic case we assume that $m_1 = m - 1$, i.e.,

$$(l, \varphi) = 0 \quad \text{for} \quad \varphi \in \mathbf{P}_{m-1}. \tag{12}$$

The general case is reduced to (12) provided that $\Pi\varphi$ is an interpolation operator.

The following inequalities are valid:

$$|(l,\varphi)| \le K\|\varphi\|_{L_2^{(m)}} \le K\|\varphi\|_{W_2^{(m)}}, \; |(l,\varphi)| \le K\|\varphi\|_{\widetilde{L}_2^{(m)}} \le K\|\varphi\|_{\widetilde{W}_2^{(m)}}. \tag{13}$$

Let us consider three problems.

Problem 1. Find $\max\limits_{\|\varphi\|_{W_2^{(m)}}=1} (l, \varphi)$.

Problem 2. Find $\min\limits_{(l,\varphi)=1} \|\varphi\|^2_{W_2^{(m)}}$.

Problem 3. Find $\min\limits_{\varphi \in W_2^{(m)}} H_\lambda(\varphi)$, where $H_\lambda(\varphi) = D(\varphi) + 2\lambda(l, \varphi)$.

These problems are reducible to one another.

Let us examine Problem 3 using a direct approach[2]. By (12) and (13), the functional $H_\lambda(\varphi)$ has a finite exact lower bound[3]:

$$H_\lambda(\varphi) \ge \left[\sqrt{D(\varphi)} - \lambda K\right]^2 - \lambda^2 K^2 \ge -\lambda^2 K^2. \tag{14}$$

From the identity

$$\frac{1}{2} H_\lambda(u_k) + \frac{1}{2} H_\lambda(u_m) - H_\lambda\left(\frac{u_k + u_m}{2}\right) = D\left(\frac{u_k - u_m}{2}\right) \tag{15}$$

it follows that for a minimizing sequence u_k the sequence $(I - \Pi)u_k$ is also minimizing, and even fundamental in $L_2^{(m)}$. Hence, there exists a unique limit of the sequence such that it is the solution of Problem 3. Next, from the identity

[2] For given l and λ this is the Neumann problem for the polyharmonic equation studied in Chap. XII of the book: Sobolev, S. L.: Introduction to the Theory of Cubature Formulas. Nauka, Moscow (1974). – *Ed.*

[3] The proof of estimate (14) is as follows:

$$H_\lambda(\varphi) \ge D(\varphi) - 2|\lambda||(l,\varphi)|$$

$$\ge D(\varphi) - 2|\lambda| K\sqrt{D(\varphi)} = \left(\sqrt{D(\varphi)} - \lambda K\right)^2 - \lambda^2 K^2. \text{ – } \textit{Ed.}$$

$$H_\lambda(\varphi) = \frac{\lambda^2}{\lambda_1^2} H_{\lambda_1}\left(\frac{\lambda_1}{\lambda}\varphi\right) \tag{16}$$

we can conclude that for varying λ the solutions of Problem 3 differ from each other only by a factor and

$$u_\lambda = \lambda u_1. \tag{17}$$

An examination of the function $\psi(\mu) = H_\lambda(\mu u_\lambda)$ leads to the conclusion that under the consideration

$$H_\lambda(u_\lambda) = \min_{u \in W_2^{(m)}} H_\lambda(u) = -d_\lambda(c_k, x^{(k)}), \tag{18}$$

the following equalities are valid[4]:

$$D(u_\lambda) = d_\lambda(c_k, x^{(k)}) \quad \text{and} \quad \lambda(l, u_\lambda) = -d_\lambda(c_k, x^{(k)}). \tag{19}$$

Clearly, if $(l, u_\lambda) = 1$, then the solution u_λ of Problem 3 is the solution of Problem 2 as well, and solutions of Problem 1 and Problem 2 differ from each other by a factor. Hence, the solutions of Problems 1 and 2 may be written as

$$u_{\mathrm{I}} = \frac{1}{d_1} u_1 \quad \text{and} \quad u_{\mathrm{II}} = -\frac{1}{\sqrt{d_1}} u_1. \tag{20}$$

In the periodic case Problems 1, 2 and 3 are solved analogously.

The search for the extremal function can now be reduced to the integration of a partial differential equation. It is convenient to use the apparatus and symbolism of the theory of generalized functions [4–6].

Solving Problem 3 with $\lambda = 1$ by classical methods of the calculus of variations, we obtain the following equation[5]:

$$\begin{gathered} 2D(u_1, \xi) + 2\int_\Omega \xi(x)\, dx - 2\sum_{k=1}^N c_k \xi(x^{(k)}) = 0, \\ D(u_1, \xi) = \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u_1 D^\alpha \xi \, dx. \end{gathered} \tag{21}$$

Here ξ is a permissible variation, i.e., ξ is any function[6] from $W_2^{(m)}$ or $\widetilde{W}_2^{(m)}$. We rewrite equation (21) in the form

$$-D(u_1, \xi) = \int_\Omega \Big[1 - \sum_{k=1}^N c_k \delta(x - x^{(k)})\Big] \xi(x)\, dx, \tag{22}$$

[4] It is sufficient to use the equality $\psi'(1) = 0$. – *Ed.*

[5] By the definition of u_1, we have $\frac{d}{d\lambda} H_1(u_1 + \lambda\xi)\big|_{\lambda=0} = 0$. From this equality it follows that (21) holds. – *Ed.*

[6] For $m > n/2$ the values $\xi(x^{(k)})$ are defined in view of the embedding theorem. – *Ed.*

where $\delta(x)$ is the Dirac delta function.

By a slight modification of classical arguments, we obtain the equation

$$\Delta^m u_1 = (-1)^{m+1}\Big[1 - \sum_{k=1}^{N} c_k \delta(x - x^{(k)})\Big] \tag{23}$$

with the boundary conditions

$$B_k(u_1)\Big|_{x\in\partial\Omega} = 0 \quad \text{for} \quad k = 0, 1, \ldots, m-1 \tag{24}$$

in the nonperiodic case. In the periodic case, the boundary conditions are absent.

In the nonperiodic case, as follows from (23), the equality holds[7]:

$$u_1(x) = u_1^*(x) + \frac{(-1)^{m+1}\Gamma(n/2)2^{-2m}}{\Gamma(n/2+m)\Gamma(m+1)}\, r^{2m}$$

$$-(-1)^{m+1}\sum_{k=1}^{N} c_k \frac{i^{n-1}2^{-2m+1}\pi^{-n/2+1}}{\Gamma(m)\Gamma(m-\frac{n}{2}+1)} r_k^{2m-n} \begin{cases} \frac{1}{2}, & \text{if} \quad n \text{ odd}, \\ \dfrac{\ln r_k}{\pi i}, & \text{if} \quad n \text{ even}, \end{cases} \tag{25}$$

where $r = |x|$, $r_k = |x - x^{(k)}|$, $\Delta^m u_1^* = 0$, and u_1^* is chosen so that (24) is satisfied.

To find u_1^* we can use the method of integral equations, the net method or any other method. To find u_1 it is convenient to apply any direct approach such as, for example, the Ritz method.

Let us consider the periodic case. Let $|\Omega_0| = 1$ and let $u^{(k)}$ be the periodic solution of the equation

$$\Delta^m u^{(k)} = (-1)^{m+1}\left[1 - \delta(x - x^{(k)})\right], \quad \text{where} \quad x \in \Omega_0, \quad x^{(k)} \in \Omega_0. \tag{26}$$

Then

$$u_1 = \sum_{k=1}^{N} c_k u^{(k)}. \tag{27}$$

Let x and y be coordinate vector columns; $x = Hy$. Let $\Lambda(x)$ be the periodic function with periods $H\beta$ and let $\Lambda(x)$ be equal to $1 - \delta(x - x^{(k)})$ in the paralleloid Ω_0. Then $M(y) = \Lambda(Hy)$ is a periodic function with the integer periods β and its value in the basic cube is given by the equality $M(y) = \big[1 - \delta(y - y^{(k)})\big]$, $-1/2 < y \le 1/2$, $y^{(k)} = H^{-1}x^{(k)}$. We expand the function $M(y)$ in the generalized Fourier series

[7] Also the equality $u_1(x) = u_1^*(x) + (-1)^{m+1}G_{m,n}(x) * l(x)$ holds, where $G_{m,n}(x)$ is the fundamental solution of the polyharmonic operator Δ^m. The explicit expression for $G_{m,n}(x)$ can be found in the book: Sobolev, S. L.: Introduction to the Theory of Cubature Formulas. Nauka, Moscow (1974), pp. 520–521. – *Ed.*

$$M(y) = \sum_{\gamma \neq 0} e^{i2\pi(\gamma, y - y^{(k)})}, \tag{28}$$

where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ ranges over all integer vectors. From (28) it follows that

$$\Lambda(x) = \sum_{\gamma \neq 0} e^{i2\pi(\gamma, H^{-1}(x - x^{(k)}))} = \sum_{\gamma \neq 0} e^{i2\pi(H^{-1*}\gamma, x - x^{(k)})}, \tag{29}$$

whence

$$u^{(k)} = -\left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} e^{i2\pi(H^{-1*}\gamma, x - x^{(k)})}. \tag{30}$$

Formulas (27) and (30) provide an easy calculation of the sought for maximum of values of (l, φ).

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8. On Interpolation of Functions of n Variables*

S. L. Sobolev

In order to find an approximate representation of a function $\varphi(x)$ of n variables by elements of a certain finite collection, it is possible to use values of this function at some finite set of points

$$x^{(k)}, \quad k = 1, 2, \ldots, N. \tag{1}$$

The corresponding problem is called the *interpolation problem*, and the points $x^{(k)}$ the *interpolation nodes*. Most usual is the interpolation by means of a linear combination of some set of functions:

$$\varphi(x) \cong \sum_{\nu=1}^{M} a_\nu \varphi_\nu(x). \tag{2}$$

The $\varphi_\nu(x)$ are frequently taken as all monomials x^α of degree at most m (x^α denotes $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$). The number of such monomials equals

$$M = \frac{(m+n)!}{m!n!}.$$

The values of a function $\varphi(x)$ at the points $x^{(k)}$ form an N-dimensional row vector

$$\varphi^k = \varphi(x^{(k)}), \quad k = 1, 2, \ldots, N. \tag{3}$$

Let us index all integer vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with non-negative entries and such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \le m$. The set of values of all monomials ${x^{(k)}}^{\alpha^{(j)}}$ forms a matrix

$$S = (S_{jk}) = \left({x^{(k)}}^{\alpha^{(j)}} \right) \tag{4}$$

with N columns and M rows.

* Dokl. Akad. Nauk SSSR, **137**, 778–781 (1961)

An arbitrary polynomial $Q = \sum_{j=1}^{M} a_j x^{\alpha^{(j)}}$ is considered to be equivalent to the row vector $a = (a_1, a_2, \dots, a_M)$. Clearly, the values of this polynomial at the points $x^{(k)}$ may be arranged as the vector

$$Q^{(k)} = aS. \tag{5}$$

The *interpolation problem* is to solve equation (5) with respect to a vector a for a given $Q^{(k)}$, i.e., to recover the polynomial Q by its values at the points $x^{(k)}$.

The solution of equation (5) and the polynomial Q are uniquely defined if[1] $r(S) = M$, which is possible only for $N \geq M$. In this case, there exists at least one right inverse matrix S_d^{-1} to S. Then

$$a = aSS_d^{-1} = Q^{(k)} S_d^{-1}. \tag{6}$$

Every such matrix S_d^{-1} is called an *interpolation matrix*. For each polynomial Q, we have

$$Q = Q^{(k)} S_d^{-1} x^{\boldsymbol{\alpha}}, \tag{7}$$

where $x^{\boldsymbol{\alpha}}$ is the vector $(x^{\alpha^{(1)}}, x^{\alpha^{(2)}}, \dots, x^{\alpha^{(M)}})$. Formula (7) is called the *interpolation formula with the nodes* $x^{(1)}$, ..., $x^{(N)}$. If $N > M$, then there are infinite sets of the interpolation matrices and the interpolation formulas. The given vector $Q^{(k)}$ can serve as the vector of values of the polynomial provided that

$$r\begin{pmatrix} S \\ Q^{(k)} \end{pmatrix} = r(S). \tag{8}$$

This is the solvability condition for equation (5).

However, the right side of (7) has a meaning for an arbitrary vector $Q^{(k)}$ as well. Substituting instead of the vector $Q^{(k)}$ the vector $\varphi^{(k)}$, we have the polynomial

$$P_\varphi = \varphi^{(k)} S_d^{-1} x^{\boldsymbol{\alpha}} = \sum_{k=1}^{M} C_k(x) \varphi\big(x^{(k)}\big). \tag{9}$$

The polynomial P_φ is called the *interpolation polynomial for the function* φ. It can happen that the solution of (5) exists for any $Q^{(k)}$; this means that $r(S) = N$ and, hence, $N \leq M$. In this case, there exists at least one left inverse matrix S_g^{-1} to S, and

$$Q^{(k)} S_g^{-1} S = Q^{(k)}. \tag{10}$$

We do not consider the cases where $r(S) < N$ and $r(S) < M$.

For $r(S) = M = N$, there exists a unique solution of (5) and the interpolation problem is classical. For an example of a solution of the classical

[1] Here, $r(S)$ is the rank of the matrix S. – *Ed.*

interpolation problem, we can mention the interpolation formula with nodes at points of a certain parallelepiped structure, which is called a *Newtonian system*.

Let s be the n-dimensional integer vector with non-negative entries and let $s - \mathbf{1}$ be the vector $(s_1 - 1, s_2 - 1, \dots, s_n - 1)$. Every s such that $|s| \le m$ is in one-to-one correspondence with a parallelepiped Π_s, consisting of the points of some cubic lattice with mesh-size h and lengths of the edges given by $s_1 h$, $s_2 h$, …, $s_n h$, respectively. The inequality $s^{(1)} < s^{(2)}$ will be taken to mean that $s_k^{(1)} \le s_k^{(2)}$ for all k, and that for at least one k, the strict inequality $s_k^{(1)} < s_k^{(2)}$ holds.

We say that the *parallelepipeds Π_s form a structure*, if:

Property 1. From $s^{(1)} < s^{(2)}$ it follows that $\Pi_{s^{(1)}} \subset \Pi_{s^{(2)}}$.

Let the points x_k of the parallelepiped Π_s be given by $x_k = \xi_{s_k} + lh$, $l = 0, 1, \dots, s_k$. From Property 1, it follows that the ξ_{s_k} cannot depend on the other entries of s.

Let the system of polynomials P_s be given by the formula

$$P_s(x) = \frac{\Gamma\bigl((x - \xi_{s-1})/h + 1\bigr)}{\Gamma\bigl((x - \xi_{s-1})/h - s + 1\bigr)} \equiv \prod_{k=1}^{n} \prod_{l=0}^{s_k - 1} \left(\frac{x_k - \xi_{s_k - 1}}{h} - l \right). \tag{11}$$

Those k for which $s_k = 0$ are not included in the product. The polynomial $P_s(x)$ is the product of generalized factorials with respect to each variable. It becomes zero at all points belonging to any parallelepiped Π_{s^*} if at least one of the inequalities $s_k^* < s_k$ is satisfied.

Furthermore, let the operator

$$\Delta_s = \Delta_1^{s_1} \Delta_2^{s_2} \dots \Delta_n^{s_n} \tag{12}$$

be the product of the difference operators of orders s_k with respect to the variables x_k computed for the given nodes

$$x_k = \xi_{s_k - 1} + lh, \quad l = 0, 1, \dots, s_k - 1. \tag{13}$$

It is easy to verify that

$$\Delta_s P_q = 0 \quad \text{for} \quad s \ne q \quad \text{and} \quad \Delta_s P_q = s! \quad \text{for} \quad s = q. \tag{14}$$

Each polynomial Q of degree m can be uniquely represented as

$$Q = \sum_{j=1}^{M} a_{s_j} P_{s_j}.$$

By (14), these coefficients a_s are defined from the formulas

$$a_s = \frac{1}{s!} \Delta_s Q, \tag{15}$$

i.e., they are found by the values of Q at all the nodes of the structure. In this event the interpolation problem is classical, since the number of nodes is precisely equal to M and $r(S) = M$.

For an arbitrary function φ, we have the interpolation polynomial defined from values of φ at the nodes of the structure by

$$P_\varphi = \sum_{|s|\le m} \frac{\Delta_s \varphi}{s!} P_s(x). \tag{16}$$

An important problem in the interpolation theory is to find the maximum of the error of the interpolation formula $\varphi(x) \cong P_\varphi(x)$ in a given class of functions. The value of this error at a fixed point z is a functional defined as

$$(j,\varphi) \equiv \varphi(z) - P_\varphi(z) = \varphi(z) - \sum_{k=1}^{N} C_k(z)\varphi\big(x^{(k)}\big), \tag{17}$$

where $C_k(z) = S_d^{-1} z^{\boldsymbol{\alpha}}$. The coefficients $C_k(z)$ are obviously connected by the linear conditions

$$(j, x^{\alpha^{(s)}}) = 0 \quad \text{for} \quad s = 1, 2, \ldots, M. \tag{18}$$

The functional (j,φ) is bounded and linear in the space $W_2^{(m)}(\Omega)$ of functions for which the mth-order derivatives are square integrable[2]. Therefore it is convenient to consider the maximum of this functional on the unit sphere of the space $W_2^{(m)}(\Omega)$.

Problem 1. Find

$$\max_{\|\varphi|W_2^{(m)}(\Omega)\|=1} (j,\varphi). \tag{19}$$

Problem 1 is solved in the same way as the problem of finding the maximal error of the cubature formula. Repeating the reasoning of article [1], we find that the corresponding *extremal function* is the solution of the equation

$$\Delta^m u = (-1)^m \left[\delta(x-z) - \sum_{k=1}^{N} C_k(z)\delta(x - x^{(k)})\right], \tag{20}$$

which satisfies the conditions

$$B^t(u)\Big|_{x\in\partial\,\Omega} = 0 \quad \text{for} \quad t = 1, 2, \ldots, m. \tag{21}$$

Thus, the extremal function and the maximum of the error depend upon in which domain Ω the functions under consideration are given.

We have the general estimate

[2] For $m > n/2$. – *Ed.*

$$|(j,\varphi)| \leq K(\Omega)\, \| \varphi \mid W_2^{(m)}(\Omega) \|. \tag{22}$$

Let $\Omega_2 \subset \Omega_1$. Then an extremal function in Ω_1 is also defined in Ω_2 and satisfies there the same equation (20), but the function is probably not an extremal function in Ω_2. In addition, its norm in the domain Ω_2 is less than 1. Thus,

$$\max_{\|\varphi | W_2^{(m)}(\Omega_1)\|=1} (j,\varphi) \leq \max_{\|\varphi | W_2^{(m)}(\Omega_2)\|=1} (j,\varphi). \tag{23}$$

Consequently, the constant $K(\Omega)$ decreases as the domain Ω expands. We show that $K(\Omega)$ tends to some limit under an unlimited expansion of the domain. For this, it is sufficient to establish that the equation (20) has a solution belonging to $W_2^{(m)}$ in the whole space.

Let $G(x)$ be an elementary solution of the equation

$$\Delta^m G = (-1)^m \delta(x). \tag{24}$$

It is known that[3]

$$G(x) = (-1)^m \varkappa_{m,n} |x|^{2m-n} \begin{cases} 1, & \text{if} \quad n \quad \text{odd} \quad \text{or} \quad n > 2m, \\ \ln|x|, & \text{if} \quad n \quad \text{even} \quad \text{and} \quad n \leq 2m. \end{cases} \tag{25}$$

We show that the function

$$\psi(x) = G(x-z) - \sum_{k=1}^{N} C_k(z) G(x - x^{(k)}) \tag{26}$$

belongs to $W_2^{(m)}$ and, hence, $\psi(x)$ is a unique extremal element in $W_2^{(m)}$ for the functional (j,φ).

For this, we expand the mth-order derivatives of the function $G(x^{(0)} - x)$ into a Taylor series with respect to powers of x in a ball of radius $A > \max\{|z|, |x^{(k)}|\}$. We have

$$\begin{aligned} D^\beta G(x^{(0)} - x) &= \sum_{|\alpha| \leq m} D^{\alpha+\beta} G(x^{(0)}) \frac{(-x)^\alpha}{\alpha!} + R_\beta(x, x^{(0)}) \\ &= Q_\beta(x, x^{(0)}) + R_\beta(x, x^{(0)}), \end{aligned} \tag{27}$$

where $Q_\beta(x, x^{(0)})$ is a polynomial in x of degree m, and $R_\beta(x, x^{(0)})$ satisfies the inequality

$$|R_\beta(x, x^{(0)})| \leq K\, |x^{(0)}|^{-n} \ln |x^{(0)}|, \tag{28}$$

for sufficiently large $|x^{(0)}|$. By condition (18), we have

[3] The explicit expressions for $\varkappa_{m,n}$ can be found, for example, in the book: Sobolev, S. L.: Introduction to the Theory of Cubature Formulas. Nauka, Moscow (1974), pp. 520–521. – *Ed.*

$$\bigl(j, D^\beta G(x^{(0)} - x)\bigr) = \bigl(j, R_\beta(x, x^{(0)})\bigr),$$

whence[4]

$$\begin{aligned}\bigl|\bigl(j, D^\beta G(x^{(0)} - x)\bigr)\bigr| &= \bigl|D^\beta \psi(x^{(0)})\bigr| \\ &\le K\left(1 + \sum_{k=1}^{N} |C_k(z)|\right) |x^{(0)}|^{-n} \ln |x^{(0)}|. \end{aligned} \tag{29}$$

From inequality (29) the convergence of the integral $\int |D^\beta \psi(y)|^2 dy$ and the boundedness of the norm $\|\psi \mid W_2^{(m)}\|$ follow.

The problem of finding the optimal interpolation formula for given nodes $x^{(k)}$ is also interesting when $N > M$. In this event, for minimizing the error it stands to reason to use the arbitrariness in defining S_d^{-1}.

References

1. Sobolev, S. L.: Formulas of mechanical cubatures in n-dimensional space. Dokl. Akad. Nauk SSSR, **137**, 527–530 (1961)[5]

[4] The equality in (29) follows from (17) and (26). – *Ed.*
[5] Paper [7] of Part II of this book. – *Ed.*

9. Various Types of Convergence of Cubature and Quadrature Formulas*

S. L. Sobolev

The error functional of the cubature formula

$$(l, f) \equiv \int\limits_{\Omega} f(x)\, dx - \sum_{k=1}^{N} c_k f(x^{(k)}) \tag{1}$$

can be studied in various topologies.

S. M. Nikol'skii [1], the author [2], and others studied the problem of the maximum of (l, f) on the unit sphere of a Banach space B:

$$\max_{\|f\|_B = 1} (l, f) = d(l). \tag{2}$$

From this point of view, the study of cubature formulas depending on N reduces to the study of the convergence of $d(l^{(N)})$ for the corresponding functionals $(l^{(N)}, f)$.

Another equally frequent approach to convergence of cubature and quadrature formulas is the convergence in proximity order. In this note we adopt the latter point of view.

In (1) we can consider instead of the numerical function f an abstract function with values in a certain Banach space X or topological space τ:

$$f(x) \in X \quad \text{or} \quad f(x) \in \tau. \tag{3}$$

In this case, the function f itself, mapping R^n onto X or τ, is a member of a certain Banach space B or, more generally, topological space T:

$$f(R^n \to X) \in B \quad \text{or} \quad f(R^n \to \tau) \in T. \tag{4}$$

The error functional (l, f) then becomes an *error operator*, mapping B or T onto X or τ, since both

* Dokl. Akad. Nauk SSSR, **146**, 41–42 (1962)

$$\int_{\Omega} f(x)\,dx \quad \text{and} \quad \sum_{k=1}^{N} c_k f(x^{(k)})$$

are members of B or T. In this event, the convergence of cubature formulas to 0 is characterized by the convergence to 0 of the operators $l^{(N)}$.

It is convenient to take as X the space of countably dimensional vectors

$$a \in X, \quad a = (a_1, a_2, \dots, a_n, \dots), \tag{5}$$

with any Banach norm, for example, the l_2 or m norm. In this case an abstract function f with values in X or τ has the form

$$f = (f_1, f_2, \dots, f_n, \dots). \tag{6}$$

The *convergence in proximity order* is the convergence of $l^{(N)}$ to 0 defined by the condition:

For a given k there is $N(k)$ such that

$$(l^{(N)}, f) = (0, 0, \dots, f_{k+1}, f_{k+2}, \dots) \quad \text{for} \quad N \geq N(k). \tag{7}$$

Let operators $l^{(N)}$ be bounded in the norm, i.e.,

$$\|(l^{(N)}, f)\|_X \leq K \|f\|_B.$$

Then the convergence in proximity order is a special case of the weak convergence.

In the linear space of countably dimensional vectors, instead of a norm we introduce a topology by defining neighborhoods of zero $\mathfrak{B}_\alpha$ as vectors of the form

$$(0, 0, \dots, a_{\alpha+1}, a_{\alpha+2}, \dots). \tag{8}$$

The neighborhoods of zero in the corresponding topology for the linear space of functions f are defined as

$$B_\alpha = (0, 0, \dots, f_{\alpha+1}, f_{\alpha+2}, \dots). \tag{9}$$

In this case, the convergence in proximity order is the uniform convergence. For any given neighborhood $\mathfrak{B}_\alpha$, we can find corresponding $N(\alpha)$ such that

$$(l^{(N)}, f) \in \mathfrak{B}_\alpha \quad \text{for} \quad N > N(\alpha). \tag{10}$$

Let us consider two examples.

Example 1. Let $f(x, y)$ be an analytic function of two variables (x, y); in a neighborhood about zero, $f(x, y)$ decomposes in the convergent Maclaurin power series

$$f(x, y) = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \dots. \tag{11}$$

We identify the function f with the vector function

$$(a_0,\ a_{10}x + a_{01}y,\ a_{20}x^2 + a_{11}xy + a_{02}y^2, \dots)$$

whose entries are the homogeneous polynomials constituting series (11).

The convergence of $l^{(N)}$ to 0 in proximity order means that for polynomials of an arbitrarily high degree $(l^{(N)}, f) = 0$, where N is sufficiently large.

Let $f_h(x, y) = f(hx, hy)$. From $\|f_h\|_X \le Ah^\alpha$ for $f \in \mathfrak{B}_\alpha$ it follows that the set of functions from the unit sphere in X, which are in a smaller neighborhood of zero $\mathfrak{B}_\alpha$, appears for sufficiently small h in a smaller neighborhood of zero in the sense of topology of X as well. One may say that a neighborhood in B is sliced in a countable set of neighborhoods in T.

Example 2. Let the function $f(\vartheta, \varphi)$ be given on the unit sphere of the three-dimensional space and belong to $W_2^{(2)}$, i.e., for example,

$$\|f\|_{W_2^{(2)}} = \int\limits_S (\Delta f)^2 \, dS + \Big(\int\limits_S f \, dS \Big)^2 < \infty. \tag{12}$$

Let us decompose f into a series of spherical harmonics

$$f = \sum_{n=0}^{\infty} Y_n(\vartheta, \varphi). \tag{13}$$

We identify f with an abstract vector function

$$(Y_0(\vartheta, \varphi), Y_1(\vartheta, \varphi), \dots, Y_n(\vartheta, \varphi), \dots). \tag{14}$$

In this case the convergence in proximity order means that for arbitrarily many spherical harmonics the equality $(l^{(N)}, f) = 0$ holds, where N is sufficiently large.

For values on the sphere of an analytic function $f(x, y, z)$ of three variables, the neighborhoods of zero $\mathfrak{B}_\alpha$ are again sliced under the substitution $f_h(x, y, z) = f(hx, hy, hz)$ into ε-neighborhoods in the topology B. In this event, there is a feature that some of f may occasionally fall into a closer neighborhood of zero than Ah^α.

References

1. Nikol'skii, S. M.: Quadrature Formulas. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow (1958)
2. Sobolev, S. L.: Formulas of mechanical cubatures in n-dimensional space. Dokl. Akad. Nauk SSSR, **137**, 527–530 (1961)[1]

[1] Paper [7] of Part II of this book. – *Ed.*

10. Cubature Formulas on the Sphere Invariant under Finite Groups of Rotations*

S. L. Sobolev

A cubature formula on the surface of the sphere

$$(l,f) = \int_S f(\vartheta,\varphi)\,dS - \sum_{k=1}^{N} c_k f(x^{(k)}) \cong 0 \tag{1}$$

is called *invariant* under transformations of a certain group G of sphere rotations if

$$\Big(l, f\big(\vartheta_1(\vartheta,\varphi), \varphi_1(\vartheta,\varphi)\big)\Big) = \Big(l, f(\vartheta,\varphi)\Big), \tag{2}$$

where

$$\vartheta_1(\vartheta,\varphi), \quad \varphi_1(\vartheta,\varphi) \tag{3}$$

is a substitution in G.

L. A. Lyusternik and V. A. Ditkin [1, 2] have considered formulas with nodes at the vertices of an icosahedron and centers of its faces. We will show how to construct cubature formulas which are invariant under the groups of rotations of the sphere corresponding to a regular polyhedron and are valid for as many spherical harmonics as possible [3].

Theorem 1. *Let a cubature formula be invariant under G. Then it is exact for all harmonics of a given degree if and only if it is exact for all invariant harmonics $Y_n^*(\vartheta,\varphi)$ of this degree n, i.e., for those hormonics which are unchanged under rotations of the sphere belonging to G:*

$$Y_n^*\big(\vartheta_1(\vartheta,\varphi), \varphi_1(\vartheta,\varphi)\big) = Y_n^*(\vartheta,\varphi). \tag{4}$$

The proof is based on the formula

$$(l,f) = (l,f_G), \tag{5}$$

where f_G is the mean of the function f over the group[1] G:

* Dokl. Akad. Nauk SSSR, **146**, 310–313 (1962)

[1] In this paper, M is the order of G. – *Ed.*

$$f_G(x) = \frac{1}{M} \sum_{g \in G} f(gx). \tag{6}$$

Let $S(n)$ be the number of invariant harmonics of degree n. This number may be computed using the representation theory of groups, as was pointed out to the author by D. K. Faddeev.

The spherical harmonics of degree n form a $(2n+1)$-dimensional space, whose basis may be chosen to be

$$e^{im\varphi} P_n^{(|m|)}(\cos\vartheta), \quad m = 0, \pm 1, \dots, \pm n. \tag{7}$$

The group of rotations of the sphere induces the group of linear substitutions which acts on harmonics (7) and is a linear representation of the former group.

Every representation decomposes into irreducible representations on subspaces of lower dimensionality. Among them some are identity representations. The number $S(n)$ of linearly independent invariant harmonics coincides with the number of such one-dimensional identity representations included in the representation A.

The traces of the matrices of irreducible representations (the so-called characters of the representation) constitute M-dimensional vectors. It is well known that characters of distinct irreducible representations are orthogonal:

$$\sum_{k=1}^{M} \chi\big(A_k^{(j)}\big)\overline{\chi}\big(A_k^{(s)}\big) = \begin{cases} M, & A^{(j)} \sim A^{(s)}, \\ 0, & A^{(j)} \nsim A^{(s)}. \end{cases} \tag{8}$$

Obviously, all characters of the identity representations equal 1. Hence, for the number $S(n)$ we get the formula

$$S(n) = \frac{1}{M} \sum_{k=1}^{M} \chi(A_k), \tag{9}$$

where A_k are the matrices representing the group rotations.

Similar matrices have the same trace; and the rotations by the same angle about corresponding elements are similar.

There are t_1 vertices, t_2 faces, and t_3 edges in a regular polyhedron. At the vertices, q_1 of elements meet; the faces are regular q_2-gons; and the edges are the axes of rotations of order $q_3 = 2$. Obviously,

$$t_1 q_1 = t_2 q_2 = t_3 q_3 = M, \tag{10}$$

while also $\dfrac{1}{2}[t_1(q_1 - 1) + t_2(q_2 - 1) + t_3(q_3 - 1)] + 1 = M$; whence

$$t_1 + t_2 + t_3 = M + 2. \tag{11}$$

The sums of the traces of all rotations (including the identity) about an arbitrary vertex, center of face, or midpoint of edge are equal to[2]

$$\sum_{k=0}^{q_j-1} \sum_{m=-n}^{n} e^{i2\pi mk/q_j} = q_j \left(2\left[\frac{n}{q_j}\right]+1\right), \quad j=1,2,3. \tag{12}$$

Summing these equalities over all the axes of rotations, we note that under this summation the identity rotation is counted $M+2$ times and each other rotation twice. Considering this and using (9) and (10), we come to the following theorem.

Theorem 2.

$$S(n) = \left[\frac{n}{q_1}\right] + \left[\frac{n}{q_2}\right] + \left[\frac{k}{q_3}\right] - n + 1. \tag{13}$$

A simple computation leads to the corollary:

$$S\left(\frac{M}{2} - n - 1\right) + S(n) = 1 \quad \text{for} \quad 0 \le n \le \frac{M}{2} - 1. \tag{14}$$

Let us transform (13) into another form. Let Q^* be the set of those q_j for which $n \neq 0 \pmod{q_j}$. Expressing in (13) the integral part through its fractional part, we find[3]

$$S(n) = 1 + \frac{1}{M}\Big(2n - \sum_{q_j \in Q^*} t_j\Big) - \sum_{q_j \in Q^*} \left(\left\{\frac{n}{q_j}\right\} - \frac{1}{q_j}\right). \tag{15}$$

Since $0 \le \sum_{q_j \in Q^*} \left(\left\{\frac{n}{q_j}\right\} - \frac{1}{q_j}\right) < 1$, (15) implies

$$S(n) = \left[1 + \frac{1}{M}\Big(2n - \sum_{q_j \in Q^*} t_j\Big)\right]. \tag{16}$$

Hence,

$$S(n) = \begin{cases} \left[\dfrac{2n+1}{M}\right] & \text{for} \quad 2n+1 \le \sum_{q_j \in Q^*} t_j, \\ \left[\dfrac{2n+1}{M}\right] + 1 & \text{for} \quad 2n+1 > \sum_{q_j \in Q^*} t_j. \end{cases} \tag{17}$$

From (17) it follows that

$$S\left(n + \frac{M}{2}\right) = S(n) + 1. \tag{18}$$

Formula (17) admits a simple interpretation based on the following theorem.

[2] Here $[\cdot]$ denotes the integral part of a number. – *Ed.*

[3] Here $\{\cdot\}$ denotes the fractional part of a number. – *Ed.*

Theorem 3. *Let* $\{(x_k, y_k, z_k) \mid k = 1, 2, \ldots, 2n+1\}$ *be orthogonal coordinate systems with the same origin, and let the directions of the* z_k*-axes be all distinct. Then the set of functions*

$$\zeta_k^n = (x_k + iy_k)^n, \quad k = 1, 2, \ldots, 2n+1, \tag{19}$$

constitute a basis for the space of spherical functions of degree n.

Proof. Let us introduce the complex variable $\mathbf{z} = x + iy$, mapping the sphere onto the plane by stereographic projection[4]. For ζ_k^n we get the formula

$$\zeta_k^n = \frac{2^n \mathbf{z}_k^n}{(1 + |\mathbf{z}_k|^2)^n}, \tag{20}$$

where

$$\mathbf{z}_k = \frac{a_k \mathbf{z} - \bar{c}_k}{c_k \mathbf{z} + \bar{a}_k}, \quad |a_k|^2 + |c_k|^2 = 1. \tag{21}$$

Linear-fractional transformation (21) corresponds to rotation of the sphere which brings the z_k-axis to the z-axis of the initial coordinate system.

From (20) it follows that

$$\zeta_k^n = \sum_{m=-n}^{n} a_k^{n-m} \bar{c}_k^{n+m} R_n^{(|m|)}(\mathbf{z}\bar{\mathbf{z}}), \tag{22}$$

where the function

$$R_n^{(|m|)}(\mathbf{z}\bar{\mathbf{z}}) = c_n^{(m)} e^{im\varphi} P_n^{(|m|)}(\cos\vartheta) \tag{23}$$

differs only by the constant multiplier from the element of basis (7). Thus, the vector

$$(a_k^{2n}, a_k^{2n-1}\bar{c}_k, \ldots, \bar{c}_k^{2n}) \tag{24}$$

corresponds to the function ζ_k^n in basis (7). From (24) and the known formula for the Vandermonde determinant, Theorem 3 follows. □

From (24) it follows that for any of the $2n+2$ harmonics of the form ζ_k^n with different z_k-directions we have the equality

$$\sum_{k=1}^{2n+2} \frac{\zeta_k^n}{\prod_{j \neq k} (a_k \bar{c}_j - a_j \bar{c}_k)} = 0. \tag{25}$$

Consider the set

$$\{g_\alpha x^{(k)} \mid g_\alpha \in G\}, \tag{26}$$

[4] Issuing half-lines from the south pole of the sphere, S. L. Sobolev maps the latter in one-to-one fashion onto the plane passing through the equator, i.e., $\mathbf{z} = \tan \frac{\vartheta}{2} e^{i\varphi}$. – *Ed.*

where

$$x^{(1)},\, x^{(2)},\, \ldots,\, x^{([\frac{2n+1}{M}])} \tag{27}$$

are points on the unit sphere which are not equivalent under G. The points of set (26) together with an arbitrary system of points

$$x^{(2n+1)},\, x^{(2n)},\, \ldots,\, x^{(M[\frac{2n+1}{M}]+1)}, \tag{28}$$

which are independent of (26), constitute a system of $2n+1$ independent directions of the z_k-axis. Hence, the corresponding functions $\zeta_k^n = (x_k + iy_k)^n$ constitute a basis for the space of spherical harmonics of degree n.

Clearly, the functions

$$\frac{1}{M} \sum_{g_\alpha \in G} \Big(\zeta(g_\alpha x^{(k)})\Big)^n, \quad k = 1, 2, \ldots, \left[\frac{2n+1}{M}\right], \tag{29}$$

are linearly independent invariant harmonics of degree n.

Let $(2n+1) \leq \sum\limits_{q_j \in Q^*} t_j$. Then we can choose as (28) all those directions of the axes which correspond to $q_j \in Q^*$. In this event, all invariant spherical harmonics of degree n are exhausted by functions (29), since the mean over the group of each of (29) is zero. The proof of the first half of formula (17) is complete.

For $(2n+1) > \sum\limits_{q_j \in Q^*} t_j$ we can choose as (28) a system of mutually equivalent points, and the mean of any of the corresponding ζ_k^n taken over the group G yields one more invariant harmonic of degree n.

Let the group G^* be generated by rotations and reflections. For G^* we have the following theorem.

Theorem 4. *The group G^* has no invariant harmonics of odd degree n. The set of G^*-invariant harmonics of even degree n coincides with the set of G-invariant harmonics of degree n.*

In conclusion we present the table of values of $S(n)$, $0 \leq n < M/4$, for the groups G_{IV}, G_{VIII}, and G_{XX} of rotations of the tetrahedron, octahedron, and icosahedron.

n	S_{IV}	S_{VIII}	S_{XX}	n	S_{XX}	n	S_{XX}
0	1	1	1	6	1	11	0
1	0	0	0	7	0	12	1
2	0	0	0	8	0	13	0
3		0	0	9	0	14	0
4		1	0	10	1		
5		0	0				

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[5] Paper [9] of Part II of this book. – *Ed.*

11. The Number of Nodes in Cubature Formulas on the Sphere*

S. L. Sobolev

In the preceding notes [1, 2] we considered cubature formulas on the sphere convergent in proximity order in the space of all series of spherical functions. In the present note we try to estimate asymptotically the gain obtained.

Theorem 1. *Let a system of functions*

$$\psi_1(x),\ \psi_2(x), \dots, \psi_K(x) \tag{1}$$

be given with integrals

$$\int\limits_{\Omega} \psi_j(x)\, dx = b_j, \quad j = 1, 2, \dots, K. \tag{2}$$

Then the formula

$$(l, f) \equiv \int\limits_{\Omega} f(x)\, dx - \sum_{k=1}^{N} c_k f(x^{(k)}) \cong 0 \tag{3}$$

with a given system of nodes is exact for all functions (1) if and only if the formula is exact for all those linear combinations

$$a_1\psi_1 + a_2\psi_2 + \cdots + a_K\psi_K, \tag{4}$$

which vanish at the nodes $x^{(k)}$.

Theorem 1 establishes the duality between the problems of interpolation and numeric integration. It is proved by comparing the system of equations

$$\sum_{k=1}^{N} \psi_j(x^{(k)}) c_k = b_j, \quad j = 1, 2, \dots, K; \quad \Longleftrightarrow \quad A\mathbf{c} = \mathbf{b} \tag{5}$$

* Dokl. Akad. Nauk SSSR, **146**, 770–773 (1962)

for finding the coefficients c_k of cubature formula (3), and the system

$$\sum_{j=1}^{K} a_j \psi_j(x^{(k)}) = 0, \quad k = 1, 2, \ldots, N; \quad \Longleftrightarrow \quad \mathbf{a}A = 0 \tag{6}$$

for finding the coefficients a_1, a_2, ..., a_K of linear combination (4). The symbol A denotes the matrix $A = \big(\psi_j(x^{(k)})\big)$.

Remark. If the rank of A equals K then system (6) has no nontrivial solutions. Hence, there exist coefficients c_k such that (3) is valid for all ψ_j from set (1).

Theorem 2. *Let*

$$\{x_r^{(1)}, x_r^{(2)}, \ldots, x_r^{(N(r))} \mid r = 1, 2, \ldots\} \tag{7}$$

be a sequence of systems of nodes for cubature formulas (3)*. For all systems (7) beginning with a certain $r > R$ to admit of cubature formulas (3) exact for all ψ_j in given finite set (1) of analytic functions, it is sufficient that the following condition holds: there exists a domain $\Omega_0 \subset \Omega$ for which nodes* (7) *form an ε-net for any $\varepsilon > 0$, beginning with a certain $r > r(\varepsilon)$.*

Proof. The idea of the proof is to consider Kth-order determinants of the matrix A. These determinants are values of the function Δ of K variables:

$$\Delta(x^{(1)}, x^{(2)}, \ldots, x^{(K)}) = \det \begin{bmatrix} \psi_1(x^{(1)}) & \ldots & \psi_1(x^{(K)}) \\ \ldots & \ldots & \ldots \\ \psi_K(x^{(1)}) & \ldots & \psi_K(x^{(K)}) \end{bmatrix} \tag{8}$$

for different particular values of $x^{(1)}$, $x^{(2)}$, ..., $x^{(K)}$.

In the domain $\Omega_0 \times \cdots \times \Omega_0$ there exist points where determinant (8) is not zero. Since nodes (7) fall in an arbitrarily small neighborhood of any such point and $\Delta(x^{(1)}, x^{(2)}, \ldots, x^{(K)})$ is an analytic function, the matrix contains nonzero determinants. By virtue of our Remark, this implies Theorem 2. □

As we have seen, the study of cubature formulas invariant under a group of rotations can be confined to those harmonics which are invariant under the same group. In trying to satisfy all conditions (5) we then have only the coefficients c_k for nonequivalent nodes. The number L of such nodes is greater than N/M, where N is the total number of nodes and M is the order of the group.

As n increases, the number $\sigma(n)$ or $\sigma^*(n)$ of invariant harmonics up to the given degree n grows slower than $L(n)$. Therefore, roughly speaking,

$$L(n) = \sigma(n) \quad \text{or} \quad L(n) = \sigma^*(n). \tag{9}$$

If the symmetry was not invoked and all the parameters taken into account were used, we would have in general the equality

$$(n+1)^2 = 3N. \tag{10}$$

Here $(n+1)^2$ is the total number of spherical harmonics up to degree n, and $3N$ is the number of degrees of freedom in formula (3): besides the coefficients c_k there are two parameters to specify each point $x^{(k)}$. For small N, formulas invariant under the icosahedron group do give such an advantage. For large N the advantage is less. It is convenient to estimate the advantage comparing the functions $N(L)$ and $n(\sigma)$ or $n(\sigma^*)$.

Let us compute $N(L)$ for the two nets obtained by projecting onto the sphere triangular nets symmetrically located on all the faces of the invariant polyhedron (see Figs. 1 and 2).

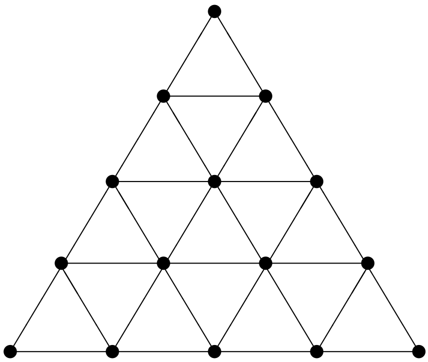

Fig. 1.

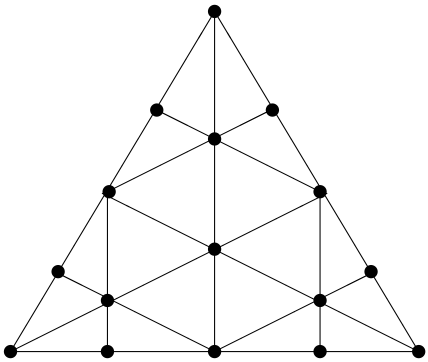

Fig. 2.

For the first type (see Fig. 1) there are k nodes on each side of the triangle. In this event, for the full rotation group and $k = 6s + r$, $r < 6$, we get

$$N = \frac{M}{6}k^2 + 2, \quad L = \begin{cases} (s+1)(k-3s)+1 & \text{for} \quad r = 0, \\ (s+1)(k-3s) & \text{for} \quad r > 0, \end{cases} \tag{11}$$

where M is the order of the rotation group of the polyhedron, equal to half the total order of the group of symmetries.

For the second type (see Fig. 2), with $k = 2s + r$ and $r < 2$:

$$N = \frac{M}{2}k^2 + 2, \quad L = \begin{cases} (s+1)^2 & \text{for} \quad r = 0, \\ (s+1)(s+2) & \text{for} \quad r = 1. \end{cases} \tag{12}$$

This yields, for example, for $k = 6s + 5$ in the first case,

$$N(L) = 2M\big(L - \sqrt{3}\,L^{1/2} + \dots\big), \tag{13}$$

and for $k = 2s + 1$ in the second case,

$$N(L) = 2M\bigl(L - 2\,L^{1/2} + \dots\bigr). \tag{14}$$

Computing σ and σ^* for

$$n = \frac{KM}{2} - 1 \tag{15}$$

in the manner indicated in preceding note [2], we get

$$\sigma(K\frac{M}{2} - 1) = K\sigma(\frac{M}{2} - 1) + \frac{K(K-1)}{2}\frac{M}{2}, \tag{16}$$

$$\sigma^*(K\frac{M}{2} - 1) = K\sigma^*(\frac{M}{2} - 1) + \frac{K(K-1)}{2}\frac{M}{4}, \tag{17}$$

$$\sigma(M/2 - 1) = M/4, \quad \sigma^*(M/2-1) = \begin{cases} 5 & \text{for} \quad G_{VIII}, \\ 11 & \text{for} \quad G_{XX}. \end{cases}$$

Hence, $\sigma(KM/2 - 1) = K^2M/4$, which means that $[n(\sigma) + 1]^2 = M\sigma$, and further that

$$(n^* + 1)^2 = 2M\bigl(\sigma^* - \sqrt{49/30}\,\sigma^{*1/2} + \dots\bigr) = 2M\bigl(\sigma^* - 1.3\,\sigma^{*1/2} + \dots\bigr) \tag{18}$$

for the icosahedron group and

$$(n^* + 1)^2 = 2M\bigl(\sigma^* - \sqrt{4/3}\,\sigma^{*1/2} + \dots\bigr) = 2M\bigl(\sigma^* - 1.16\,\sigma^{*1/2} + \dots\bigr) \tag{19}$$

for the octahedron group.

The comparison of (13) and (14) with (18) and (19) for large L exhibits the gain obtained by using invariant formulas of the type described. It is apparent that this gain is not large.

Setting $L = \sigma^*$ for the small values of n, we tabulate the functions $n(N)$, $(n(N) + 1)^2$, and $L(N)$ for the formulas of the first and second types which are invariant under the group of octahedron and icosahedron.

G^I_{VIII}				G^{II}_{VIII}				G^I_{XX}				G^{II}_{XX}			
N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L
6	3	16	1	14	5	26	2	12	5	36	1	32	9	100	2
18	5	32	2	50	9	100	4	42	9	100	2	122	15	256	4
38	7	64	3	110	11	144	6	92	11	144	3	272	19	400	6
66	9	100	4	194	15	256	9	162	15	258	4	482	25	676	9
102	11	144	5					262	17	324	5				
146	13	196	7												
198	15	256	8												
258	17	324	10												

Formulas with four points for the icosahedron group were given in the paper of V. A. Ditkin and L. A. Lyusternik [3].

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[1] Paper [9] of Part II of this book. – *Ed.*
[2] Paper [10] of Part II of this book. – *Ed.*

12. Certain Questions of the Theory of Cubature Formulas*

S. L. Sobolev

Let $l(x)$ be a generalized function such that

$$l(x) = \chi_\Omega(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}), \tag{1}$$

where $\chi_\Omega(x)$ is the characteristic function of the domain Ω. The values of the functional

$$(l, \varphi) = \int l(x)\varphi(x)\,dx \tag{2}$$

are the errors of a certain cubature formula, and we discuss this functional keeping in mind this formula. Suppose that

$$(l, x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{3}$$

and introduce the Lax norm

$$\|l \mid L_2^{(-m)}\| = \inf_{\varphi \neq 0} \frac{|(l, \varphi)|}{\|\varphi \mid L_2^{(m)}\|}. \tag{4}$$

Here $m > n/2$ and $\|\varphi \mid L_2^{(m)}\|^2 = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2\,dx$. The best is a cubature formula whose error functional has a lesser norm.

The explicit expression of the norm of $l(x)$ can be obtained by solving a variation problem for the Euler equation

$$\Delta^m u(x) = (-1)^m l(x) \tag{5}$$

with the corresponding boundary conditions. The equality holds

$$\|l \mid L_2^{(-m)}\| = \frac{|(l(x), u(x))|}{\|u \mid L_2^{(m)}\|}.$$

* Proc. Joint Soviet-American Sympos. Partial Differential Equations (Novosibirsk, 1963). Inst. Mat., Akad. Nauk SSSR Sibirsk. Otdel., Novosibirsk (1963), 8 p.

Theorem 1. *In the norm*

$$\left\|\varphi \mid V_2^{(m)}(\Omega)\right\| = \inf_{\substack{\widetilde{\varphi}(x)=\varphi(x)\\ x\in\Omega}} \left\|\widetilde{\varphi} \mid L_2^{(m)}(R^n)\right\| \tag{6}$$

the required solution of (5) *can be explicitly written as*

$$u_0(x) = G(x) * l(x), \tag{7}$$

where

$$G(x) = (-1)^m \varkappa_{m,n} |x|^{2m-n} \begin{cases} \ln|x|, & \textit{if} \quad n \quad \textit{even and} \quad n \le 2m, \\ 1, & \textit{if} \quad n \quad \textit{odd or} \quad n > 2m. \end{cases} \tag{8}$$

The proof of Theorem 1 is based on an estimate of convergence of the integrals

$$\int D^\alpha G(x-y)\,dx \quad \text{for} \quad |\alpha| < m.$$

In the periodic case the problems under study can be also solved explicitly. Let H be the matrix of periods,

$$H = \big(\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n\big), \tag{9}$$

$$\det H = |H| = 1. \tag{10}$$

Let us consider the set of functions such that

$$\varphi(x + H\beta) = \varphi(x), \tag{11}$$

where $x \in R^n$ is a coordinate column vector and β is an arbitrary integer column vector. We denote by Ω_0 the fundamental parallelohedron of the matrix H. It means that

$$\sum_\beta \chi_{\Omega_0}(x + H\beta) = 1. \tag{12}$$

Let the error functional of the cubature formula be defined as the generalized function

$$\widehat{l}(x) = \chi_{\Omega_0}(x) - \delta(x). \tag{13}$$

Then we have the next theorem.

Theorem 2. *The norm of* $\widehat{l}(x)$ *can be written as*

$$\|\widehat{l}(x) \mid \widetilde{L}_2^{(-m)}\| = \frac{|(\widehat{l}(x), u_0(x))|}{\|u_0(x) \mid \widetilde{L}_2^{(m)}\|}, \tag{14}$$

where $u_0(x)$ *is the periodic solution of equation* (5) *such that*

$$u_0(x) = -\left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma\ne 0} \frac{1}{|H^{-1*}\gamma|^{2m}}\, e^{i2\pi H^{-1}x\cdot\gamma}. \tag{15}$$

From Theorem 2 it follows that

$$\|\widehat{l}(x) \mid \widetilde{L}_2^{(-m)}\|^2 = \frac{1}{(2\pi)^{2m}} \sum_{\gamma\neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} = \frac{1}{(2m)!} B^2_{n,m}(H^{-1*}). \tag{16}$$

The vectors $H^{-1*}\gamma$ are the nodes of the lattice dual to the lattice with nodes $H\beta$, and $|H^{-1*}\gamma|$ is the distance of the nodes to the coordinate origin.

For large m, i.e., for many times differentiable functions, the term corresponding to the shortest of the distances mentioned dominates in (16):

$$B^2_{n,m}(H^{-1*}) \approx \frac{s}{r^{2m}_{\min}}. \tag{17}$$

Here s is the number of nodes of the lattice $H^{-1*}\gamma$ at minimal distance from the coordinate origin. Therefore, the optimal lattice is given by the nodes $H\beta$, for which the vectors $H^{-1*}\gamma$ constitute the lattice corresponding to the densest packing of balls in n-dimensional space.

For the bounded domain of integration and for the given lattice of nodes hH with a small mesh-size h, we construct cubature formulas with uniform boundary layer; they are obtained by summing cubature formulas for all elementary cells.

Assume that

$$l_0(x) = \chi_\Omega\left(\frac{x}{h}\right) - \sum_{|\beta'|\leq L} c[\beta']\delta\left(\frac{x}{h} - H\beta'\right), \tag{18}$$

and let $(l_0(x), x^\alpha) = 0$ for $|\alpha| \leq m$. Let B_1 be the set of all β such that $l_0(x - hH\beta)$ is supported in the interior of the domain Ω. We compose the sum

$$l_1(x) = \sum_{\beta\in B_1} l_0(x - hH\beta) = \chi_{\Omega^*}(x) - \sum_{\beta'\in B} c^*[\beta']\delta(x - hH\beta'), \tag{19}$$

where B is the set of all β' such that $hH\beta' \in \Omega$. The equality holds

$$\chi_\Omega(x) - \chi_{\Omega^*}(x) = \sum_{\beta\in B\setminus B_1} \chi_\Omega(x)\chi_{\Omega_0}(x - hH\beta). \tag{20}$$

For each β from $B \setminus B_1$ we consider a cubature formula with the error functional defined by

$$l_\beta(x) = \chi_\Omega(x)\chi_{\Omega_0}(x - hH\beta) - \sum_{\substack{|\beta'|\leq L\\ H(\beta+\beta')\in\Omega}} c^{\beta'}[\beta]\delta(x - H\beta - H\beta'). \tag{21}$$

Suppose that $\sup\limits_{\beta,\beta'} |c^{\beta'}[\beta]| \leq A$, and let

$$l_2(x) = \sum_{\beta\in B\setminus B_1} l_\beta(\frac{x}{h}). \tag{22}$$

We refer to the cubature formula with the error functional $l(x) = l_1(x) + l_2(x)$ as the *normal cubature formula*. Let

$$m(x) = \sum_{|\beta'|\leq L} d[\beta']\delta(x - hH\beta'), \tag{23}$$

$$(m(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m. \tag{24}$$

Then we consider the sum

$$M(x) = \sum_{\beta\in B\setminus B_1} m(x - hH\beta) = \sum_{\beta} F[\beta]\delta(x - hH\beta). \tag{25}$$

This sum is equal to zero at all nodes $hH\beta$ of the lattice lying at a distance greater than $2Lh$ from the boundary of Ω. Hence, the discrete function $F[\beta]$ is supported inside a certain boundary layer of the boundary of Ω.

Theorem 3. *Let the generalized function* $M(x) = \sum\limits_{\beta} F[\beta]\delta(x-hH\beta)$ *be equal to zero at all points* $hH\beta$ *lying at a distance greater than* $2Lh$ *from the boundary of* Ω, *and let* $(M(x), x^\alpha) = 0$ *for* $|\alpha| < m$. *Then* $M(x)$ *can be written as*

$$M(x) = \sum_{\beta\in B\setminus B_1} m_\beta(x), \tag{26}$$

where $m_\beta(x) = \sum\limits_{\beta'} F^\beta[\beta']\delta(x - hH(\beta + \beta'))$ *for* $\beta \in B \setminus B_1$, $(m_\beta(x), x^\alpha) = 0$ *for* $|\alpha| < m$, *and the set* $B \setminus B_1$ *consists of the nodes in a certain expanded boundary layer with width* KL.

We call the function expanded like (26) the *normal homogeneous boundary layer with the order* m.

Corollary. *Two normal cubature formulas differ from each other by a normal boundary layer with the order* m.

Theorem 3 is proved by using a special technique of the partial summation over each variable in turn and the replacement of the integration domain by an approximate domain with the coordinate planes taken for its boundaries.

There is an analogy between operators orthogonal to x^α, $|\alpha| \leq m$, and the differential operators with constant coefficients $L(D) = \sum\limits_{\gamma} a_\gamma D^\gamma$, where $|\gamma| > m$. The integral over the volume of differential expressions with such operators is expressed as a surface integral containing derivatives of order higher than $m - 1$.

Theorem 4. *In the space* $V_2^{(m)}(\Omega)$ *the value of the extremal function of the normal cubature formula of order* m *at any interior point of* Ω *tends to the value of the periodic extremal function with the lattice* hH. *From above the errors of the normal cubature formula can be estimated as*

$$|(l, \varphi)| \leq h^m B_{n,m}(H^{-1*})\sqrt{|\Omega|}\ \|\varphi \mid V_2^{(m)}(\Omega)\| + O(h^{m+1}).$$

Proof. The extremal function of a normal cubature formula can be expanded in the sum

$$G(x) * \Big(l_1(x) + l_2(x)\Big) = \sum_{\beta \in B_1} G(x) * l_0(x - hH\beta) + \sum_{\beta \in B \setminus B_1} G(x) * l_\beta(x - hH\beta).$$

Each term of the first sum on the right side decreases as $|x|/h \to \infty$ not slower than $|H\beta|^{-n-1}$, and each term of the second sum decreases not slower than $|H\beta|^{-n}$. Therefore, as $h \to 0$, the first sum converges to the periodic solution of the equation $\Delta^m u_0(x) = (-1)^m \big[1 - h^n \sum_\gamma \delta(x - hH\gamma)\big]$ absolutely, and the second sum tends to zero. □

From Theorem 4 it follows that the quality of normal cubature formulas is determined mainly by the properties of the lattice. Therefore, for large m, the optimal lattice is again dual to the lattice with nodes that are the centers of the balls constituting a densest packing in n-dimensional space.

It is convenient to construct normal cubature formulas using the Fourier transform.

Theorem 5. *For the given generalized function $l(x)$ to be orthogonal to all polynomials of degree $m - 1$, it is necessary and sufficient that its Fourier image $\widetilde{l}(p)$ would have a zero of multiplicity m at the coordinate origin and the integrals $\int l(x) x^\alpha dx$ for $|\alpha| < m$ would have meaning.*

The proof is elementary.

Rejecting the second requirement of Theorem 5, we can construct normal cubature formulas for infinite domains such as a half-space and s-faced solid angles with rational faces. Also, we can construct boundary layers for polyhedral domains. In fact, such cubature formulas have been constructed in the cases listed.

Theorem 6. *In the case of a polyhedron, the cubature formula with boundary layer coinciding in a neighborhood of each s-faced solid angle of the polyhedron with the boundary layer constructed for the corresponding infinite solid angle is a normal cubature formula.*

13. A Method for Calculating the Coefficients in Mechanical Cubature Formulas*

S. L. Sobolev

For a given set of nodes, it is often possible to construct a mechanical cubature formula that is exact for arbitrary polynomials of a given degree by means of the Fourier transform. Let the problem require seeking for c_k such that the functional

$$(l,\varphi) = \int\limits_{\Omega} \varphi(x)\,dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}) \equiv \int l(x)\varphi(x)\,dx \tag{1}$$

vanishes at all polynomials of degree $m-1$. Here the generalized function $l(x)$ is defined by

$$l(x) = \chi_{\Omega}(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}), \tag{2}$$

where $\chi_{\Omega}(x)$ is the characteristic function of the domain Ω, and $\delta(x - x^{(k)})$ is the generalized Dirac delta function.

Theorem 1. *For the functional $l(x)$ to vanish at all polynomials of degree $m-1$, it is necessary and sufficient that its Fourier transform $\widetilde{l}(p)$ has a zero of multiplicity m at the origin.*

Proof. From the condition of the theorem it follows that for all α with $|\alpha| \leq m-1$, we have

$$l(x) * x^{\alpha} = 0, \tag{3}$$

where $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Since the Fourier transform of the convolution is transformed into a product of the Fourier images and the Fourier transform of x^{α} is $(2\pi)^{n/2} D^{\alpha}\delta(p)$, the equality holds

$$D^{\alpha}\widetilde{l}(p) = 0. \tag{4}$$

Formula (4) means that there exists a zero of multiplicity m of the function $\widetilde{l}(p)$ at the origin. □

* Dokl. Akad. Nauk SSSR, **150**, 1238–1241 (1963)

It is convenient to apply Theorem 1 to that domain Ω for which the Fourier transform of the characteristic function $\chi_\Omega(x)$ can be calculated in a finite form. For example, ellipsoids or polyhedrons possess this property. In this case, the use of Theorem 1 leads to a system of linear equations for c_k. This system is generally underdetermined, i.e., the number of its equations is less than the number of its unknowns. By finding the solutions of this system, we construct the required coefficients.

Theorem 1 was formulated for bounded domains. If the domain Ω is unbounded, then in general integrals (3) do not exist. However, the Fourier transform of the characteristic function $\chi_\Omega(x)$ sometimes also retains its sense in the case of unbounded domains.

The generalized function $l(x)$ is called a *function of order* m if it has a generalized Fourier transform $\widetilde{l}(p)$, which is an m times continuously differentiable function in a neighborhood about the coordinate origin, and there is a zero of order m of $\widetilde{l}(p)$ at the coordinate origin. Also an arbitrary linear combination of functions of order m is a function of order m.

In certain cases the use of generalized functions of order m makes it possible to calculate coefficients for cubature formulas of degree $m-1$. Let us give an example.

Let functions of one variable be defined on the infinite interval as follows:

$$\psi_0(x) = 1, \quad \Phi_0(x) = \sum_{k=-\infty}^{+\infty} \delta(x-k), \quad \psi_1(x) = \begin{cases} 1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0, \end{cases}$$

$$\Phi_1(x) = \frac{1}{2}\delta(x) + \sum_{k=1}^{\infty} \delta(x-k),$$

$$\chi_0(x) = \sum_{k=-\infty}^{+\infty} \delta\Big(x - k - \frac{1}{2}\Big), \qquad \chi_1(x) = \sum_{k=0}^{\infty} \delta\Big(x - k - \frac{1}{2}\Big).$$

In n-dimensional space the characteristic functions of coordinate s-faced solid angles $\Omega^{(j_1,\ldots,j_s)}$ can be represented in the form

$$\chi^{(j_1,j_2,\ldots,j_s)}(x) = \psi_1(x_{j_1})\psi_1(x_{j_2})\ldots\psi_1(x_{j_s}). \tag{5}$$

Using a linear transformation we may write the characteristic functions of the s-faced solid angles between hyperplanes in arbitrary directions as a product like (5).

The characteristic function of the parallelepiped $\{x \mid 0 < x_j < a_j\}$ can be written as a sum of functions like (5). For example, in two dimensions we obtain

$$\chi_\Omega(x,y) = \chi^{(1,2)}(x,y) + \chi^{(1,2)}(a_1 - x, y) + \chi^{(1,2)}(x, a_2 - y)$$
$$+\chi^{(1,2)}(a_1 - x, a_2 - y) - \chi^{(1)}(x,y) - \chi^{(1)}(a_1 - x, y)$$

$$-\chi^{(2)}(x,y) - \chi^{(2)}(x, a_2 - y) + 1.$$

There is an analogous formula for an arbitrary dimension n.

Let all differences

$$\chi^{(j_1,j_2,\dots,j_s)}(x) - \sum_{k\in K} c_k \delta(x - x^{(k)})$$

be generalized functions of order m. Then there exists a linear combinations of these functions such that it is an error functional of a certain cubature formula in a bounded parallelepiped. This cubature formula is exact for all polynomials of degree $m-1$.

Instead of the parallelepiped, we can consider an arbitrary convex polyhedron with rational faces and seek for a cubature formula of order m for each unbounded s-faced solid angle $\Omega^{(j_1,j_2,\dots,j_s)}$ of the polyhedron in the form

$$\chi^{(j_1,j_2,\dots,j_s)}(x) - c\sum_{k\in K_1} \delta(x - x^{(k)}) - \sum_{k\in K_2} c_k\delta(x - x^{(k)}).$$

Here nodes $x^{(k)}$ are members of some parallelepipedal system of points. For $k \in K_1$ the nodes $x^{(k)}$ are all points of the lattice which lie in the interior of the domain $\Omega^{(j_1,j_2,\dots,j_s)}$, and for $k \in K_2$ the nodes $x^{(k)}$ are points of a boundary layer which lie in $\Omega^{(j_1,j_2,\dots,j_s)}$, at a finite distance from the boundary of the domain. In future publications we will establish that the choice of c and c_k may be realized in such a way that the corresponding cubature formula is close to an optimal formula in a known sense.

It is convenient to define the coefficients for the nodes in the boundary layer by distinguishing these nodes according to their order. We call a set $K_2^{(r)}$ a *boundary layer of order* r if it consists of points at distance r from the coordinate planes. We have:

$$K_2^{(n)} \cup \ldots \cup K_2^{(2)} \cup K_2^{(1)} = K_2.$$

The coefficients c_k for the nodes in $K_2^{(r)}$ are defined in such a way that they are common for all domains

$$\Omega^{(j_1,j_2,\dots,j_r)},\ \Omega^{(j_1,j_2,\dots,j_{r+1})},\ \dots,\ \Omega^{(1,2,\dots,n)}.$$

The Fourier transform makes it possible to calculate all coefficients for nodes in boundary layers of order r. The cubature formulas so obtained are regular in a certain sense, which we shall indicate later.

Theorem 2. *Let Ω be an arbitrary bounded convex polyhedron with rational faces. For error functional*

$$\chi_\Omega(x) - c\sum_{k\in K_1} \delta(x - x^{(k)}) - \sum_{k\in K_2} c_k\delta(x - x^{(k)})$$

to vanish at all polynomials of degree $m-1$, it is sufficient that the coefficients for nodes in the boundary layer K_2 are the same as the coefficients for nodes in unbounded boundary layers of s-faced solid angles formed by the boundaries of the polyhedron.

Given an optimal periodic lattice in three-dimensional space, we have carried out the computation of the coefficients for nodes in the simplest boundary layers.

In [1] it was established that for a lattice defined by a lattice matrix H with $|H| = 1$ and for the space of periodic functions $\widetilde{L}_2^{(m)}$ the error has the bound

$$|(l, \varphi)|^2 \leq \frac{1}{(2\pi)^{2m}} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} \left\| \varphi \mid L_2^{(m)} \right\|^2.$$

Hence, the constant

$$\sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} \tag{6}$$

gives the quality measure of the lattice. In a later publication we will prove that the same constant (6) also estimates the quality of the lattice in the nonperiodic case.

From (6) it follows that at m large, only the first term $1/r_{\min}^{2m}$ of the total sum (6) is significant. Here, $r_{\min}$ is the shortest distance between points of the lattice with the lattice matrix H^{-1*}, i.e., $r_{\min}$ is the maximal diameter of the disjoint spheres centered at the nodes of the lattice with the matrix H^{-1*}. From this it follows that the optimal lattice is that for which the diameter $r_{\min}$ is the largest. Hence, the volumes of the spheres are the largest as well. In a large domain for different H with $|H| = 1$ the number of spheres is constant and numerically equal to the volume V of the domain. Hence, the optimal lattice H^{-1*} must be the lattice *with the closest packing of spheres* in n dimensions[1].

From this it follows that for $n = 3$ the optimal H^{-1} is the face-centered cubic lattice, and the optimal H is the centered cubic lattice. For the optimal lattice we have constructed formulas for the two-faced and three-faced solid angles.

In calculating the coefficients of such formulas, the method is based on the Fourier transform of the functions Φ_j, ψ_j, and χ_j under consideration. The formulas for these transforms are

$$\widetilde{\Phi}_0(p) = \sqrt{2\pi} \sum_{k=-\infty}^{+\infty} \delta(p - 2k\pi),$$

[1] The algorithm for finding the closest packing was proposed by G. F. Voronoi. For small dimensions the description of the closest packing can be found in Chap. II of the book: Sobolev, S. L.: Introduction to the Theory of Cubature Formulas. Nauka, Moscow (1974). – *Ed.*

$$\widetilde{\chi}_0(p) = \sqrt{2\pi} \sum_{k=-\infty}^{+\infty} (-1)^k \delta(p - 2k\pi),$$

$$\widetilde{\psi}_0(p) = \sqrt{2\pi}\delta(p),$$

$$\widetilde{\Phi}_1(p) = \sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{+\infty} \delta(p - 2k\pi) + \frac{i}{2\sqrt{2\pi}} \cot\frac{p}{2},$$

$$\widetilde{\chi}_1(p) = \sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{+\infty} (-1)^k \delta(p - 2k\pi) + \frac{i}{2\sqrt{2\pi}\sin\dfrac{p}{2}},$$

$$\widetilde{\psi}_1(p) = \sqrt{\frac{\pi}{2}}\delta(p) + \frac{i}{\sqrt{2\pi}p}.$$

L. V. Voitisek has carried out numerical calculations of the coefficients for the boundary layer in such formulas. For the three-faced solid angle $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, one may write the formula for the error functional in the following way:

$$l(x) = \chi^{(1,2,3)}(x) - \frac{1}{2}\,\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) - \frac{1}{2}\,\chi_1(x_1)\chi_1(x_2)\chi_1(x_3)$$

$$-\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) \sum_{j_3=1}^{3} \sum_{k=0}^{2} \frac{\alpha_{2k}}{\Phi_1(x_{j_3})}\,\delta(x_{j_3} - k)$$

$$-\chi_1(x_1)\chi_1(x_2)\chi_1(x_3) \sum_{j_3=1}^{3} \sum_{k=0}^{1} \frac{\alpha_{2k+1}}{\chi_1(x_{j_3})}\,\delta(x_{j_3} - k - \frac{1}{2})$$

$$-\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) \sum_{j_2=1}^{3} \sum_{j_1=j_2}^{3} \sum_{k,l=0}^{2} \frac{\alpha_{2k,2l}}{\Phi_1(x_{j_1})\Phi_1(x_{j_2})}\,\delta(x_{j_1} - k)\delta(x_{j_2} - l)$$

$$-\chi_1(x_1)\chi_1(x_2)\chi_1(x_3) \sum_{j_2=1}^{3} \sum_{j_1=j_2}^{3} \sum_{k,l=0}^{2} \frac{\alpha_{2k+1,2l+1}}{\chi_1(x_{j_1})\chi_1(x_{j_2})}\delta(x_{j_1} - k - \frac{1}{2})\delta(x_{j_2} - l - \frac{1}{2})$$

$$- \sum_{k_1,k_2,k_3=0}^{2} \alpha_{2k_1,2k_2,2k_3}\delta(x_1 - k_1)\delta(x_2 - k_2)\delta(x_3 - k_3)$$

$$- \sum_{k_1,k_2,k_3=0}^{1} \alpha_{2k_1+1,2k_2+1,2k_3+1}\delta(x_1 - k_1 - \frac{1}{2})\delta(x_2 - k_2 - \frac{1}{2})\delta(x_3 - k_3 - \frac{1}{2}).$$

It turns out that, for the coefficients α, we may take the following values:

$\alpha_0 = -0.850694444 \cdot 10^{-1}$	$\alpha_{000} = -0.335894549 \cdot 10^{-2}$
$\alpha_2 = -0.116666667$	$\alpha_{002} = \alpha_{020} = \alpha_{200} = 0.309787330 \cdot 10^{-2}$
$\alpha_4 = -0.93750000 \cdot 10^{-2}$	$\alpha_{022} = \alpha_{202} = \alpha_{220} = 0.527777778 \cdot 10^{-1}$
$\alpha_1 = +0.160416667$	$\alpha_{222} = \alpha_{224} = \alpha_{242} = \alpha_{442} = \alpha_{333} = 0$
$\alpha_3 = +0.506944444 \cdot 10^{-1}$	$\alpha_{004} = \alpha_{040} = \alpha_{400} = 0.969939054 \cdot 10^{-1}$
$\alpha_{00} = +0.318612557 \cdot 10^{-1}$	$\alpha_{024} = \alpha_{042} = \alpha_{204} = \alpha_{402} =$
$\alpha_{02} = \alpha_{20} = 0.633680555 \cdot 10^{-1}$	$\alpha_{240} = \alpha_{420} = 0.63964836 \cdot 10^{-2}$
$\alpha_{22} = +0.361111111 \cdot 10^{-1}$	$\alpha_{440} = \alpha_{404} = \alpha_{044} = 0.210458262 \cdot 10^{-3}$
$\alpha_{04} = \alpha_{40} = 0.372902200 \cdot 10^{-1}$	$\alpha_{442} = \alpha_{424} = \alpha_{244} = 0.969509547 \cdot 10^{-2}$
$\alpha_{24} = \alpha_{42} = 0.607638888 \cdot 10^{-1}$	$\alpha_{444} = 0.927847402 \cdot 10^{-2}$
$\alpha_{44} = +0.320818866 \cdot 10^{-2}$	$\alpha_{111} = 0.114626736$
$\alpha_{11} = -6.127097801$	$\alpha_{113} = \alpha_{131} = \alpha_{311} = 0.247395833 \cdot 10^{-1}$
$\alpha_{13} = \alpha_{31} = -0.444299768 \cdot 10^{-1}$	$\alpha_{133} = \alpha_{313} = \alpha_{331} = 0.742187498 \cdot 10^{-2}$
$\alpha_{33} = +0.484664352 \cdot 10^{-2}$	$\alpha_{333} = 0.$

By the same method a boundary layer is computed near a boundary separating domains with lattices of different densities, if such domains exist.

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[2] Paper [7] of Part II of this book. – *Ed.*

14. On the Rate of Convergence of Cubature Formulas*

S. L. Sobolev

The subject of the present note is the estimation of the norm of the error of cubature formulas in a domain Ω of n independent variables. We consider the error of a cubature formula as a linear functional of the form

$$l(x) = \chi_\Omega(x) - \sum_{t=1}^{N} c_t \delta(x - x^{(t)}) \tag{1}$$

in the space $L_2^{(m)}(R^n)$. The function $\varphi(x)$ with domain R^n is a member of $L_2^{(m)}(R^n)$ provided that it has all derivatives up to order m locally integrable and the norm

$$\| \varphi \mid L_2^{(m)} \| = \left\{ \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx \right\}^{1/2} \tag{2}$$

is finite. The space $L_2^{(m)}(R^n)$ is the quotient space of $W_2^{(m)}$ over the space of polynomials of degree $m-1$. By $\chi_\Omega(x)$ in (1) we denote the characteristic function of the domain Ω, the symbol $|\alpha|$ for a vector α with integer entries means $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$. Here it is necessary to assume that

$$m > n/2 \tag{3}$$

and

$$(l(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m. \tag{4}$$

Let $|\Omega|$ be the volume of the domain Ω and let

$$\frac{|\Omega|}{N} = h^n. \tag{5}$$

* Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)

Theorem 1. *There exists a constant K_1 depending only on m and n such that*

$$\|l \mid L_2^{(m)*}\| \geq K_1\sqrt{|\Omega|}h^m. \tag{6}$$

Proof. The proof of this theorem is based on estimates similar to those given by N. S. Bakhvalov in the proof of theorems like Theorem 1 in other functional spaces.

Let

$$Q_j = \left\{x \mid \left|x_s - x_s^{(j,0)}\right| < k/2,\ s = 1, 2, \ldots, n\right\}$$

be a cube with the edge length k, and require that Q_j does not contain the nodes $x^{(t)}$ of the error (1) on its boundary or near it to a distance ηk, where $\eta > 0$. If

$$(l(x), \chi_{Q_j}(x)) > \eta_2 k^n, \tag{7}$$

where $\eta_2 > 0$, then Q_j is called a *cube with insufficient data for the error $l(x)$*.

Lemma 1. *Let Q_j, $j = 1, 2, \ldots, N_1$, be a system of disjoint cubes with insufficient data for the error $l(x)$ in the domain Ω, and let the sum of volumes Q_j be greater than some positive constant*

$$\sum_{j=1}^{N_1} |Q_j| > |\Omega_1|.$$

Then the norm of the error $l(x)$ satisfies the inequality

$$\|l(x) \mid L_2^{(m)*}\| \geq K_1\sqrt{|\Omega_1|}k^m. \tag{8}$$

Proof. As in the paper by N. S. Bakhvalov, the proof of (8) consists of a direct estimate of the error $l(x)$ on a certain function consisting of a sum of “hats” over each cube of a system with insufficient data for the error $l(x)$. □

Theorem 1 follows from Lemma 1. Let us cover the domain Ω with a system of cubes generating a cubic lattice with side $k_1 = 2^{-1/n}h$ and consider all cubes with the edge length $k < (1 - \eta_3)k_1$, concentric with the cubes of this lattice. The number of these concentric cubes is obviously no less than $2N$. Since they do not contain common points, at least half of them contain none of the nodes $x^{(t)}$, and hence they are cubes with insufficient data for the error $l(x)$. Their total volume $|\Omega_1|$ is greater than $(1 - \eta_3)^n|\Omega|/2$, from which Theorem 1 follows. □

Lemma 1 also shows that the main source of the error of the cubature formula is the irregularity of distribution of the nodes $x^{(t)}$, and this distribution cannot be made perfect.

The estimate given by Theorem 1 is attainable, as the following theorem shows.

Theorem 2. *Let the error* $l(x)$ *be written as the sum*

$$l(x) = \sum_{\gamma} l_\gamma \left(\frac{x}{h} - \gamma \right), \tag{9}$$

where γ *ranges over the points of the integral lattice, and let each* $l_\gamma(y)$ *satisfy the conditions*

$$(l_\gamma(y), y^\alpha) = 0 \quad for \quad |\alpha| \le m, \tag{10}$$

$$\|l_\gamma(y) \mid L_2^{(m)*}\| \le A, \tag{11}$$

$$\operatorname{supp} l_\gamma(y) \subset \{y \mid |y| \le L\}, \tag{12}$$

where $\operatorname{supp} l_\gamma(y)$ *denotes the support of* $l_\gamma(y)$*. Then for the norm of* $l(x)$ *the following inequality is valid:*

$$\|l \mid L_2^{(m)*}\| \le K_2 h^m. \tag{13}$$

Here the constant K_2 *depends on the domain* Ω*, the numbers* A *and* L*, but* K_2 *does not depend on the functionals* $l_\gamma(y)$*.*

Proof. Before proving this theorem, it is useful to note that for a domain with piece-wise smooth boundaries and given numbers A and L, it is always possible to construct for sufficiently small h an infinite set of functionals permitting representation (9). Indeed, we may always decompose the domain Ω in the union of cells

$$\Omega = \bigcup_{\gamma} \Omega_\gamma, \tag{14}$$

where Ω_γ is a cell lying at a distance not greater than Lh from the point $x = h\gamma$:

$$\operatorname{dist}(\Omega_\gamma, h\gamma) \le Lh. \tag{15}$$

The characteristic function $\chi_{\Omega_\gamma}(x)$ may be written as

$$\chi_{\Omega_\gamma}(x) = \chi_{\Omega_\gamma^*}(\frac{x}{h} - \gamma), \tag{16}$$

where $\chi_{\Omega_\gamma^*}(y)$ is the characteristic function of some bounded domain Ω_γ^*. By the classical method of extrapolation, we can construct in the domain Ω_γ^* a cubature formula, which is exact for all polynomials of degree $m-1$ and has the error functional

$$l_\gamma(y) = \chi_{\Omega_\gamma^*}(y) - \sum_{|\gamma'| \le L} c_\gamma^{(\gamma')} \delta(y - \gamma'). \tag{17}$$

The nodes of $l_\gamma(y)$ are those points of the lattice where

$$h(\gamma + \gamma') \in \Omega. \tag{18}$$

In this case, the error $l(x)$ defined by (9) satisfies all the conditions of Theorem 2.

Let us point out the idea of the proof of Theorem 2. As has been established [1–3], the norm of the error $l(x)$ may be expressed by means of a solution of the polyharmonic equation

$$\Delta^m u = (-1)^m l(x). \tag{19}$$

For the norm of $l(x)$ the equality holds

$$\|l \mid L_2^{(m)*}\| = \frac{|(l,u)|}{\|u \mid L_2^{(m)}\|} = \|u \mid L_2^{(m)}\|. \tag{20}$$

To find the solution u of (19) it is convenient to use the elementary solution of the polyharmonic equation

$$G(x) = (-1)^m \varkappa_{m,n} |x|^{2m-n} \begin{cases} 1, & \text{for} \quad n \quad \text{odd or} \quad n > 2m, \\ \ln|x|, & \text{for} \quad n \quad \text{even and} \quad n \le 2m. \end{cases} \tag{21}$$

In this case, we apply the known formula for the inner product:

$$(\phi, \psi) = \Big(\phi(x) * \psi(-x)\Big)\Big|_{x=0}. \tag{22}$$

From (20) and (22) it follows that

$$\|l \mid L_2^{(m)*}\|^2 = (l,u) = l(x) * G(x) * l(-x)\Big|_{x=0}. \tag{23}$$

By virtue of the fact that $l(x)$ and $l(-x)$ are finite generalized functions, the triple convolution on the right side of (23) is associative and commutative. Substituting in (23) the expressions for $l(x)$ and $l(-x)$ from (9), we have

$$\begin{aligned} \|l \mid L_2^{(m)*}\|^2 &\le \sum_{\gamma_1}\sum_{\gamma_2} \left| l_{\gamma_1}\left(\frac{x}{h} - \gamma_1\right) * G(x) * l_{\gamma_2}\left(-\frac{x}{h} + \gamma_2\right)\Big|_{x=0} \right| \\ &= \sum_{\gamma_1}\sum_{\gamma_2} \left| G(x) * \left(l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(-\frac{x}{h}\right)\right)\Big|_{x=h(\gamma_1+\gamma_2)} \right|. \end{aligned} \tag{24}$$

It is not difficult to establish the equalities

$$l_1\left(\frac{x}{h}\right) * l_2\left(\frac{x}{h}\right) = h^n l_3\left(\frac{x}{h}\right), \tag{25}$$

where $l_3(y) = l_1(y) * l_2(y)$, and

$$\left\| l\left(\frac{x}{h}\right) \mid L_2^{(m)*} \right\| = h^{n/2+m} \|l(y) \mid L_2^{(m)*}\|, \tag{26}$$

$$\|\varphi(x) \mid L_2^{(m)}(hx \in \Omega)\| = h^{n/2-m} \|\psi(y) \mid L_2^{(m)}(y \in \Omega)\|, \tag{27}$$

where $\psi(y) = \varphi(hy)$, i.e., $\varphi(x) = \psi(x/h)$. The convolution $l_3(y)$ possesses the properties

$$\|l_3(y) \mid L_2^{(m)*}\| \leq \|l_1(y) \mid L_2^{(m)*}\| \|l_2(y) \mid L_2^{(m)*}\| \leq A^2; \tag{28}$$

$$(l_3(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \leq 2m+1; \tag{29}$$

$$\operatorname{supp} l_3(y) \subset \{y \mid |y| \leq 2L\}. \tag{30}$$

Lemma 2. *The following estimate holds:*

$$\left|G(x) * l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(\frac{x}{h}\right)\right| \leq K \frac{A^2 h^{2n+2m+2}}{(h^2+|x|^2)^{n/2+1}}, \tag{31}$$

where the constant K does not depend on h, l_{γ_1}, and l_{γ_2}.

Proof. For $|x| \leq 3Lh$ inequality (31) follows from (25)–(30). In order to prove (31) for $|x| \geq 3Lh$, we expand $G(x-y)$ in a power series in y in a neighborhood about the point $y = 0$:

$$G(x-y) = \sum_{|\alpha|<2m+2} \frac{(-y)^\alpha}{\alpha!} D^\alpha G(x) + R_{2m+2}(x,y). \tag{32}$$

It is obvious that for $|y| \leq 2L$ the function $R_{2m+2}(x,y)$ satisfies the inequality

$$|D_y^\alpha R_{2m+2}(x,y)| \leq K\,|x|^{-n-2}. \tag{33}$$

Hence,

$$\|R_{2m+2}(x,y) \mid L_2^{(m)*}(|y| \leq 2Lh)\| \leq K\,h^{m-n/2}|x|^{-n-2}. \tag{34}$$

Since

$$G(x) * l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(\frac{x}{h}\right) = \int l_3(\frac{y}{h}) G(x-y)\,dy = (l_3(\frac{y}{h}), G(x-y)),$$

where $l_3 = l_{\gamma_1} * l_{\gamma_2}$, (31) follows from (26)–(30) and (34). □

Theorem 2 is obtained from Lemma 2 and (24). The double sum (24) may be estimated from above by applying the integral criterion. □

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[1] Paper [7] of Part II of this book. – *Ed.*

15. Theory of Cubature Formulas*

S. L. Sobolev

The connection between functional analysis and computational mathematics, completely realized in the recent two decades, is too broad to discuss it as a whole. Therefore, in this report I restrict myself only to one aspect of this connection, the theory of approximate integration of functions of many independent variables. In this area, it is possible to obtain a number of important results by applying functional analytic methods.

To each formula of mechanical cubature used for integration of the function φ from X, where X is a certain Banach space, there corresponds the linear error functional defined by

$$(l, \varphi) = \int\limits_{\Omega} \varphi(x)\, dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}), \tag{1}$$

$$l(x) = \chi_{\Omega}(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}). \tag{2}$$

The subject of our study is the norm of this functional:

$$\|l \mid X^*\|. \tag{3}$$

To the different nodes $(x^{(k)})$ and the coefficients (c_k) there correspond different cubature formulas. It is important to study them, and to minimize the norm $\|l \mid X^*\|$.

Of course, in practical questions the value of the error (l, φ) for each individual function is more important than the norm of $l(x)$. For any continuous function this error tends to zero. There is always a weak convergence. However, it is difficult to estimate the error, and therefore it is useful to apply the formula with the least norm of the error functional in the space X^*.

* Wiss. Z. Hochsch. Architektur Bauwesen Weimar. Jahrgang 12 (1965). Heft 5/6, S. 537–546. (III International Colloquium on Application of Mathematics in Engineering, June 27 – July 4, 1965, Weimar)

The choice of the space X, where we consider a cubature formula, is the matter of the insight of a researcher and his intuition pointing to what properties of the function, e.g., its smoothness, we should pay the greatest attention. This choice also determines the quality of chosen cubature formulas.

However, following the traditions accepted in mathematics from A. M. Lyapunov's times, a problem stated mathematically has to be mathematically solved in strict terms.

The norm of the error functional $l(x)$ characterizes the degree of approximation of the functional $\chi_\Omega(x)$, the characteristic function of the domain Ω, by the functionals of specific form

$$\sum_{k=1}^{N} c_k \delta(x - x^{(k)}) = R_N(x), \tag{4}$$

as N increases. The study of $l(x)$ reduces to the study of different functionals of form (4). To minimize the norm of the error functional, we can change:

a) the coefficients c_k for given nodes $x^{(k)}$ and N;
b) the disposition of the nodes $x^{(k)}$ for fixed N;
c) the number N of the nodes.

These three problems, composing three steps of the search for the best integration formulas, are the particular problems of such general problems of functional analysis as approximations in the functional space. It is difficult to point out traditional approaches to solving this problem because of its enormous complexity, and generally speaking, choice of a solution method is significantly based on the intuition of the researcher.

Many scientists have offered different approaches. The study of this branch of mathematics has often resembled a list of more or less successful prescriptions such as the formulas of Simpson, Gregory, Gauss, Chebyshev, and others. It seems that in our time the state of the problem has begun to change, and the general methods of functional analysis are changing the theory of cubature formulas in front of our eyes.

As we have already mentioned, the main problem in question is the problem about the error functionals with the least norms. However, somewhat later we will also consider certain questions about the rate of convergence to zero of the error of the formula for individual functions. We will show that there is an essential difference in the estimates obtained under the two approaches to the problem.

It turns out that $|(l, \varphi)|$ for an individual function φ is significantly less than $\|l \mid X^*\| \|\varphi \mid X\|$ in a large number of cases, and in particular, in the Hilbert spaces that we study.

Besides the Banach spaces X, it is also convenient to consider approximate integration in certain countably normed spaces, for example, in the spaces of infinitely differentiable functions, which often appear in applications. We will also discuss this question.

Recently, approximate integration in different functional spaces X has been studied in the literature. Many papers are devoted to approximate integration over cubes of periodic functions that have some derivatives integrable with degrees exceeding 1. Also, there are some results in other spaces. However, it is not my task to discuss all such results here, and the major part of my presentation is devoted to the mechanical cubature formulas in the spaces $L_2^{(m)}$ of functions defined on the whole Euclidean n-dimensional space R^n, whose derivatives of order m are square integrable. The norm squares of such functions may be written as

$$\|\varphi \mid L_2^{(m)}\|^2 = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx, \tag{5}$$

with an integral that is invariant under orthogonal transformations of the variable x from R^n. In (5), as commonly accepted today, α is a vector with integer nonnegative entries, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, and $D^\alpha \varphi$ stands for the derivative

$$\frac{\partial^m \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

It is surely assumed that the functional $l(x)$ has to be defined on $L_2^{(m)}$. Hence, for all polynomials of degree $m-1$ the values of $l(x)$ must be equal to zero.

Also, the number m must satisfy the inequality $m > n/2$, for the value of function $\varphi(x)$ at a fixed point to be the linear functional in $L_2^{(m)}$ which is defined on the whole space. In other words, for $m > n/2$ the following embedding holds

$$L_2^{(m)} \subset C, \quad \|\varphi \mid C\| \le K \|\varphi \mid L_2^{(m)}\|. \tag{6}$$

Besides the functions φ with domain R^n, we also consider the periodic functions φ defined in the bounded domain Ω_0 with fixed periods.

Assuming that x is a column vector, and writing the periods of the function $\varphi(x)$ in the form of columns of the square matrix H of periods, we write the periodicity condition for the function $\varphi(x)$ as

$$\varphi(x + H\gamma) = \varphi(x), \quad \forall x \in R^n, \tag{7}$$

with γ an arbitrary column vector with integer entries. In this case the integration domain is chosen as the fundamental domain Ω_0, i.e., in such a way that

$$\sum_{\gamma} \chi_{\Omega_0}(x + H\gamma) = 1. \tag{8}$$

It is often assumed that the volume of the fundamental domain is equal to 1:

$$\det H = |H| = 1. \tag{9}$$

For a domain with the volume h^n, the matrix of periods may be written as hH.

We consider the approximate integration formula over an arbitrary domain Ω as the approximation of the functional $\chi_\Omega(x)$ in the space $L_2^{(m)*}$. As we know from the Calderon theory, an arbitrary function from $L_2^{(m)}(\Omega)$ can be continued to the whole space R^n. Hence, in $L_2^{(m)}(\Omega)$ we can introduce the new norm

$$\left\|\varphi \mid V_2^{(m)}(\Omega)\right\| = \inf_{\substack{\overline{\varphi}(x)=\varphi(x)\\ x\in\Omega}} \left\|\overline{\varphi} \mid L_2^{(m)}(R^n)\right\|. \tag{10}$$

It is easy to see that

$$\left\|l \mid V_2^{(m)*}(\Omega)\right\| = \left\|l \mid L_2^{(m)*}(R^n)\right\|, \tag{11}$$

which simplifies our problem.

The computation of the $L_2^{(m)*}$-norm of a given functional is the simplest problem of variations calculus on the minimum of the quadratic form $\left\|l \mid \ L_2^{(m)*}\right\|^2$. It can be reduced in a classical fashion to the solution of the partial differential equation

$$\Delta^m u(x) = (-1)^m l(x) \tag{12}$$

in $L_2^{(m)}$. Problem (12) turns out to be solvable because of the conditions

$$(l, x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{13}$$

where x^α denotes the product $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Equality (13) expresses the orthogonality of $l(x)$ to all polynomials of degree $m-1$, which we mentioned above.

Under condition (13) the solution of equation (12) may be explicitly expressed through the fundamental solution

$$G_{m,n}(x) = \varkappa_{m,n}|x|^{2m-n} \begin{cases} \ln|x|, & \text{if} \quad n \quad \text{even} \quad \text{and} \quad n \le 2m, \\ 1, & \text{if} \quad n \quad \text{odd} \quad \text{or} \quad n > 2m, \end{cases} \tag{14}$$

of the polyharmonic equation. The equality holds

$$u(x) = (-1)^m G_{m,n}(x) * l(x) \equiv G(x) * l(x). \tag{15}$$

As usual, the symbol $*$ stands for the convolution of generalized (or regular) functions

$$\varphi(x) * \psi(x) = \int \varphi(x-y)\psi(y)\, dy. \tag{16}$$

Using the convolution of two regular functions (or, the convolution of a generalized function with a regular one), their inner product can be written as

$$(\varphi, \psi) = \left[\varphi(x) * \overline{\psi}(-x)\right]\Big|_{x=0}. \tag{17}$$

From (16) we obtain the explicit expression for the norm of $l(x)$:

$$\|l \mid L_2^{(m)*}\|^2 = (l, u) = (l(x) * G(x), l(x)) = \big[l(x) * G(x) * l(-x)\big]\Big|_{x=0}. \tag{18}$$

Let us consider the periodic case with the matrix hH of periods in more detail. Put

$$l_0(x) = 1 - h^n \sum_\gamma \delta(x - hH\gamma), \tag{19}$$

where γ ranges over the set of all possible vectors with integer entries. Such error functional is convenient for integration of functions with periods $hH\Lambda$ over the fundamental domain Ω_0, where Λ is a diagonal integer matrix. It is convenient to map the fundamental domain Ω_0 onto the torus in R^n. In this case the periods of the function $\varphi(x)$ are multiples of periods of the generalized function $l_0(x)$. Moreover, for integration of compactly-supported functions in R^n, it is convenient to use the same error functionals, and hence, the same cubature formula.

In the case of periodic functions, generating on the torus Ω_0 the space $\widetilde{L}_2^{(m)}(\Omega_0)$, the norm of $l_0(x)$ is again computed using the solution of equation (13). This is convenient to do, using the Fourier expansions for generalized functions, studied by L. Schwartz, and also by I. M. Gelfand and G. E. Shilov. The solution of the equation

$$\Delta^m u_0(x) = (-1)^m l_0(x) \tag{20}$$

in the class of functions that are members of $\widetilde{L}_2^{(m)}(\Omega_0)$ and have a zero average over Ω_0, can be written as the Fourier series

$$u_0(x) = -\left(\frac{h}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} e^{i2\pi H^{-1}x\cdot\gamma/h}. \tag{21}$$

From (21) it follows that in the periodic case the norm of the functional $l_0(x)$ may be written as

$$\|l_0 \mid \widetilde{L}_2^{(m)*}\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega_0|}, \tag{22}$$

where $B_{2m}(H^{-1*})$ stands for the expression

$$\begin{aligned} B_{2m}(H^{-1*}) &= (-1)^{m-1} \frac{(2m)!}{(2\pi)^{2m}} \zeta(H^{-1*} \mid 2m) \\ &= (-1)^{m-1}(2m)! \left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}}. \end{aligned} \tag{23}$$

Here $\zeta(H^{-1*} \mid 2m)$ is the known Epstein zeta function of the quadratic form $\psi(\gamma) = |H^{-1*}\gamma|^{2m}$.

The same parameters are involved in the estimate of the values of (l_0, φ) for an arbitrary compactly-supported function $\varphi(x)$ from $L_2^{(m)}(R^n)$. Using the Green identity and the Cauchy–Bunyakovskii–Schwarz inequality, we obtain the estimate

$$\big|(l_0, \varphi)\big| \le h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{S(\varphi)}\big(1 + O(h)\big), \tag{24}$$

where $S(\varphi)$ is a finite volume of a support of the function φ.

The main idea of our theory is to connect the value of the norm of the linear functional with the uniformity of its distribution. In what follows we expose the specifics of this idea and systematically perform it. Let

$$h = \left(\frac{|\Omega|}{N}\right)^{1/n}. \tag{25}$$

Theorem 1. *Let a system of cubes Ω_j in the domain Ω be of total volume $|\Omega'|$, the length of the edge of Ω_j be equal to K, and the part in Ω_j of the functional $l(x)$ be insufficiently defined, i.e., let this part integrate the identity over Ω_j with the significant nonpositive error*

$$\int \chi_{\Omega_j}(x) l(x)\, dx > qK^n, \quad q > 0. \tag{26}$$

Then the functional $l(x)$ has a norm satisfying the condition

$$\big\| l \mid L_2^{(m)*} \big\| \ge \eta K^n \sqrt{|\Omega'|}, \tag{27}$$

where η is a positive constant.

Since the cubes with edges $h/2$, not containing the nodes $x^{(k)}$ of $l(x)$, always occupy more than half of the volume Ω and the functional $l(x)$ is insufficiently defined in each of these cubes, then for no cubature formula in $L_2^{(m)}$ is it possible to obtain the norm of the functional $l(x)$ less than Kh^m:

$$\big\| l \mid L_2^{(m)*} \big\| \ge Kh^m. \tag{28}$$

We omit the proof. It consists of the construction of a certain special function φ from $L_2^{(m)}$, for which the value (l, φ) is greater than $\eta K h^m \sqrt{|\Omega'|} \big\| \varphi \mid L_2^{(m)} \big\|$.

By definition, the functional $l(x)$ is *completely equidistributed over Ω*, if it may be written as

$$l(x) = \sum_{hH\gamma \in \Omega} l_\gamma \left(\frac{x}{h} - H\gamma\right), \tag{29}$$

where the support of $l_\gamma(y)$ lies in the ball of radius L,

$$\operatorname{supp} l_\gamma(y) \subset \{y : |y| \le L\}, \tag{30}$$

$l_\gamma(y)$ is orthogonal to all polynomials of degree $m-1$,

$$(l_\gamma(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{31}$$

and the norm of $l_\gamma(y)$ in C^* is bounded by a constant A, the same for all γ,

$$\|l_\gamma(y) \mid C^*\| \leq A. \tag{32}$$

Theorem 1 solves the question about the order of the norm of the error, while the following theorem establishes the attainability of this order.

Theorem 2. *The norm of the completely equidistributed functional $l(x)$ satisfies the inequality*

$$\left\|l(x) \mid L_2^{(m)*}\right\| \leq Kh^m. \tag{33}$$

Proof. The proof is based on the estimate of the quadratic form

$$\left\|l \mid L_2^{(m)*}\right\|^2 = \sum_{hH\gamma, hH\gamma' \in \Omega} \left(l_\gamma\left(\frac{x}{h} - H\gamma\right), l_{\gamma'}\left(\frac{x}{h} - H\gamma'\right) * G(x)\right). \tag{34}$$

It turns out that under the conditions

$$(l_\gamma(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \leq s_1, \quad (l_{\gamma'}(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \leq s_2, \tag{35}$$

where

$$s_1 + s_2 > 2m - n, \tag{36}$$

the following estimate holds:

$$\left|G(x) * l_\gamma\left(\frac{x}{h}\right) * l_{\gamma'}\left(\frac{x}{h}\right)\right| \leq K \frac{A^2 h^{2n+s_1+s_2}}{(h^2 + |x|^2)^{-m+(n+s_1+s_2)/2}}, \tag{37}$$

where the constant K does not depend on h, l_γ, $l_{\gamma'}$. Formula (37) means that the functionals $l_\gamma\left(x/h - H\gamma\right)$ and $l_{\gamma'}\left(x/h - H\gamma'\right)$ become "more orthogonal" in the sense of the inner product $\left(l_\gamma\left(x/h - H\gamma\right), l_{\gamma'}\left(x/h - H\gamma'\right) * G(x)\right)$ when their supports move away from each other. Using (37) and applying the integral majorant we estimate the right side of (34) and prove (33). □

Theorems 1 and 2 establish the order of convergence of cubature formulas. In the next part of the presentation we establish the principal term in the expansion of the norm of the error functional with the given lattice of nodes $hH\gamma$. Also, we consider cubature formulas where this optimal value of the norm is attained. The corresponding conclusions are based on several theorems.

Theorem 3. *Let $l_*(y)$ satisfy conditions* (30), (32), *and*

$$(l_*(y), y^\alpha) = 0 \quad \textit{for} \quad |\alpha| < 2m + 2. \tag{38}$$

Also, let

$$\sum_\gamma l_* \left(\frac{x}{h} - H\gamma\right) = l_0(x) = 1 - h^n \sum_\gamma \delta(x - hH\gamma). \tag{39}$$

Then the solution of equation (20) *can be written as*

$$u_0(x) = \sum_\gamma \Big(l_*(\frac{y}{h}), G(x - hH\gamma - y)\Big) + c. \tag{40}$$

Proof. For example, the functional $l_*(y)$ with the conditions of Theorem 3 may be constructed by integration of the interpolation formula.

To prove (40), it suffices to establish the estimate

$$\Big| \Big(l_*(\frac{y}{h}), G(x-y)\Big) \Big| = \Big|G(x) * l_*(\frac{x}{h})\Big| \le K|x|^{-n-1}, \tag{41}$$

from where the convergence of the series on the right side of (40) follows, and to use the uniqueness of solution (20) to within an additive constant. □

The functional of the form

$$k(y) = \sum_{|\gamma|<L} c_\gamma \delta(y - H\gamma) \tag{42}$$

with bounded coefficients and a bounded support, consisting of points of the lattice, is called the *point functional of order* s provided that

$$(k(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < s. \tag{43}$$

Theorem 4 (Summation by parts). *Let Ω be a domain with a smooth boundary and let $k(y)$ be a point functional of order s. Then the sum*

$$m(x) = \sum_{hH\gamma\in\Omega} k\left(\frac{x}{h} - H\gamma\right) \tag{44}$$

may be written as

$$m(x) = \sum_{hH\gamma\in B_2} k_\gamma \left(\frac{x}{h} - H\gamma\right), \tag{45}$$

where B_2 is the set of points that lie at a distance less than Lh from the boundary Γ of Ω, and each functional $k_\gamma(y)$ is the point functional of order $s-1$.

Proof. The proof is based on an easily established expansion of an arbitrary point functional of order s in a sum of differences in each variable of certain functionals of order $s-1$:

$$k(y) = \sum_{j=1}^n \widehat{\Delta}_j k_j(y). \tag{46}$$

□

By definition, the error functional $l(x)$ is an *error functional with regular boundary layer, width* L*, order* m*, and estimate* A, if it may be written down as

$$l(x) = \sum_{hH\gamma \in B_1} l_* \left(\frac{x}{h} - H\gamma \right) + \sum_{hH\gamma \in B_2} l_\gamma \left(\frac{x}{h} - H\gamma \right), \tag{47}$$

in which $B_1 = \Omega \setminus B_2$, all l_γ and l_* satisfy (30)–(32), and l_* satisfies (38).

Let

$$(l_1(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < m + 1. \tag{48}$$

Whence and from Theorem 4 it follows that

$$\sum_{hH\gamma \in B_1} l_1 \left(\frac{x}{h} - H\gamma \right) - \sum_{hH\gamma \in B_1} l_* \left(\frac{x}{h} - H\gamma \right) = \sum_{hH\gamma \in B_2} l_\gamma \left(\frac{x}{h} - H\gamma \right), \tag{49}$$

where l_γ has order m. Hence, in (47) we could use just (48) instead of (38) for l_*.

From (47) and (19) it follows that $l(x) = l_0(x) + l_2(x)$, where

$$l_2(x) = \sum_{hH\gamma \in B_2} l_\gamma \left(\frac{x}{h} - H\gamma \right) - \sum_{hH\gamma \notin \Omega} l_* \left(\frac{x}{h} - H\gamma \right). \tag{50}$$

All coefficients c_γ of the error functional with regular boundary layer are equal to h^n at the points that lie at a distance greater than $2Lh$ from the boundary Γ of Ω, since in this condition $c_\gamma = \sum_{\gamma'} c^*_{\gamma'}$, where $c^*_{\gamma'}$ are the coefficients of $l_*(y)$.

Theorem 5. *The extremal function* $u(x)$ *of the functional* $l(x)$ *with regular boundary layer may be written as*

$$u(x) = G(x) * l(x) + P_{m-1}(x),$$

where $P_{m-1}(x)$ *is a polynomial of degree* $m - 1$ *and*

$$\begin{aligned} G(x) * l(x) &= \sum_{hH\gamma \in B_1} G(x) * l_* \left(\frac{x}{h} - H\gamma \right) \\ &+ \sum_{hH\gamma \in B_2} G(x) * l_\gamma \left(\frac{x}{h} - H\gamma \right) = u_0(x) + \sum_{hH\gamma \in B_3} l'_\gamma \left(\frac{x}{h} - H\gamma \right) * G(x) \\ &+ \sum_{hH\gamma \in B_4} l_* \left(\frac{x}{h} - H\gamma \right) * G(x) = u_0(x) - w(x). \end{aligned} \tag{51}$$

The set B_3 *consists of points that lie at a distant less than* Lh *from the boundary* Γ *of* Ω*, and* $B_4 = (R^n \setminus \Omega) \setminus B_3$*.*

Theorem 6. *The norm of the error functional with regular boundary layer of order m is expressed as*

$$\left\| l \mid L_2^{(m)*} \right\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega|} + O(h^{m+1}). \qquad (52)$$

Proof. By (51) we have:

$$\left\| l \mid L_2^{(m)*} \right\|^2 = (l, u) = (l, u_0 - w) = (l, u_0) - (l, w). \qquad (53)$$

The direct computation of (l, u_0) gives

$$(l, u_0) = h^{2m} \frac{|B_{2m}(H^{-1*})|}{(2m)!} |\Omega_0| (1 + O(h)). \qquad (54)$$

The second term (l, w) may be written as

$$(l, w) = \sum_{hH\gamma \in B_1 \cup B_2} \sum_{hH\gamma' \in B_3 \cup B_4} \left[l_\gamma \left(\frac{x}{h} - H\gamma \right) * G(x) * l'_{\gamma'} \left(\frac{x}{h} - H\gamma' \right) \right], \qquad (55)$$

where $hH\gamma$ ranges over the nodes lying in Ω, i.e., the set of nodes from $B_1 \cup B_2$, and $hH\gamma'$ ranges over the set of nodes from $B_3 \cup B_4$.

Using (37) and applying the integral majorant, we have the final result, i.e., the estimate

$$|(l, w)| \leq K h^{2m+1}, \qquad (56)$$

where K depends only on L and A.

Theorem 6 immediately follows from (54) and (56). □

In Theorem 6 we establish the principal term of the norm of the error functional with regular boundary layer. As we see in the proof, we cannot change the value of this principal term by increasing the order of $l_*(y)$ from m up to any other number. In the following theorems we establish directly that the norm of the optimal error functional with the given lattice of nodes $hH\gamma$ has the same principal term.

Theorem (Babuška). *For the given lattice of nodes $hH\gamma$, coefficients $c^{(0)}[\gamma]$ of optimal error functional*

$$l^{(0)}(x) = \chi_\Omega(x) - \sum_{hH\gamma \in \Omega} c^{(0)}[\gamma] \delta(x - hH\gamma) \qquad (57)$$

are such that the solution $u(x)$ of (12) vanishes at all nodes of the formula:

$$u(hH\gamma) = 0 \quad \textit{for} \quad hH\gamma \in \Omega. \qquad (58)$$

Proof. Equalities (58) are equivalent to the fact that the convolution

$$G(x) * l^{(0)}(x) \tag{59}$$

coincides with a certain polynomial of degree $m-1$ at all nodes $hH\gamma$ from Ω. If (58) would not hold, then on the set of all $c[\gamma]$, subject to the conditions

$$\sum_{hH\gamma\in\Omega} c[\gamma]\,(hH\gamma)^{\alpha} = \int\limits_{\Omega} x^{\alpha}\,dx \quad \text{for} \quad |\alpha| < m, \tag{60}$$

there would exist directions such that the directional derivative of the polynomial

$$\psi(c) = [l(x) * G(x) * l(-x)]\Big|_{x=0} \tag{61}$$

of the second degree with respect to the variables $c[\gamma]$ would be nonzero, which is impossible at the minimum point. □

Theorem 7. *The difference of the norm square of an arbitrary error functional $l(x)$ in $L_2^{(m)*}$ and the norm square of the optimal error functional $l^{(0)}(x)$ is expressed as*

$$\begin{aligned} &\left\|l \mid L_2^{(m)*}\right\|^2 - \left\|l^{(0)} \mid L_2^{(m)*}\right\|^2 \\ &= \sum_{\substack{hH\beta\in\Omega \\ hH\beta'\in\Omega}} G\big(hH(\beta-\beta')\big)\big(c[\beta]-c^{(0)}[\beta]\big)\big(c[\beta']-c^{(0)}[\beta']\big). \end{aligned} \tag{62}$$

In other words, this difference is a quadratic form with respect to the differences of coefficients of the cubature formulas under consideration, and the matrix of this quadratic form has the elements $G\,(hH(\beta-\beta'))$.

Theorem 7 means that the difference between $\psi(c)$ and its minimum is the second-order value expressed by the quadratic form $\Xi(c[\beta]-c^{(0)}[\beta])$ of increments of the independent variables.

In view of Theorem 7, the deviation of the norm of the given error functional from the least possible one reduces to the study of the quadratic form of a large number of independent variables.

We use (62) to show that the norm of the optimal error functional differs from the norm of an error functional with regular boundary layer by a value of higher order of smallness.

Let us assume that the coefficients $c[\beta]$ and $c^{(0)}[\beta]$ are given not only for $hH\beta \in \Omega$, but also for all β, moreover, let

$$c[\beta] = c^{(0)}[\beta] = 0 \quad \text{for} \quad hH\beta \notin \Omega. \tag{63}$$

Then, as it is easy to show, the form under consideration is reduced to the infinite convolution

$$\left\|l \mid L_2^{(m)*}\right\|^2 - \left\|l^{(0)} \mid L_2^{(m)*}\right\|^2 = \Xi\big(c[\beta]-c^{(0)}[\beta]\big)$$

$$= \left[(c[\beta] - c^{(0)}[\beta]) * G(hH\beta) * (c[-\beta] - c^{(0)}[-\beta]) \right] \Big|_{\beta=0} \tag{64}$$

with respect to the discrete argument β. The quadratic form $\Xi(c[\beta] - c^{(0)}[\beta])$ is the generalization of the form studied above,

$$\int \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u(x) D^\alpha v(x)\, dx = \left[l_1\left(\frac{x}{h}\right) * G(x) * l_2\left(-\frac{x}{h}\right) \right] \Big|_{x=0}, \tag{65}$$

and it is the discrete analogue of (65).

Theorem 4 serves as the beginning of the theory of functions defined on a lattice. Let us continue this theory. For our goals it is necessary to develop the theory of the form $\Lambda(U, V)$, in the same way by which we develop the theory of the corresponding integral form (65) using the polyharmonic equation.

First we consider the discrete potential

$$U^*[\beta] = \sum_{\beta'} G[hH(\beta - \beta')]c[\beta'] = G[hH\beta] * c[\beta], \tag{66}$$

which is similar to the polyharmonic potential. It is useful to note that, by Babuška's theorem, for the given functional $l(x)$, values of the potential

$$U[\beta] = G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) \tag{67}$$

are known for $hH\beta \in \Omega$.

Indeed, it is easy to see that for $hH\beta \in \Omega$,

$$G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) = \left[G(x) * \left(l(x) - l^{(0)}(x) \right) \right] \Big|_{x=hH\beta}. \tag{68}$$

However, $G(x) * l^{(0)}(x) \Big|_{x=hH\beta} = P_{m-1}(hH\beta)$, and therefore

$$G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) = u(hH\beta) - P_{m-1}(hH\beta) \quad \text{for} \quad hH\beta \in \Omega. \tag{69}$$

Theorem 8. *The operator of convolution with $G[hH\beta]$, i.e., the discrete potential, has a difference inverse operator $L_h[\beta]$ such that*

$$L_h[\beta] * G[hH\beta] = \delta[\beta] = \begin{cases} 1 & \textit{for} \quad \beta = 0, \\ 0 & \textit{for} \quad \beta \neq 0; \end{cases} \tag{70}$$

$$L_h[\beta] * [\beta]^\alpha = 0 \quad \textit{for} \quad |\alpha| < 2m - 1; \tag{71}$$

$$|L_h[\beta]| \le e^{-\eta|\beta|/h}, \tag{72}$$

where η is a positive constant.

Thus, the operation $L_h[\beta]*$ is an analogue of the usual discrete difference polyharmonic operator $\widehat{\Delta}^m$ with the only difference that the function $L_h[\beta]$ is not compactly-supported, but rather decreases exponentially at infinity.

The proof of Theorem 8 is based on an application of the Fourier transform, and in the case of functions of a discrete argument it reduces to the search for such periodic functions of the variable x with periods $2\pi h^{-1}H^{-1}$, for which $G[hH\beta]$ and $L_h[\beta]$ are the Fourier coefficients.

Let $U[\beta]$ and $V[\beta]$ be compactly-supported functions. Let us compose the bilinear form

$$\Lambda(U,V) = U[\beta] * L_h[\beta] * V[-\beta]\Big|_{\beta=0}. \tag{73}$$

Theorem 9. *The form $\Lambda(U,V)$ can be extended by continuity on the space $l_2^{(m)}$ of the functions $U[\beta]$ and $V[\beta]$ of the discrete argument with square summable differences of order m. The corresponding quadratic form $\Lambda(U,U)$ satisfies the inequalities*

$$0 < M_1 \| U \mid l_2^{(m)} \| \le \Lambda(U,U) \le M_2 \| U \mid l_2^{(m)} \| < \infty. \tag{74}$$

The proof of this theorem can be conducted by study of the Fourier transform in detail. We can also prove it directly using the following expansion of the operator $L_1[\beta]$:

$$L[\beta] \equiv L_1[\beta] = (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} L^\alpha[\beta] * L^\alpha[-\beta]. \tag{75}$$

Equality (75) is the generalization of the known formula

$$\Delta^m = (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha(x) * D^\alpha(-x). \tag{76}$$

From (75) it follows that the form $\Lambda(U,U)$ expands in the following sum of squares:

$$\Lambda(U,U) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \big| L^\alpha(U,U) \big|^2. \tag{77}$$

First the expansion (77) is established for all compactly-supported functions, and then for all functions from $l_2^{(m)}$. The operator $L^\alpha[\beta]*$ turns out to be an analogue of the finite difference Δ^α of order α.

The proof of (77) is based on the following lemma, first proved for differentiable functions from $L_2^{(m)}$, and then for difference functions from $l_2^{(m)}$.

Lemma (on the density of compactly-supported functions). *Each function φ from $L_2^{(m)}$ can be expressed as the limit of a sequence of compactly-supported functions:*

$$\varphi(x) + P_{m-1}(x) = \lim_{\substack{\eta\to\infty \\ L_2^{(m)}}} \Big[\varphi(x) + P_{m-1}(x)\Big] \xi\left(\frac{\ln |x|}{\ln \eta}\right), \tag{78}$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$ and $\xi(\lambda)$ is a truncator, i.e., a function with the properties

$$\xi(\lambda) = \begin{cases} 0 & \text{for} \quad \lambda > 1, \\ 1 & \text{for} \quad \lambda < 1/2, \end{cases} \tag{79}$$

$|\xi(\lambda)| \le 1$, and moreover $\xi(\lambda)$ has continuous derivatives of all orders. A property like (78) holds for the members of $l_2^{(m)}$ as well.

Using the form Λ, let us introduce the Hilbert space $\mathfrak{S}$ with the inner product

$$\{U, V\} = U[\beta] * L_h[\beta] * V[-\beta]\Big|_{\beta=0}, \tag{80}$$

defined for every pair of compactly-supported functions. Applying the lemma on the density of compactly-supported functions we establish the formula

$$\{U, V\} = \lim_{\eta\to\infty} \{U_\eta, V_\eta\} = (U[\beta], L_h * V[\beta]) \tag{81}$$

in the case when at least one of $U[\beta]$ and $V[\beta]$ is compactly-supported.

Theorem 10. *Let $\varrho[\beta]$ relate with $V[\beta]$ by*

$$V[\beta] = G[hH\beta] * \varrho[\beta], \tag{82}$$

and $\varrho[\beta] * [\beta]^\alpha = 0$ for $|\alpha| < m$. Then[1]

$$\varXi(\varrho[\beta]) = \Lambda(V[\beta], V[\beta]). \tag{83}$$

Equality (83) is the generalization of the usual equality

$$xFx^* = yF^{-1}y^*, \quad y = xF, \tag{84}$$

from the theory of quadratic forms on a finite number of variables. Equality (84) is proved by accurate testing of the validity of the associative law in the formula

$$yF^{-1}FF^{-1}y^* = yF^{-1}y^* \tag{85}$$

for symmetric finite matrices.

The continuous analogue of (83) may be written as

$$\int\!\!\int \varrho(x)\varrho(y)G(x-y)\,dxdy = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!}\Big|D^\alpha u(x)\Big|^2\,dx, \tag{86}$$

[1] Here $\varXi(\varrho[\beta]) = \varrho[\beta] * G[hH\beta] * \varrho[-\beta]\Big|_{\beta=0}$. – *Ed.*

where

$$u(x) = G(x) * \varrho(x). \tag{87}$$

Equality (86) is valid for any compactly-supported function $\varrho(x)$ such that

$$(\varrho(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{88}$$

and we used this when deriving formula (18) for the norm of the error functional.

Theorem 11. *Let $U^*[\beta]$ be a discrete potential with the compactly-supported density $\varrho[\beta]$,*

$$U^*[\beta] = G[hH\beta] * \varrho[\beta], \tag{89}$$

$$\operatorname{supp} \varrho[\beta] = \Omega. \tag{90}$$

The $\Lambda(U^*, U^*)$ is less than $\Lambda(U, U)$ for all those functions $U[\beta]$, which take the same values as $U^*[\beta]$ at the points of Ω.

Proof. Let $W[\beta]$ vanish at the points of Ω and be a member of $l_2^{(m)}$. Then this function is orthogonal to $U^*[\beta]$ in the inner product of $\mathfrak{S}$. Indeed, for $W[\beta] \in l_2^{(m)}$ we have

$$\Lambda(U^*, W) = (U^*[\beta] * L_h[\beta], W[\beta]) = \sum_\beta (U^*[\beta] * L_h[\beta]) \cdot W[\beta].$$

In the last sum all terms are equal to zero, since the function $W[\beta]$ vanishes at the points of Ω, and the convolution $L_h[\beta] * U^*[\beta]$ is equal to zero at the others. Hence, $\Lambda(U^*, W) = 0$ and

$$\Lambda(U^* + W, U^* + W) = \Lambda(U^*, U^*) + \Lambda(U^*, W) + \Lambda(W, U^*) + \Lambda(W, W). \tag{91}$$

From this it follows that

$$\Lambda(U^* + W, U^* + W) \geq \Lambda(U^*, U^*). \tag{92}$$

However, $U^* + W$ is an arbitrary function from $l_2^{(m)}$ coinciding with U^* at the points of Ω. □

In order to complete the estimate of the norm of the optimal error functional for the given lattice of nodes, we note that by Theorem 11 and (69) the following inequality holds:

$$\Lambda(U^*, U^*) \leq \Lambda(U[\beta], U[\beta]), \tag{93}$$

where $U[\beta] = u(hH\beta)$, and $u(x)$ is the extremal function of a cubature formula with regular boundary layer of order m. This very important inequality allows us to prove that

$$\left\| l \mid L_2^{(m)*} \right\|^2 - \left\| l^{(0)} \mid L_2^{(m)*} \right\|^2 = O(h^{2m+1}), \tag{94}$$

where $l^{(0)}(x)$ is the optimal error functional, and $l(x)$ is an arbitrary error functional with regular boundary layer of order m.

In many respects the derivation of estimate (94) is similar to the one carried out above when estimating the norm $\left\| l \mid L_2^{(m)*} \right\|$ of an error functional with regular boundary layer. From (52) it follows that it suffices to establish (94) for the functional with regular boundary layer of order $2m+2$, which we are going to do.

Studying the form $\Lambda(U,U)$, let us make use of (51):

$$\Lambda(U,U) = (L_h[\beta] * U[\beta], U[\beta]) = \sum_\beta \Big(\sum_{\beta'} L_h[\beta - \beta'] U[\beta'] \Big) U[\beta]$$

$$= \sum_\beta \Big(\sum_{\beta'} L_h[\beta - \beta'] (u_0(hH\beta') - w(hH\beta')) \Big) U[\beta]. \tag{95}$$

Since the function $u_0(hH\beta')$ is constant at the points $hH\beta'$, and the operator $L_h[\beta - \beta']$ is orthogonal to the constant, we obtain

$$\Lambda(U,U) = - \sum_\beta \Big(\sum_{\beta'} L_h[\beta - \beta'] w(hH\beta') \Big) U[\beta]$$

$$= -\Lambda(U,W) = -\{U,W\}. \tag{96}$$

The inner product $\{U,W\}$ can be estimated in the same way as the corresponding inner product in $L_2^{(m)}$. We obtain

$$\{U,W\} = \sum_\gamma \sum_{\gamma'} L_h[\beta] * U_\gamma[\beta] * W_{\gamma'}[-\beta] \Big|_{\beta=0}, \tag{97}$$

where

$$U_\gamma[\beta] = G(x) * l_\gamma(\frac{x}{h}) \Big|_{x=hH\beta}, \quad W_{\gamma'}[\beta] = G(x) * l_{\gamma'}(\frac{x}{h}) \Big|_{x=hH\beta}, \tag{98}$$

and

$$\begin{aligned} (l_\gamma\ (x), x^\alpha) &= 0 \quad \text{for} \quad |\alpha| < 2m+1, \\ (l_{\gamma'}(x), x^\alpha) &= 0 \quad \text{for} \quad |\alpha| \le 2m+1. \end{aligned} \tag{99}$$

The convolution $L_h[\beta] * U_\gamma[\beta]$ can be written as

$$L_h[\beta] * U_\gamma[\beta] = \sum_{\beta'} L_h[\beta - \beta'] \Big(l_\gamma(\frac{y}{h}), G(hH\beta' - y) \Big)$$

$$= \Big(l_\gamma(\frac{y}{h}), \sum_{\beta'} L_h[\beta - \beta'] G(hH\beta' - y) \Big).$$

It turns out that the function $\psi(y) = \sum_{\beta'} L_h[\beta - \beta']G(hH\beta' - y)$ is regular in y and decreases exponentially at infinity. Hence,

$$|L_h[\beta] * U_\gamma[\beta]| \leq K h^n e^{-\eta|\beta|/h}. \tag{100}$$

By the assumption that the order of the boundary layer is equal to $2m+1$, we obtain

$$|W_{\gamma'}(x)| \leq K \begin{cases} h^{2m+n+1}|x|^{-n-1} & \text{for} \quad |x| \geq L'h, \\ h^{2m} \quad \text{for} \quad |x| \leq Lh. \end{cases} \tag{101}$$

Comparing (100) and (101), and performing some natural computations, we obtain an estimate similar to (37):

$$\left|U_\gamma[\beta] * L_h[\beta] * W_{\gamma'}[-\beta]\right| \leq K \begin{cases} h^{2m+n} & \text{for} \quad |hH\beta| \geq L'h, \\ h^{2m+2n+1} & \text{for} \quad |hH\beta| \leq Lh. \end{cases} \tag{102}$$

In the same way as above, we obtain

$$\left|U[\beta] * L_h[\beta] * W[-\beta]\right|\Big|_{\beta=0} \leq K h^{2m+1}. \tag{103}$$

The next theorem follows from (103).

Theorem 12. *For the given lattice of nodes the norm of the optimal error functional is expressed as*

$$\left\|l^{(0)} \mid L_2^{(m)*}\right\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega|} + O(h^{m+1}). \tag{104}$$

In particular, Theorem 12 tells us that the formulas with regular boundary layer are asymptotically optimal.

For each particular computation a practical error of the cubature formula can happen to be far away from the estimate that follows from (104). The fact is that for each given value of the mesh-size h of the lattice, there is an extremal function $u_0(x/h)$ such that

$$(l, u_0) = \|l \mid X^*\| \cdot \|u_0 \mid X\|. \tag{105}$$

In this case, the sequence $\dfrac{1}{\|u_0(x/h) \mid X\|} u_0(x/h)$ is noncompact, and moreover it has no condensation point $w(x)$ in $L_2^{(m)}$. On the contrary, we establish that for every φ from $L_2^{(m)}$ the estimate $|(l, \varphi)|$ as $h \to 0$ is always substantially better than the one that follows from the inequality $|(l, \varphi)| \leq \|l \mid L_2^{(m)*}\| \cdot \|\varphi \mid L_2^{(m)}\|$.

For function φ from $L_2^{(m)}$ we have

$$(l, \varphi) = h^m \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u_0\Big(\frac{x}{h}\Big) D^\alpha \varphi(x)\, dx. \tag{106}$$

The right side of this formula can be conveniently estimated by considering the integrals over each cell Ω_γ, corresponding to the periods $hH\gamma$ of the function u_0. Using the Cauchy–Bunyakovskii–Schwarz inequality, we obtain

$$|(l, \varphi)| \le h^m \sum_{hH\gamma \in \Omega} \Big(\int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha u_0\Big(\frac{x}{h}\Big)\Big|^2 dx \Big)^{1/2}$$

$$\times \Big(\int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 dx \Big)^{1/2}. \tag{107}$$

One can show that the sum on the right side of (107) has a limit as $h \to 0$. Passing to the limit, we obtain the estimate

$$|(l, \varphi)| \le h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}}\, \|\varphi \mid L_1^{(m)}(\Omega)\| \Big(1 + \eta(h)\Big), \tag{108}$$

where $\eta(h) \to 0$ as $h \to 0$, and $\|\varphi \mid L_1^{(m)}(\Omega)\|$ denotes the integral

$$\|\varphi \mid L_1^{(m)}(\Omega)\| = \int_\Omega \Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 \Big)^{1/2} dx. \tag{109}$$

Estimate (108) is stronger than when it follows from (104). Indeed, from the Cauchy–Bunyakovskii–Schwarz inequality we obtain

$$\|\varphi \mid L_1^{(m)}(\Omega)\| \le \Big\{ \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 dx \Big\}^{1/2} \Big\{ \int_\Omega dx \Big\}^{1/2}$$

$$= \sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}(\Omega)\|. \tag{110}$$

The equality in (110) can only occur in such a case if

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 = 1. \tag{111}$$

Since the extremal function of $l(x)$ does not satisfy (111), we see that the maximum of $|(l, \varphi)|$ taken over the solutions of (111) is less than the maximum of $|(l, \varphi)|$ taken over $L_2^{(m)}(\Omega)$. Hence, estimate (108) is always more advantageous than the estimate in the norm.

It is possible that estimate (108) is also nonoptimal. It would be quite interesting from the theoretical point of view to find the exact estimate of the functional $K(\varphi) = \lim\limits_{h \to 0} h^{-m} |(l, \varphi)|$ in the whole $L_2^{(m)}(\Omega)$. As we have seen,

$$K(\varphi) \leq \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}}\, \|\varphi \mid L_1^{(m)}(\Omega)\|. \tag{112}$$

Now we study the question about the rate of convergence to zero of the error of cubature formulas in classes of infinitely differentiable functions. Let us consider the set of such functions, which are periodic with the matrix H of periods and define a space with a sequence of norms:

$$\|\varphi \mid L_2^{(m)}(\Omega_0)\|, \quad m = \left[\frac{n}{2}\right] + 1, \left[\frac{n}{2}\right] + 2, \ldots . \tag{113}$$

As we observed above, on this space the simplest error functional

$$l_0(x) = 1 - h^n \sum_{\gamma} \delta(x - hH\gamma) = 1 - \Phi_0(h^{-1}H^{-1}x),$$

where

$$\Phi_0(h^{-1}H^{-1}x) = h^n \sum_{\gamma} \delta(x - hH\gamma), \tag{114}$$

has an infinite order of exactness, and its norm in any space $L_2^{(m)*}(\Omega_0)$ can be explicitly estimated. Thus, for each fixed h and for all m the following inequalities hold:

$$|(l_0, \varphi)| \leq h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}(\Omega_0)\|. \tag{115}$$

Estimating the order of the right side in (115) for different classes of functions, and finding every time the best value of m, we obtain the estimate $|(l_0, \varphi)|$ in terms of h.

Let us consider the Gevrey classes of functions, where the growth of the mth-order derivatives obeys the following conditions

$$\left\| \frac{D^\alpha \varphi}{\alpha!} \right\| \leq K e^{A|m|^\beta} \quad \text{for} \quad |\alpha| = m, \tag{116}$$

where K, A, and β are independent of m, and $m = 1, 2, \ldots$ There are particular cases of Gevrey's classes such as quasi-analytic functions, functions regular in a certain strip of the complex plane surrounding the set of real values of x, and, finally, entire functions of a particular order and type. As we have seen, these classes are characterized by the pair of numbers A and β. For the Gevrey classes we have the inequality

$$\|\varphi \mid L_2^{(m)}(\Omega_0)\| \leq K e^{A|m|^\beta}. \tag{117}$$

Further, from the explicit expression for $\dfrac{|B_{2m}(H^{-1*})|}{(2m)!}$ we obtain

$$\frac{|B_{2m}(H^{-1*})|}{(2m)!} = \left(\frac{1}{2\pi}\right)^{2m} \frac{1}{r_{\min}^{2m}} \Big(1 + \eta(m)\Big), \tag{118}$$

where $r_{\min}$ denotes the shortest distance between the nodes of the lattice with the matrix H^{-1*} of periods. Formulas (117) and (118) lead to the following corollaries:

a) for sufficiently large m the best optimal lattice with the matrix H^{-1*} is the one whose nodes coincide with the centers of balls from the densest packing. The theory of such lattices has been a subject of numerous studies in the geometry of numbers;
b) from (115) it follows that the best estimate has the form

$$|(l_0, \varphi)| \leq K e^{-Bh^{-\sigma}}, \tag{119}$$

where B and σ are expressed through A and β by formulas, which we do not discuss now. For example, we obtain the estimate of form (119) if $\varphi(x)$ is an analytic function with a given radius of convergence at each point x of the integration domain.

From (119) it follows that beginning with a certain h the corresponding cubature formulas are completely exact for each trigonometric polynomial of a given degree.

In conclusion, let us discuss certain practical methods of the construction of formulas with regular boundary layer.

If Ω is a polyhedron with the rational faces, then we can construct formulas with regular boundary layer by using the Fourier transform of the error functional $l(x)$. The following theorem holds.

Theorem 13. *For the error functional $l(x)$ in a bounded domain Ω to be orthogonal to all polynomials of degree $m-1$, it is necessary and sufficient for its Fourier transform $\widetilde{l}(p)$ to have a zero of multiplicity m at the origin.*

By using Theorem 13, the boundary layer can be found with the aid of the method of undetermined coefficients. For certain cases these layers were calculated in the Novosibirsk Computer Center of the Siberian Division of the USSR Academy of Sciences.

In practice there are cases such that for a given function it is convenient to choose a grid with different mesh-sizes in different parts of its domain. For example, this occurs, when the integrand is changing rapidly in some subdomains, and slowly in others. In order to keep a good order of accuracy, it suffices to trace the fact that in different parts the coefficients of the formulas would be equal to h_1^n and h_2^n, respectively, and to introduce the boundary layer of the required order on the boundary between the subdomains. The coefficients of the cubature formulas in this layer are computed by the same method of the Fourier transform, provided that the boundary is formed by parts of the rational planes.

Conclusion

The problems of the theory of cubature formulas, when we study their error functionals in the corresponding functional spaces, can be treated as problems of functional analysis. In particular, applying certain Hilbert metrics and solving the appearing problems by the methods of variations calculus, we can obtain exact estimates of the norms of error functionals, find an optimal lattice, and find asymptotically optimal coefficients. The formulas with such nodes and coefficients are convenient for practical applications. These formulas are called formulas with regular boundary layer, and they are generalizations of the Gregory quadrature formulas for one independent variable. Apparently, in a number of cases they are also sufficiently convenient for practical applications.

16. Convergence of Approximate Integration Formulas for Functions from* $L_2^{(m)}$

S. L. Sobolev

In the papers by the author [1,2] it was established that an extremal function $u(x)$ for which the error functional attains its maximum value on the unit sphere in $L_2^{(m)}$ is a solution of the polyharmonic equation with a right side:

$$\Delta^m u = (-1)^m l(x). \tag{1}$$

Let the nodes of a cubature formula be of the form

$$x^{(\gamma)} = hH\gamma, \tag{2}$$

where $x^{(\gamma)}$ is a column vector, γ is a column vector with integer entries, H is a matrix with the unit determinant, and h is a small positive parameter. In what follows we consider periodic functions of n variables defined on a torus Ω. Let the periods of the torus Ω be multiples of the columns of the matrix hH, i.e., periods of the lattice.

Theorem 1. *All coefficients C_γ of the error*

$$l(x) = 1 - \sum_\gamma h^n C_\gamma \delta(x - x^{(\gamma)}) \tag{3}$$

with minimal $\widetilde{L}_2^{(m)*}$-norm are given by

$$C_\gamma = 1. \tag{4}$$

Proof. The proof and the formulation of Theorem 1 are known, although for other spaces. It is also known that the $\widetilde{L}_2^{(m)*}$-norm of $l(x)$ is a strictly convex function of C_γ. It means that for $l_1 \neq l_2$ and $\|l_1 \mid \widetilde{L}_2^{(m)*}\| = \|l_2 \mid \widetilde{L}_2^{(m)*}\| = C$ the inequality holds

$$\left\| \frac{l_1(x) + l_2(x)}{2} \mid \widetilde{L}_2^{(m)*} \right\| < C. \tag{5}$$

* Dokl. Akad. Nauk SSSR, **162**, 1259–1261 (1965)

If the coefficients of some functional of form (3) are not all equal, then $l(x)$ and $l(x - hH\gamma)$ do not coincide. At the same time their half-sum is an error functional with the same nodes (2), but smaller in norm. Therefore the $\widetilde{L}_2^{(m)*}$-norm of $l(x)$ cannot be minimal. The proof of Theorem 1 is complete. □

By the Fourier method, we obtain an explicit expression for the extremal function of the error

$$l_0(x) = 1 - \sum_{\gamma} h^n \delta(x - hH\gamma). \tag{6}$$

This function is

$$u(x) = -\left(\frac{h}{2\pi}\right)^{2m} \sum_{\beta \neq 0} \frac{e^{-i2\pi\beta h^{-1} H^{-1} x}}{[A(\beta)]^m}. \tag{7}$$

Here $A(\beta) = (A\beta, \beta)$ is a quadratic form with the matrix

$$A = H^{-1} H^{-1^*}. \tag{8}$$

From (7) it follows that

$$\|l_0(x) \mid \widetilde{L}_2^{(m)*}\| = \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)}, \tag{9}$$

where

$$\zeta(H^{-1*} \mid 2m) = \sum_{\gamma \neq 0} \frac{1}{[A(\gamma)]^m}. \tag{10}$$

By (9), the following estimate for the error of the cubature formula holds:

$$|(l_0, \varphi)| \leq \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)}\, \|\varphi \mid \widetilde{L}_2^{(m)}\|. \tag{11}$$

The purpose of the present note is to prove the following theorem.

Theorem 2. *For every individual function φ in $\widetilde{L}_2^{(m)}$ the estimate holds*

$$|(l_0, \varphi)| \leq \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\, \|\varphi \mid L_1^{(m)}(\Omega)\| + o(h^m), \tag{12}$$

where $o(h^m)$ depends on the function $\varphi(x)$, $h \to 0$, and

$$\|\varphi \mid L_1^{(m)}(\Omega)\| = \int_{\Omega} \Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 \Big)^{1/2} dx. \tag{13}$$

Estimate (12) *is sharper than* (11) *as $h \to 0$.*

As follows from Theorem 2, estimate (11), which is exact for the total class of periodic functions in $\widetilde{L}_2^{(m)}$, is not exact for every individual function. For each given h the equality in (11) holds for the function $\varphi(x)$ depending on h. However, for all functions except solutions of

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha \varphi)^2 = \text{const}, \tag{14}$$

we have the strict inequality

$$\|\varphi \mid L_1^{(m)}(\Omega)\| < \sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}(\Omega)\|, \tag{15}$$

and extremal function (7) does not satisfy equation (14). Hence estimate (12) is stronger than estimate (11) for every individual function and sufficiently small h.

Proof. Let us present the idea of the proof of Theorem 2. For any function $\varphi(x)$ in $\widetilde{L}_2^{(m)}$ the error (l, φ) is given by the formula

$$(l, \varphi) = \big((-1)^m \Delta^m u, \varphi\big)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha u, D^\alpha \varphi) \equiv \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u D^\alpha \varphi \, dx. \tag{16}$$

We cover the domain Ω with a system of disjoint parallelepipeds Ω_γ. The sides of each Ω_γ are given by the columns of the matrix hH, and the beginning of Ω_γ is $hH\gamma$. Using the Cauchy–Bunyakovskii–Schawrz inequality, we have

$$(l, \varphi) = \sum_\gamma \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha \varphi D^\alpha u \, dx$$

$$\le \sum_\gamma \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx \Big\}^{1/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u)^2 \, dx \Big\}^{1/2}$$

$$= \Big(\frac{h}{2\pi}\Big)^m \sqrt{\zeta(H^{-1*} \mid 2m)} \sum_\gamma h^{n/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx \Big\}^{1/2}. \tag{17}$$

To prove estimate (12) it remains to show that

$$\sum_\gamma h^{n/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 dx \Big\}^{1/2}$$

$$= \int_\Omega \Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \Big\}^{1/2} dx + o(1). \tag{18}$$

Consider the function

$$f_\gamma(\lambda) = \int\limits_{\Omega_\gamma} \Bigg(\bigg[\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2} - \lambda\Bigg)^2 dx. \qquad (19)$$

Clearly, the function $f_\gamma(\lambda)$ is a positive quadratic trinomial in λ, and

$$f_\gamma(\lambda) = h^n\lambda^2 - 2\lambda \int\limits_{\Omega_\gamma} \bigg[\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2} dx + \int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\, dx. \quad (20)$$

For every positive trinomial $f(\lambda) = a\lambda^2 - 2b\lambda + c$ the equality holds

$$a \min f(\lambda) = ac - b^2. \qquad (21)$$

Let

$$\min f_\gamma(\lambda) = f_\gamma(\lambda_\gamma) = \varepsilon_\gamma. \qquad (22)$$

Clearly, the function $\bigg[\sum\limits_{|\alpha|=m} \dfrac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2}$ is square integrable over Ω. Hence, it can be approximated in norm by a step function

$$\psi(x) = \lambda_\gamma \quad \text{for} \quad x \in \Omega_\gamma. \qquad (23)$$

From this it follows that the sum

$$\sum_\gamma \varepsilon_\gamma = \varepsilon = \tau_\varphi^{(m)}(h) \qquad (24)$$

tends to zero as $h \to 0$. However, from (21) and (22) it follows that

$$\varepsilon_\gamma = \int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\, dx - \frac{1}{h^n}\Bigg(\int\limits_{\Omega_\gamma} \bigg[\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2} dx\Bigg)^2, \qquad (25)$$

and hence

$$\begin{aligned}
&\sum_\gamma h^{n/2}\bigg\{\int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\, dx\bigg\}^{1/2} \\
&= \sum_\gamma h^{n/2}\Bigg\{\frac{1}{h^n}\Bigg(\int\limits_{\Omega_\gamma} \bigg[\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2} dx\Bigg)^2 + \varepsilon_\gamma\Bigg\}^{1/2} \\
&= \sum_\gamma \Bigg\{\Bigg(\int\limits_{\Omega_\gamma} \bigg[\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2\bigg]^{1/2} dx\Bigg)^2 + h^n\varepsilon_\gamma\Bigg\}^{1/2}. \qquad (26)
\end{aligned}$$

Finally, from the triangle inequality it follows that

$$\left\{\left(\int\limits_{\Omega_\gamma}\left[\sum_{|\alpha|=m}\frac{m!}{\alpha!}(D^\alpha\varphi)^2\right]^{1/2}dx\right)^2+h^n\varepsilon_\gamma\right\}^{1/2}$$

$$-\int\limits_{\Omega_\gamma}\left[\sum_{|\alpha|=m}\frac{m!}{\alpha!}(D^\alpha\varphi)^2\right]^{1/2}dx\le h^{n/2}\sqrt{\varepsilon_\gamma}, \tag{27}$$

and further

$$\sum_\gamma h^{n/2}\sqrt{\varepsilon_\gamma}\le\left(\sum_\gamma h^n\right)^{1/2}\left(\sum_\gamma\varepsilon_\gamma\right)^{1/2}=\sqrt{|\Omega|}\sqrt{\tau_\varphi^{(m)}(h)}. \tag{28}$$

Summing (27) over all γ, and using (26) and (28), we obtain (18). The proof of Theorem 2 is complete. □

References

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2. Sobolev, S. L.: Lectures on the Theory of Cubature Formulas. Part I. Novosibirsk. Gosudarstv. Univ., Novosibirsk (1964)

[1] Paper [7] of Part II of this book. – *Ed.*

17. Evaluation of Integrals of Infinitely Differentiable Functions*

S. L. Sobolev

In previous note [1] of the author it was shown that for each individual function in the class of periodic functions from $L_2^{(m)}$, the error estimate for the numerical evaluation of the integral using the method of nets is always better than the error estimate by means of the norm of the error functional. Clearly, if we consider the whole space $L_2^{(m)}$, the estimate of accuracy of a formula by means of the norm of its error functional is unimprovable. Studying countably-normed spaces of infinitely differentiable functions, we come to another similar case. Each such space is the intersection of the sequence of spaces $L_2^{(m)}$, $m = 1, 2, \ldots$. The cubature formulas with equal coefficients mentioned in the note [1] are applicable for periodic functions which are members of all spaces $L_2^{(m)}$ simultaneously. Hence, for any function belonging to all of $L_2^{(m)}$ simultaneously, we get the following sequence of estimates[1]:

$$|(l,\varphi)| \le \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}\|, \tag{1}$$

$$m = [\frac{n}{2}] + 1,\ [\frac{n}{2}] + 2,\ \ldots .$$

Given some law of growth of $L_2^{(m)}$-norm of φ we may choose for each given h the best estimate as the lower bound of all estimates (1). By this way we can, in a number of cases, establish that the convergence of (l,φ) to zero as $h \to 0$ is far more rapid than polynomial decay of errors. The present note is devoted to this question.

Lemma 1. *The function $\zeta(A \mid m)$ at large m admits the estimate*

$$\zeta(H^{-1*} \mid 2m) \le \frac{K_{\min}}{r_{\min}^{2m}} \Big(1 + o(1)\Big), \tag{2}$$

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* Dokl. Akad. Nauk SSSR, **163**, 33–35 (1965)

[1] Here Ω is an integration domain. – *Ed.*

where $r_{\min}$ *is the shortest distance between the nodes of the integer lattice* βH^{-1}*,* β *is an integer row vector, and* K_{min} *is the number of such lattice nodes which are at the distance* $r_{\min}$ *from a given one.*

The proof of Lemma 1 is easy from the definition of the Epstein zeta function $\zeta(H^{-1*} \mid 2m)$ (see [1]).

We consider classes of real infinitely differentiable periodic functions whose derivatives of order α are subject to the inequalities

$$|\frac{D^{\alpha}\varphi(x)}{\alpha!}| \leq KA^{|\alpha|}|\alpha|^{(\beta-1)|\alpha|}. \tag{3}$$

Here, $\alpha!$ denotes $\alpha_1! \dots \alpha_n!$ and the remaining notations are the same as in [1,2]. We call the number β the *order of growth* of the derivatives of φ, and the number A the *type* of this growth. When (3) is satisfied, we write

$$\varphi \in \mathfrak{K}(A, \beta). \tag{4}$$

A long time ago, the basic properties of such classes were established for functions of one independent variable, and we may translate these properties almost without any change to functions of several independent variables.

We distinguish five cases, not all of which are of interest: a) $\beta < 0$; b) $\beta = 0$; c) $0 < \beta < 1$; d) $\beta = 1$; and e) $\beta > 1$.

In case a), the class $\mathfrak{K}(A, \beta)$ contains no periodic function besides a constant function, and we shall not consider it.

In case b), i.e., for $\beta = 0$, the class $\mathfrak{K}(A, \beta)$ of periodic functions may contain only polynomials whose degree n depends on the size of the constant A: $n \leq K_1 A$.

In case c), for $0 < \beta < 1$, $\mathfrak{K}(A, \beta)$ contains entire functions of order

$$\varrho = \frac{1}{1-\beta} \tag{5}$$

and of type

$$\sigma = \frac{1-\beta}{e} A^{1/(1-\beta)}. \tag{6}$$

In case d), for $\beta = 1$, the class $\mathfrak{K}(A, \beta)$ consists of functions, analytic with a radius of convergence at each point determined by the constant A,

$$\operatorname{Im}\{x_j\} < e^{-K_2 A}, \tag{7}$$

where K_2 is some constant.

Finally, for $\beta > 1$, the class $\mathfrak{K}(A, \beta)$ consists of infinitely differentiable nonanalytic functions, and for each A the compactly-supported functions belong to it. Various quasi-analytic functions belong to the intersection $\cap_{A>0}\mathfrak{K}(A, 1)$.

Lemma 2. *Let periodic function φ of the variable x in n dimensions be in the class $\mathfrak{K}(A, \beta)$. Then the following estimates of the $L_2^{(m)}$-norms are valid:*

$$\|\varphi \mid L_2^{(m)}\| \le K_3 m^{\beta m+1/2} \left(\frac{A}{e}\right)^m, \quad m = 0, 1, 2, \dots . \tag{8}$$

Here K_3 is some constant depending on K and Ω.

Proof. The proof of this lemma is based on simple estimates. We may always assume that $\sum\limits_{|\alpha|=m} f(\alpha)$ denotes a summation as though the function $f(\alpha)$ does not possess symmetry with respect to permutations of the entries of the integer vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$. If $f(\alpha)$ possesses such symmetry, then the sum can be taken with the corresponding repetitions. Hence, the norm of the function $\|\varphi \mid L_2^{(m)}\|$ may be written as

$$\|\varphi \mid L_2^{(m)}\|^2 = \int\limits_{\Omega} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx = \int\limits_{\Omega} \sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!}\right)^2 dx. \tag{9}$$

Since

$$\sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!}\right)^2 \le A^{2m} m^{2m(\beta-1)} \sum_{|\alpha|=m} (\alpha!)^2, \tag{10}$$

we obtain

$$\|\varphi \mid L_2^{(m)}\| \le A^m m^{m(\beta-1)} \sqrt{|\Omega|} \left\{ \sum_{|\alpha|=m} (\alpha!)^2 \right\}^{1/2}. \tag{11}$$

The inequality is valid

$$\sum_{|\alpha|=m} (\alpha!)^2 \le K_h e^{-2m} m^{2m+1}. \tag{12}$$

It is proved by rearranging the sum

$$\sum_{|\alpha|=m} (\alpha!)^2 = m! \sum_{|\alpha^{(j)}|=m} (\alpha^{(j)})!. \tag{13}$$

The sum on the right side of (13) is taken over distinct integer vectors $\alpha^{(j)}$ without repetitions. Further, one establishes

$$\sum_{|\alpha^{(j)}|=m} \frac{(\alpha^{(j)})!}{(m-1)!} = n + O\left(\frac{1}{m}\right). \tag{14}$$

Inequality (12) follows immediately from (13), (14), and the Stirling formula. Finally, from (11) and (12) inequality (8) follows. □

Lemmas 1 and 2 allow us to prove the main theorem.

Theorem 1. *For each periodic function* φ *of the class* $\mathfrak{K}(A,\beta)$, $\beta > 0$, *and for the cubature formula with nodes at the points* $hH\gamma$ *the following estimate of the error of this cubature formula holds*

$$|(l,\varphi)| \leq K h^{-1/2} \exp\left[-\frac{\beta}{e}\left(\frac{2\pi e r_{\min}}{Ah}\right)^{1/\beta}\right]. \tag{15}$$

Proof. The proof of Theorem 1 is based on determining the minimum with respect to m of the function on the right side of the inequality

$$\begin{aligned} &\|l \mid L_2^{(m)*}\| \cdot \|\varphi \mid L_2^{(m)}\| \\ &\leq \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)} K m^{\beta m+1/2}\left(\frac{A}{e}\right)^m, \end{aligned} \tag{16}$$

which is easy to carry out in an elementary way. Incidentally, one establishes which m is optimal for a given value of h, namely,

$$m = \frac{1}{e}\left(\frac{2\pi e r_{\min}}{Ah}\right)^{1/\beta}, \tag{17}$$

as required. □

It is interesting to note the case which does not follow directly from Theorem 1, namely, when $\beta = 0$. A direct estimate in this case gives Theorem 2.

Theorem 2. *For a sufficiently small mesh-size* h*, the cubature formula with a fixed lattice of nodes* $hH\gamma$ *and equal coefficients is exact for any trigonometric polynomial.*

References

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2. Sobolev, S. L.: On the rate of convergence of cubature formulas. Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)[3]

[2] Paper [16] of Part II of this book. – *Ed.*
[3] Paper [14] of Part II of this book. – *Ed.*

18. Cubature Formulas with Regular Boundary Layer*

S. L. Sobolev

In one of the previous notes [1] we considered error functionals of cubature formulas of the form[1]

$$l(x) = \chi_\Omega(x) - \sum_{j=1}^{N} c_j \delta(x - x^{(j)}) = \sum_\gamma l_\gamma \left(\frac{x}{h} - \gamma \right), \tag{1}$$

where $l_\gamma(x)$ are functionals with bounded supports and finite norms in $L_2^{(m)}$, and $l_\gamma(x)$ are orthogonal to all polynomials of degree m. The set of $l_\gamma(x)$, which satisfy conditions (10), (11), and (12) of [1], is denoted by $\mathfrak{R}(L, A, m+1)$.

Let H be the matrix of periods of some lattice with determinant equal to 1, $|H| = 1$. Formula (1) is evidently equivalent to the equality

$$l(x) = \sum_\gamma l_\gamma \left(\frac{x}{h} - H\gamma \right). \tag{2}$$

In what follows, we consider formulas for which:

a) the nodes of all $l_\gamma(x)$ are located at the points $hH\gamma'$,

$$l_\gamma(x) = \chi_\gamma(x) - \sum_{\gamma'} c_\gamma^{\gamma'} \delta(x - hH\gamma'), \quad \sum \chi_\gamma \left(\frac{x}{h} - H\gamma \right) = \chi_\Omega(x); \tag{3}$$

b) all errors $l_\gamma(x)$ are members of $\mathfrak{R}(L, A, s)$,

$$l_\gamma(x) \in \mathfrak{R}(L, A, s); \tag{4}$$

c) for all points $hH\gamma$ such that $\operatorname{dist}(hH\gamma, \Gamma) > 2Lh$ errors $l_\gamma(x)$ coincide,

$$l_\gamma(x) = l_0(x). \tag{5}$$

Under these conditions we call $l(x)$ the *error with regular boundary layer of order m*.

The purpose of our note is to establish the following theorem.

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* Dokl. Akad. Nauk SSSR, **163**, 587–590 (1965)

[1] Here Ω is a domain with piece-wise smooth boundary. – *Ed.*

Theorem. *Let $l(x)$ be an error with regular boundary layer of order m. Then the following equality holds:*

$$\|l \mid L_2^{(m)}\| = \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\sqrt{|\Omega|} + O(h^{m+1}) \quad as \quad h \to 0. \tag{6}$$

The proof is based on a series of auxiliary lemmas, which are given here.

Lemma 1. *Let $l(x)$ be an error with regular boundary layer. Then for all nodes $hH\gamma$ in Ω at a distance not less than Lh from the boundary Γ of Ω, coefficients c_γ are all equal to h^n.*

Proof. Indeed, for the node $hH\gamma$ under consideration we have

$$c_\gamma = h^n \sum_{|\gamma'|<L} c^{(0)}_{\gamma-\gamma'}. \tag{7}$$

Here $c_\gamma^{(0)}$ are coefficients of $l_0(x)$. From the conditions that the volume of Ω_0 equals 1 and $(l_0(x), 1) = 0$ it follows that

$$c_\gamma = h^n, \tag{8}$$

as required. □

We call the set of nodes $hH\gamma$ at which $c_\gamma \neq h^n$ the *boundary layer*. If only nodes in the interior of Ω are used in integration, then the boundary layer is *interior*. If we also use nodes in the exterior of Ω in approximating the functional $\chi_\Omega(x)$ in $L_2^{(m)}$, then we can get a two-sided boundary layer which is comprised by nodes $hH\gamma$ at a distance not greater than Lh from the boundary Γ, or else an exterior boundary layer which is comprised by nodes $hH\gamma$ in the exterior of Ω at a distance not greater than $2Lh$ from the boundary Γ. Of course, for a function $\varphi(x)$ in $L_2^{(m)}(\Omega)$ only formulas with interior boundary layer make sense. We call the number $2L$ the *width of the boundary layer.*

Let $m_\gamma(x) = \sum c_\gamma^{\gamma'} \delta(x - hH\gamma')$. We call such functionals the *narrow-like functionals.* Further, let $m_\gamma(x) \in \mathfrak{R}(L, A, s)$ and

$$m(x) = \sum_{\gamma \in B_j} m_\gamma(x - hH\gamma), \tag{9}$$

where γ ranges over some set B_j, and $\{hH\gamma| \ \gamma \in B_j\}$ is a boundary layer of width L. Then we call functional (9) a *zero's error with a boundary layer of order s.* A zero's error with a boundary layer is exterior, interior, or two-sided according to the location of its support. The width of this functional is introduced analogously to the width of a boundary layer, and it is, generally speaking, equal to $3L$, but it may also be less. In all that follows it may be taken equal to $2L$.

Lemma 2. *Let $l^{(1)}(x)$ and $l^{(2)}(x)$ be errors with regular boundary layer of orders $s^{(1)}$ and $s^{(2)}$, respectively. The difference of $l^{(1)}(x)$ and $l^{(2)}(x)$ is a zero's error with a boundary layer of order*

$$\min(s^{(1)}, s^{(2)}) - 1. \tag{10}$$

The proof of Lemma 2 is based on an auxiliary lemma.

Lemma 3. *Let $m(x)$ be a compactly-supported functional of the form*

$$m(x) = \sum_{|H\gamma|<L} c[\gamma]\delta(x - hH\gamma), \tag{11}$$

and let $m(x)$ be orthogonal to all polynomials of degree s:

$$(m(x), x^\alpha) = \sum c[\gamma](hH\gamma)^\alpha = 0 \quad \textit{for} \quad |\alpha| \le s.$$

Then $m(x)$ admits the equivalent representation as follows[2]*:*

$$m(x) = \sum_{j=1}^{n} \big(M_j(x + hH\delta_j) - M_j(x)\big), \tag{12}$$

where $(M_j(x), x^\alpha) = 0$ for $|\alpha| \le s - 1$ and $\operatorname{supp} M_j(x)$ is a subset of the smallest parallelepiped, with edges parallel to the columns of H, containing $\operatorname{supp} m(x)$.

Proof. Lemma 3 is proved by the method of induction on the number n of independent variables. We have to establish that the coefficients $c[\gamma]$ may be written as

$$c[\gamma] = \sum_{j=1}^{n} \widehat{\Delta}_j c_j[\gamma], \tag{13}$$

where $\sum c_j[\gamma]\gamma^\alpha = 0$ for $|\alpha| \le s - 1$ and $\widehat{\Delta}_j \varphi[\gamma] = \varphi[\gamma + \delta_j] - \varphi[\gamma]$.

Let us show that for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ the equality holds

$$c[\gamma] = c_n[\gamma_1, \ldots, \gamma_{n-1}, \gamma_n+1] - c_n[\gamma_1, \ldots, \gamma_{n-1}, \gamma_n] + c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}], \tag{14}$$

where the function $c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]$ is orthogonal to all polynomials in the variables $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ of degree s, and the function $c_n[\gamma_1, \gamma_2, \ldots, \gamma_n]$ is orthogonal to polynomials of degree $s - 1$ and $\operatorname{supp} c_n[\gamma]$ is a subset of the smallest parallelepiped containing the support of $c[\gamma]$. From this Lemma 3 follows. □

[2] Here $\delta_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0)$. – *Ed.*

Functions $c_n[\gamma_1, \gamma_2, \dots, \gamma_n]$ and $c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ may be written as

$$c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}] = \sum_{\gamma_n'=-L}^{L} c[\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n']; \tag{15}$$

$$c_n[\gamma] = \begin{cases} 0 \quad \text{for} \quad |\gamma_n| \geq L, \\ \sum\limits_{\gamma_n'=-L}^{\gamma_n-1} c[\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n'] - (\gamma_n + L) c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]. \end{cases} \tag{16}$$

Hence, formula (14) and the orthogonality of $c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ to polynomials in the variables $(\gamma_1, \gamma_2, \dots, \gamma_{n-1})$ of degree s are clear. The orthogonality of $c_n[\gamma_1, \gamma_2, \dots, \gamma_n]$ to all polynomials of degree $s-1$ follows from the well-known formula for summation by parts:

$$\sum_\gamma [\varphi(\gamma + \delta_n) - \varphi(\gamma)]\psi(\gamma) = \sum_\gamma \varphi(\gamma)[\psi(\gamma) - \psi(\gamma - \delta_n)]. \tag{17}$$

It suffices to note that $x_n^{\alpha_n} = \dfrac{1}{\alpha_n + 1} \widehat{\Delta}_n B_{\alpha_n+1}(x_n)$, where B_{α_n+1} is the Bernoulli polynomial of degree $\alpha_n + 1$, and use (14).

Lemma 3 can be also proved in a different way, namely, by passing to the Fourier transform. This is its dual statement.

Lemma 3a. *Let Z be the class of rational functions $\Psi(z)$ of the form $\dfrac{P(z)}{z^{\mathbf{k}}}$, where $P(z)$ is a polynomial in $z = (z_1, \dots, z_n)$, and $z^{\mathbf{k}} = z_1^{k_1} \dots z_n^{k_n}$; $\mathbf{k} = (k_1, k_2, \dots, k_n)$. Every function $\varphi(z)$ with a zero of multiplicity m at the point $(1, 1, \dots, 1)$ may be written down as*

$$\varphi(z) = \sum_{j=1}^{n} (z_j - 1)\varphi_j(z), \tag{18}$$

where the functions φ_j are members of the same class Z with the polynomials $P_j(z)$, and they have zeros of multiplicity $m-1$ at the point $(1, 1, \dots, 1)$. Also, the degree of the polynomial $P_j(z)$ in the variable z_j does not exceed the degree of $P(z)$ and $\mathbf{k}_j \leq \mathbf{k}$.

It seems the proof in the above text is no longer than any possible proof of Lemma 3a, especially if we take into account the necessity to establish their equivalence.

Lemma 3a and Lemma 3 are the particular examples of lemmas on the expansion of an analytic function with a root of multiplicity m at a given point $z^{(0)}$ into the sum

$$\varphi(z) = \sum (z_j - z^{(0)})\varphi_j(z), \tag{19}$$

and of dual lemmas on the corresponding expansion of generalized functions ψ in $K^{(s)}$, where $(\psi(x) * x^\alpha) = 0$ for $|\alpha| \leq s$.

Essentially, the theorem of L. Schwartz on the representation of every generalized function in the form of a differential operator on a continuous function is like Lemma 3.

Corollary. *Let $m_0(x)$ be a zero's error from $\mathfrak{R}(L, A, s+1)$. Then the sum*

$$\sum_{hH\gamma\in\Omega} m_0(x - hH\gamma) = M_0(x), \tag{20}$$

is a zero's error with boundary layer of order s.

This corollary is obtained immediately, if we replace m_0 in the left side of (20) by its expansion into the sum (11) and change the order of summation.

Lemma 2 follows immediately from Lemma 3.

Corollary to Lemma 2. *Each functional $l(x)$ with a boundary layer of order $s \geq m$ may be written as*

$$l(x) = \sum_{hH\gamma\in\Omega} l^* \left(\frac{x}{h} - H\gamma\right) + \sum_{\gamma\in B} l^{**}_\gamma \left(\frac{x}{h} - H\gamma\right), \tag{21}$$

where B is a boundary layer of width L,

$$l^* \left(\frac{x}{h} - H\gamma\right) \in \mathfrak{R}(L, A, s_1). \tag{22}$$

The order s_1 in (22) *can be each number greater than s.*

To prove (21) for given s_1 it suffices to expand the difference between $l(x)$ and any given error functional with regular boundary layer of order s_1 into a sum like (9).

Let $l(x)$ be a linear functional with regular boundary layer of order $s \geq m$. The equality is valid,

$$l(x) = 1 - \Phi_0(h^{-1}H^{-1}x) - l^{(1)}(x), \tag{23}$$

where $l^{(1)}(x)$ is a functional with exterior regular boundary layer for the domain $\overline{\Omega} = R^n \backslash \Omega$. As was established in [1], the extremal function for $l(x)$ has the form

$$u(x) = l(x) * G(x). \tag{24}$$

From (23) and (24) it follows that

$$u(x) = u_0(x) + C - l^{(1)}(x) * G(x), \tag{25}$$

where $u_0(x)$ is the elementary solution of the extremal problem in the periodic case.

Let us write down the norm of $l(x)$ explicitly,

$$\|l(x)\|^2 = \big(l(x), u(x)\big) = \big(l(x), u_0(x)\big) - l(x) * G(x) * l^{(1)}(-x)\Big|_{x=0}. \qquad (26)$$

Replacing $l(x)$ and $l^{(1)}(x)$ by their expansions into sums like (21) and repeating the estimates mentioned in [1], we obtain

$$l(x) * G(x) * l^{(1)}(-x)\Big|_{x=0} = O(h^{2m+1}). \qquad (27)$$

By a direct calculation one can also show that

$$(l(x), u_0(x)) = \frac{h^{2m}}{(2\pi)^{2m}}\zeta(H^{-1*} \mid 2m)|\Omega| + O(h^{2m+1}). \qquad (28)$$

The main theorem follows from (26)–(28).

References

1. Sobolev, S. L.: On the rate order of convergence of cubature formulas. Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)[3]

[3] Paper [14] of Part II of this book. – *Ed.*

19. A Difference Analogue of the Polyharmonic Equation*

S. L. Sobolev

Let x be a column vector in the n-dimensional space R^n, and let β be a column vector with integer entries. The generalized function

$$\psi(x) = \sum_{\beta} \varphi[\beta]\delta(x - \beta)$$

is called a *lattice* function, and the function $\varphi[\beta]$ of integer variables β is called a *discrete* function. We denote the set of lattice functions by P, and the set of discrete functions by R. To an arbitrary continuous function $\varphi(x)$, there correspond lattice and discrete functions defined by

$$\begin{gathered} \Omega^{CR}[\beta \mid \varphi] = \varphi(\beta); \\ \Omega^{CP}(x \mid \varphi) = \sum_{\beta} \varphi[\beta]\delta(x - \beta) = \varphi(x)\Phi_0(x) = \psi(x), \end{gathered} \tag{1}$$

where $\Phi_0(x) = \sum_{\beta} \delta(x - \beta)$. Equality (1) may be also written as

$$\psi(x) = \Omega^{RP}(x \mid \varphi) \quad \text{and} \quad \varphi[\beta] = \Omega^{PR}[\beta \mid \psi]. \tag{2}$$

Also let us introduce the mappings

$$\Omega^{PC}(x \mid \varphi) \quad \text{and} \quad \Omega^{RC}(x \mid \varphi), \tag{3}$$

where

$$\Omega^{PC} = (\Omega^{CP})^{-1} \quad \text{and} \quad \Omega^{RC} = (\Omega^{CR})^{-1}. \tag{4}$$

Mappings (3) and (4) are not uniquely defined. Indeed, we may explicitly express Ω^{PC} as a convolution

$$\Omega^{PC}(x \mid \varphi) = \varphi(x) * \Lambda(x),$$

* Dokl. Akad. Nauk SSSR, **164**, 54–57 (1965)

where $\Lambda(x)$ is an arbitrary solution of the equation $\Lambda(x)\Phi_0(x) = \delta(x)$; i.e.,

$$\Lambda(\beta) = 1 \quad \text{for} \quad \beta = 0 \quad \text{and} \quad \Lambda(\beta) = 0 \quad \text{for} \quad \beta \neq 0.$$

It is surely assumed that the convolution of $\Lambda(x)$ with the function $\varphi(x)$ exists.

Below we expose the spaces containing the product and the convolution of two functions from C or P. As regards the inner product

$$(\varphi, \psi) = c \in R^1, \tag{5}$$

it exists provided that φ and ψ are not simultaneously members of P.

a) $\varphi \cdot \psi = \chi$		
$\varphi \backslash \psi$	C	P
C	C	P
P	P	

b) $\varphi * \psi = \chi$		
$\varphi \backslash \psi$	C	P
C	C	C
P	C	P

(6)

On the set of functions in R there are defined conventional operations. Obviously, convolution is applicable not for an arbitrary pair of functions, but only for sufficiently rapidly decreasing ones. The mappings Ω of the spaces C, P, and R preserve the validity of (5) and (6a) in all cases, and the validity of (6b) everywhere except for the case $\varphi \in C$ and $\psi \in C$.

Also let us introduce the space Φ of functions decreasing at infinity sufficiently rapidly. For example, compactly-supported functions are members of Φ. Let Π be the space of periodic functions with integer periods, and let T be the space of functions defined on the torus Ω_0 obtained by identifying all points of R^n that differ by an integer vector. Thus, we can consider the mappings

$$\Xi^{\Phi\Pi}, \quad \Xi^{\Pi\Phi}, \quad \Xi^{T\Pi}, \quad \Xi^{\Pi T}, \quad \Xi^{\Phi T}, \quad \text{and} \quad \Xi^{T\Phi}.$$

The mapping $\Xi^{\Pi T}$ sends a periodic function $\varphi(p)$ with domain R^n into a function defined on the torus with the same values. On the other hand, the mapping $\Xi^{T\Pi}$ sends T into Π.

The mapping $\Xi^{\Phi\Pi}$ sends $\varphi(p) \in \Phi$ into the function

$$\psi(p) \equiv \Xi^{\Phi\Pi}(p \mid \varphi) = \sum_{\gamma} \varphi(p - \gamma) = \varphi(p) * \Phi_0(p),$$

where the vector γ has integer entries. Further, $\Xi^{\Phi T} = \Xi^{\Phi\Pi}\Xi^{\Pi T}$.

The inner product (φ, ψ) of the functions $\varphi \in \Phi \cup \Pi$ and $\psi \in \Phi \cup \Pi$ has sense provided that φ and ψ are not simultaneously members of Π.

Below we expose the spaces containing the product and the convolution of functions under consideration.

a) $\varphi \cdot \psi = \chi$		
$\varphi \backslash \psi$	Φ	Π
Φ	Φ	Φ
Π	Φ	Π

b) $\varphi * \psi = \chi$		
$\varphi \backslash \psi$	Φ	Π
Φ	Φ	Π
Π	Π	

Certainly, the product of functions in Π is defined only in the case when neither φ nor ψ is a generalized function. Otherwise, we need the special hypotheses for defining the product, and we shall not consider this case.

The mappings Ξ of the spaces Φ, Π, and P preserve the binary operations and the inner product

$$(\varphi, \psi) = \int \varphi\psi \, dp,$$

except for the cases when the operation is the product $\varphi\psi = \chi$ or the operation is the inner product (φ, ψ) with $\varphi \in \Phi$ and $\psi \in \Phi$.

The Fourier transform and the inverse Fourier transform of functions with domain R^n are defined by the formulas

$$\widetilde{f}(p) = \int e^{i2\pi px} f(x)\, dx \quad \text{and} \quad \widecheck{f}(x) = \int e^{-i2\pi px} f(p)\, dp, \tag{7}$$

respectively. In particular, (7) holds for an arbitrary function $f(x)$ in $L_2^{(m)}$. The Fourier transform defined by (7) is a unitary operator. As well known, using the weak continuity of the Fourier transform, we may extend it to the space of generalized functions.

Theorem 1. *The following equalities are valid:*

$$\widecheck{\widetilde{f}}(x) = \widetilde{\widecheck{f}}(x) \quad \textit{and} \quad \widetilde{\widetilde{f}}(x) = \widecheck{\widecheck{f}}(x) = f(-x).$$

Theorem 2 (Parseval's identity). $\big(\widetilde{f}(p), \widetilde{\varphi}(p)\big) = \big(f(x), \varphi(x)\big)$.

Using Theorem 2, we may define the Fourier transforms for generalized functions.

Theorem 3. *The duality of the multiplication and convolution holds, i.e.,*

$$\left(\widetilde{f\varphi}\right)(p) = \widetilde{f}(p) * \widetilde{\varphi}(p) \quad \textit{and} \quad \left(\widecheck{f\varphi}\right)(p) = \widecheck{f}(p) * \widecheck{\varphi}(p).$$

The Fourier images of some simplest functions are given by the formulas:

$$\widetilde{\delta}(x) = 1,\ \widetilde{1} = \delta(p),\ \widetilde{e^{-\pi x^2}} = e^{-\pi p^2},\ \widetilde{\Phi_0(x)} = \Phi_0(p),\ \widetilde{D^\alpha(x)} = (i2\pi p)^\alpha.$$

By definition, the convolution of a function with the generalized function $D^\alpha(x)$ is the derivative of order α for the function. The equality for the Fourier transform of $\Phi_0(x)$ is the well-known Poisson formula [1].

The Fourier transform maps periodic functions into lattice functions. On the other hand, it sends lattice functions into periodic functions. The Fourier transform extends to discrete functions if we suppose that

$$\widetilde{\varphi}[\beta] = \widetilde{\varphi}(p) \equiv \Omega^{\Pi T}\big(\widetilde{\Omega^{RP}}(x \mid \varphi)\big).$$

Theorems 1, 2, and 3 remain valid for discrete functions.

Theorem 4. *The Fourier transform maps the spaces C, P, and R in a one-to-one fashion onto Φ, Π, and T, respectively. Correspondences established above for the operators Ω become analogous correspondences established above for the operators Ξ.*

All the theorems that we state above may be translated to the case of lattice functions which have singularities at the nodes $A\beta$ like the corresponding translations of the Dirac delta function. To these lattice functions there correspond periodic functions with periods γA^{-1}. In this event the definitions are as follows.

Let P_A be the space of lattice functions of the form

$$\psi(x) = \sum_\beta \varphi[\beta]\delta(x - A\beta).$$

The spaces C, P_A, and R are mapped into each other by means of the mappings

$$\Omega_A^{CP}, \quad \Omega_A^{PC}, \quad \Omega_A^{PR}, \quad \Omega_A^{RP}, \quad \Omega_A^{CR}, \quad \text{and} \quad \Omega_A^{RC},$$

whose definitions are similar to the definitions of the mappings Ω considered above. In this case,

$$\Omega_A^{CP}(x \mid \varphi) = \varphi(x)|A|^{-1}\Phi_0(A^{-1}x) \text{ and } \Omega_A^{CR}[\beta \mid \varphi] = \varphi(A\beta).$$

The remaining Ω_A are formed similarly. All Ω_A are easily expressed by means of the corresponding Ω. The mappings Ω again preserve the elementary binary operations.

Also let us consider the space Π_B of periodic functions $\varphi(p)$ with periods $B\gamma$, i.e.,

$$\varphi(p) = \varphi(p + B\gamma),$$

where B is a nonsingular matrix. The spaces Φ, Π_B, and T_B are mapped into each other by means of the mappings Ξ_B. To the space Π_B there corresponds the space T_B of functions defined on the torus. For $\Xi_B^{\Phi\Pi}$ we use the formula

$$\Xi_B^{\Phi\Pi}(p \mid \varphi) = |B| \sum_\gamma \varphi(p - B\gamma) = \varphi(p) * \Phi_0(B^{-1}p).$$

The remaining Ξ_B are formed similarly. All Ξ_B are easily expressed in elementary form by means of the corresponding Ξ.

Theorem 5. *The Fourier transform maps the spaces C, P_A, and R in a one-to-one fashion onto Φ, $\Pi_{A^{-1}}$, and $T_{A^{-1}}$, respectively. The image of the mapping Ω_A under this transform is the mapping $\Xi_{A^{-1}}$, and vice versa.*

Theorem 6. *Let the inner product and the convolution of two functions in T_B be given by the formulas*[1]

$$(\varphi(p), \psi(p)) = \frac{1}{|B|} \int \varphi(p)\overline{\psi(p)}\, dp, \quad \varphi(p) * \psi(p) = \frac{1}{|B|} \int \varphi(p-q)\psi(q)\, dq.$$

In this event, the inner product is invariant under the Fourier transform; the image of the convolution under the Fourier transform is the product of the images, and vice versa.

The polyharmonic equation

$$\Delta^m u = f \tag{8}$$

with the inverse operator $G(x) * f(x) = u(x)$, where

$$G(x) = \varkappa_{m,n} |x|^{2m-n} \begin{cases} 1, & \text{if } n \text{ odd or } n > 2m, \\ \ln|x|, & \text{if } n \text{ even and } n \le 2m, \end{cases} \tag{9}$$

is often studied using the generalized inner product

$$D(\varphi, \psi) = \int \sum_{|\alpha|=m} D^\alpha \varphi D^\alpha \psi\, dx = (-1)^m \int \varphi \Delta^m \psi\, dx = (-1)^m \int \psi \Delta^m \varphi\, dx.$$

There are various analogues of Δ^m, G, and $D(\varphi, \psi)$ for discrete functions $\varphi[\beta]$. The finite differences $\widehat{\Delta}^m[\beta]$ are often used, where[2]

$$\widehat{\Delta} * \varphi[\beta] = \sum_{j=1}^{n} \big(\delta[\beta + \delta_j] + \delta[\beta - \delta_j] - 2\delta[\beta]\big) * \varphi[\beta]$$

$$\equiv \sum_{j=1}^{n} \big[\varphi[\beta + \delta_j] + \varphi[\beta - \delta_j]\big] - 2n\varphi[\beta].$$

The inverse operator to $\widehat{\Delta}^m$ is a convolution with some discrete function which behaves like $G[\beta]$ at infinity. As regards the sum

$$\Delta(\varphi, \psi) = \sum_{\beta} (\widehat{\Delta}^\alpha \varphi[\beta], \widehat{\Delta}^\alpha \psi[\beta]),$$

[1] Here the integration is carried out over the domain of functions $\varphi(p)$ and $\psi(p)$, i.e., over the torus. – *Ed.*

[2] Here, $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0)$; $\delta[0] = 1$ and $\delta[\beta] = 0$ for $\beta \neq 0$. – *Ed.*

it plays the role of the inner product $D(\varphi, \psi)$. We give here one more generalization of these notions.

Let $G(x)$ be the fundamental solution of the polyharmonic equation, i.e., let G be given by (9). Then we assume that

$$\overset{\sqcap}{G}_{hH}(x) = \Omega_{hH}^{CP}(x \mid G) \text{ and } G_{hH}[\beta] = \Omega_{hH}^{CR}[\beta \mid G] = G(hH\beta), \tag{10}$$

where $|H| = 1$. We call the convolution $G_{hH}[\beta] * \varrho[\beta] = U[\beta]$ a *discrete potential*. This is the natural generalization of the convolution $G(x) * \varrho(x)$.

For the inverse operator $L_{hH}[\beta]$ of the convolution with $G_{hH}[\beta]$ the equality holds

$$G_{hH}[\beta] * L_{hH}[\beta] = \delta[\beta].$$

Passing to lattice functions and using the Fourier transform, we find the function $\widetilde{L}_{hH}(p)$ as

$$\widetilde{L}_{hH}(p) = \left(\left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma} \frac{h^{-n}}{\left\{ \sum_{j=1}^{n} [p_j - (h^{-1}H^{-1*}\gamma)_j]^2 \right\}^m} \right)^{-1}. \tag{11}$$

From (11) a number of theorems follow.

Theorem 7. *The convolution with $L_{hH}[\beta]$ is an orthogonal operator to all polynomials of degree $2m-1$, i.e.,*

$$L_{hH}[\beta] * \beta^{\alpha} = 0 \quad \textit{for} \quad |\alpha| < 2m.$$

Theorem 8. *The discrete function $L_{hH}[\beta]$ can be written as*

$$L_{hH}[\beta] = h^{n-2m} L_H[\beta],$$

where $L_H[\beta]$ decreases exponentially with $[\beta]$ growing, i.e., $|L_H[\beta]| \le e^{-\eta|\beta|}$.

Theorem 9. *For an arbitrary $2m$-times continuously differentiable function φ we have*

$$h^{-n} L_{hH}(x) * \varphi(x) \overset{\textit{weakly}}{\Longrightarrow} \Delta^m \varphi \quad \textit{as} \quad h \to 0.$$

Theorem 10. *The convolution $\tau_{hH} = L_{hH}(x) * G(x) = \tau_H(h^{-1}\gamma)$ decreases exponentially at infinity, i.e., there exist positive constants K and η such that*

$$|\tau_{hH}(x)| = |\tau_H(h^{-1}x)| \le K e^{-\eta|x|/h}.$$

Theorem 11. *For any two real discrete functions $\varphi[\beta]$ and $\psi[\beta]$, decreasing exponentially at infinity, the bilinear form*

$$D_{hH}(\varphi, \psi) = (\varphi[\beta], L_{hH}[\beta] * \psi[\beta]) = \varphi[\beta] * L_{hH}[\beta] * \psi[-\beta]\Big|_{\beta=0}$$

is symmetric and non-negative; moreover,

$$D_{hH}(\varphi(hH\beta), \psi(hH\beta)) \overset{weakly}{\Longrightarrow} D(\varphi, \psi) \quad as \quad h \to 0.$$

There exist positive constants m and M such that

$$m\Delta(\varphi, \varphi) \leq D_{hH}(\varphi, \varphi) \leq M\Delta(\varphi, \varphi),$$

and m and M are independent of φ.

Theorems 6–10 follow from the fact that $\widetilde{L}_{hH}(p)$ is an analytic function which is non-negative for all real p. Obviously, $\widetilde{L}_{hH}(p) = h^{n-2m}\widetilde{L}_H(hp)$. The product of $\widetilde{L}_{hH}(p)$ and $\widetilde{G}(p)$ is a regular function which is equal to h^n at the coordinate origin.

Theorem 11 follows from the fact that the ratio

$$\widetilde{L}_{hH}(p)/\left(\sum_{j=1}^{n} \sin^2 \frac{(hH^*p)_j}{2}\right)^m$$

is bounded by finite positive limits, and the forms $\Delta(\varphi, \psi)$ and $D_{hH}(\varphi, \psi)$ are reduced by the Fourier transform to the integrals

$$D_{hH}(\varphi, \psi) = \int \widetilde{L}_{hH}(p)\varphi(p)\overline{\psi}(p)\, dp$$

and

$$\Delta(\varphi, \psi) = \int \left(\sum_{j=1}^{n} \sin^2 \frac{(hHp)_j}{2}\right)^m \varphi(p)\overline{\psi}(p)\, dp.$$

From the boundedness of the ratio of the integrands it follows that the ratio of these integrals is also bounded by finite positive limits.

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10. Cubature Formulas on the Sphere Invariant under Finite Groups of Rotations*

S. L. Sobolev

A cubature formula on the surface of the sphere

$$(l,f) = \int_S f(\vartheta,\varphi)\,dS - \sum_{k=1}^{N} c_k f(x^{(k)}) \cong 0 \tag{1}$$

is called *invariant* under transformations of a certain group G of sphere rotations if

$$\Big(l, f\big(\vartheta_1(\vartheta,\varphi),\varphi_1(\vartheta,\varphi)\big)\Big) = \Big(l, f(\vartheta,\varphi)\Big), \tag{2}$$

where

$$\vartheta_1(\vartheta,\varphi), \quad \varphi_1(\vartheta,\varphi) \tag{3}$$

is a substitution in G.

L. A. Lyusternik and V. A. Ditkin [1, 2] have considered formulas with nodes at the vertices of an icosahedron and centers of its faces. We will show how to construct cubature formulas which are invariant under the groups of rotations of the sphere corresponding to a regular polyhedron and are valid for as many spherical harmonics as possible [3].

Theorem 1. *Let a cubature formula be invariant under G. Then it is exact for all harmonics of a given degree if and only if it is exact for all invariant harmonics $Y_n^*(\vartheta,\varphi)$ of this degree n, i.e., for those hormonics which are unchanged under rotations of the sphere belonging to G:*

$$Y_n^*\big(\vartheta_1(\vartheta,\varphi),\varphi_1(\vartheta,\varphi)\big) = Y_n^*(\vartheta,\varphi). \tag{4}$$

The proof is based on the formula

$$(l,f) = (l,f_G), \tag{5}$$

where f_G is the mean of the function f over the group[1] G:

* Dokl. Akad. Nauk SSSR, **146**, 310–313 (1962)
[1] In this paper, M is the order of G. – *Ed.*

$$f_G(x) = \frac{1}{M} \sum_{g \in G} f(gx). \tag{6}$$

Let $S(n)$ be the number of invariant harmonics of degree n. This number may be computed using the representation theory of groups, as was pointed out to the author by D. K. Faddeev.

The spherical harmonics of degree n form a $(2n+1)$-dimensional space, whose basis may be chosen to be

$$e^{im\varphi} P_n^{(|m|)}(\cos\vartheta), \quad m = 0, \pm 1, \dots, \pm n. \tag{7}$$

The group of rotations of the sphere induces the group of linear substitutions which acts on harmonics (7) and is a linear representation of the former group.

Every representation decomposes into irreducible representations on subspaces of lower dimensionality. Among them some are identity representations. The number $S(n)$ of linearly independent invariant harmonics coincides with the number of such one-dimensional identity representations included in the representation A.

The traces of the matrices of irreducible representations (the so-called characters of the representation) constitute M-dimensional vectors. It is well known that characters of distinct irreducible representations are orthogonal:

$$\sum_{k=1}^{M} \chi\big(A_k^{(j)}\big)\overline{\chi}\big(A_k^{(s)}\big) = \begin{cases} M, & A^{(j)} \sim A^{(s)}, \\ 0, & A^{(j)} \nsim A^{(s)}. \end{cases} \tag{8}$$

Obviously, all characters of the identity representations equal 1. Hence, for the number $S(n)$ we get the formula

$$S(n) = \frac{1}{M} \sum_{k=1}^{M} \chi(A_k), \tag{9}$$

where A_k are the matrices representing the group rotations.

Similar matrices have the same trace; and the rotations by the same angle about corresponding elements are similar.

There are t_1 vertices, t_2 faces, and t_3 edges in a regular polyhedron. At the vertices, q_1 of elements meet; the faces are regular q_2-gons; and the edges are the axes of rotations of order $q_3 = 2$. Obviously,

$$t_1 q_1 = t_2 q_2 = t_3 q_3 = M, \tag{10}$$

while also $\dfrac{1}{2}[t_1(q_1 - 1) + t_2(q_2 - 1) + t_3(q_3 - 1)] + 1 = M$; whence

$$t_1 + t_2 + t_3 = M + 2. \tag{11}$$

The sums of the traces of all rotations (including the identity) about an arbitrary vertex, center of face, or midpoint of edge are equal to[2]

$$\sum_{k=0}^{q_j-1} \sum_{m=-n}^{n} e^{i2\pi mk/q_j} = q_j \left(2\left[\frac{n}{q_j}\right]+1\right), \quad j=1,2,3. \tag{12}$$

Summing these equalities over all the axes of rotations, we note that under this summation the identity rotation is counted $M+2$ times and each other rotation twice. Considering this and using (9) and (10), we come to the following theorem.

Theorem 2.

$$S(n) = \left[\frac{n}{q_1}\right] + \left[\frac{n}{q_2}\right] + \left[\frac{k}{q_3}\right] - n + 1. \tag{13}$$

A simple computation leads to the corollary:

$$S\left(\frac{M}{2} - n - 1\right) + S(n) = 1 \quad \text{for} \quad 0 \le n \le \frac{M}{2} - 1. \tag{14}$$

Let us transform (13) into another form. Let Q^* be the set of those q_j for which $n \neq 0 \pmod{q_j}$. Expressing in (13) the integral part through its fractional part, we find[3]

$$S(n) = 1 + \frac{1}{M}\Big(2n - \sum_{q_j \in Q^*} t_j\Big) - \sum_{q_j \in Q^*} \left(\left\{\frac{n}{q_j}\right\} - \frac{1}{q_j}\right). \tag{15}$$

Since $0 \le \sum_{q_j \in Q^*} \left(\left\{\frac{n}{q_j}\right\} - \frac{1}{q_j}\right) < 1$, (15) implies

$$S(n) = \left[1 + \frac{1}{M}\Big(2n - \sum_{q_j \in Q^*} t_j\Big)\right]. \tag{16}$$

Hence,

$$S(n) = \begin{cases} \left[\dfrac{2n+1}{M}\right] & \text{for} \quad 2n+1 \le \sum\limits_{q_j \in Q^*} t_j, \\ \left[\dfrac{2n+1}{M}\right] + 1 & \text{for} \quad 2n+1 > \sum\limits_{q_j \in Q^*} t_j. \end{cases} \tag{17}$$

From (17) it follows that

$$S\left(n + \frac{M}{2}\right) = S(n) + 1. \tag{18}$$

Formula (17) admits a simple interpretation based on the following theorem.

[2] Here $[\cdot]$ denotes the integral part of a number. – *Ed.*

[3] Here $\{\cdot\}$ denotes the fractional part of a number. – *Ed.*

Theorem 3. *Let* $\{(x_k, y_k, z_k) \mid k = 1, 2, \ldots, 2n+1\}$ *be orthogonal coordinate systems with the same origin, and let the directions of the* z_k*-axes be all distinct. Then the set of functions*

$$\zeta_k^n = (x_k + iy_k)^n, \quad k = 1, 2, \ldots, 2n+1, \tag{19}$$

constitute a basis for the space of spherical functions of degree n*.*

Proof. Let us introduce the complex variable $\mathbf{z} = x + iy$, mapping the sphere onto the plane by stereographic projection[4]. For ζ_k^n we get the formula

$$\zeta_k^n = \frac{2^n \mathbf{z}_k^n}{(1 + |\mathbf{z}_k|^2)^n}, \tag{20}$$

where

$$\mathbf{z}_k = \frac{a_k \mathbf{z} - \bar{c}_k}{c_k \mathbf{z} + \bar{a}_k}, \quad |a_k|^2 + |c_k|^2 = 1. \tag{21}$$

Linear-fractional transformation (21) corresponds to rotation of the sphere which brings the z_k-axis to the z-axis of the initial coordinate system.

From (20) it follows that

$$\zeta_k^n = \sum_{m=-n}^{n} a_k^{n-m} \bar{c}_k^{n+m} R_n^{(|m|)}(\mathbf{z}\bar{\mathbf{z}}), \tag{22}$$

where the function

$$R_n^{(|m|)}(\mathbf{z}\bar{\mathbf{z}}) = c_n^{(m)} e^{im\varphi} P_n^{(|m|)}(\cos\vartheta) \tag{23}$$

differs only by the constant multiplier from the element of basis (7). Thus, the vector

$$(a_k^{2n}, a_k^{2n-1}\bar{c}_k, \ldots, \bar{c}_k^{2n}) \tag{24}$$

corresponds to the function ζ_k^n in basis (7). From (24) and the known formula for the Vandermonde determinant, Theorem 3 follows. □

From (24) it follows that for any of the $2n+2$ harmonics of the form ζ_k^n with different z_k-directions we have the equality

$$\sum_{k=1}^{2n+2} \frac{\zeta_k^n}{\prod\limits_{j \neq k} (a_k \bar{c}_j - a_j \bar{c}_k)} = 0. \tag{25}$$

Consider the set

$$\{g_\alpha x^{(k)} \mid g_\alpha \in G\}, \tag{26}$$

[4] Issuing half-lines from the south pole of the sphere, S. L. Sobolev maps the latter in one-to-one fashion onto the plane passing through the equator, i.e., $\mathbf{z} = \tan\frac{\vartheta}{2} e^{i\varphi}$. – *Ed.*

where

$$x^{(1)},\, x^{(2)},\, \ldots,\, x^{([\frac{2n+1}{M}])} \tag{27}$$

are points on the unit sphere which are not equivalent under G. The points of set (26) together with an arbitrary system of points

$$x^{(2n+1)},\, x^{(2n)},\, \ldots,\, x^{(M[\frac{2n+1}{M}]+1)}, \tag{28}$$

which are independent of (26), constitute a system of $2n+1$ independent directions of the z_k-axis. Hence, the corresponding functions $\zeta_k^n = (x_k + iy_k)^n$ constitute a basis for the space of spherical harmonics of degree n.

Clearly, the functions

$$\frac{1}{M} \sum_{g_\alpha \in G} \Big(\zeta(g_\alpha x^{(k)})\Big)^n, \quad k = 1, 2, \ldots, \left[\frac{2n+1}{M}\right], \tag{29}$$

are linearly independent invariant harmonics of degree n.

Let $(2n+1) \le \sum_{q_j \in Q^*} t_j$. Then we can choose as (28) all those directions of the axes which correspond to $q_j \in Q^*$. In this event, all invariant spherical harmonics of degree n are exhausted by functions (29), since the mean over the group of each of (29) is zero. The proof of the first half of formula (17) is complete.

For $(2n+1) > \sum_{q_j \in Q^*} t_j$ we can choose as (28) a system of mutually equivalent points, and the mean of any of the corresponding ζ_k^n taken over the group G yields one more invariant harmonic of degree n.

Let the group G^* be generated by rotations and reflections. For G^* we have the following theorem.

Theorem 4. *The group G^* has no invariant harmonics of odd degree n. The set of G^*-invariant harmonics of even degree n coincides with the set of G-invariant harmonics of degree n.*

In conclusion we present the table of values of $S(n)$, $0 \le n < M/4$, for the groups G_{IV}, G_{VIII}, and G_{XX} of rotations of the tetrahedron, octahedron, and icosahedron.

n	S_{IV}	S_{VIII}	S_{XX}	n	S_{XX}	n	S_{XX}
0	1	1	1	6	1	11	0
1	0	0	0	7	0	12	1
2	0	0	0	8	0	13	0
3		0	0	9	0	14	0
4		1	0	10	1		
5		0	0				

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[5] Paper [9] of Part II of this book. – *Ed.*

11. The Number of Nodes in Cubature Formulas on the Sphere*

S. L. Sobolev

In the preceding notes [1, 2] we considered cubature formulas on the sphere convergent in proximity order in the space of all series of spherical functions. In the present note we try to estimate asymptotically the gain obtained.

Theorem 1. *Let a system of functions*

$$\psi_1(x),\ \psi_2(x), \ldots, \psi_K(x) \tag{1}$$

be given with integrals

$$\int\limits_{\Omega} \psi_j(x)\,dx = b_j, \quad j = 1, 2, \ldots, K. \tag{2}$$

Then the formula

$$(l, f) \equiv \int\limits_{\Omega} f(x)\,dx - \sum_{k=1}^{N} c_k f(x^{(k)}) \cong 0 \tag{3}$$

with a given system of nodes is exact for all functions (1) if and only if the formula is exact for all those linear combinations

$$a_1\psi_1 + a_2\psi_2 + \cdots + a_K\psi_K, \tag{4}$$

which vanish at the nodes $x^{(k)}$.

Theorem 1 establishes the duality between the problems of interpolation and numeric integration. It is proved by comparing the system of equations

$$\sum_{k=1}^{N} \psi_j(x^{(k)}) c_k = b_j, \quad j = 1, 2, \ldots, K; \quad \Longleftrightarrow \quad A\mathbf{c} = \mathbf{b} \tag{5}$$

* Dokl. Akad. Nauk SSSR, **146**, 770–773 (1962)

for finding the coefficients c_k of cubature formula (3), and the system

$$\sum_{j=1}^{K} a_j \psi_j(x^{(k)}) = 0, \quad k = 1, 2, \ldots, N; \quad \Longleftrightarrow \quad \mathbf{a}A = 0 \tag{6}$$

for finding the coefficients a_1, a_2, ..., a_K of linear combination (4). The symbol A denotes the matrix $A = \big(\psi_j(x^{(k)})\big)$.

Remark. If the rank of A equals K then system (6) has no nontrivial solutions. Hence, there exist coefficients c_k such that (3) is valid for all ψ_j from set (1).

Theorem 2. *Let*

$$\{x_r^{(1)}, x_r^{(2)}, \ldots, x_r^{(N(r))} \mid r = 1, 2, \ldots\} \tag{7}$$

be a sequence of systems of nodes for cubature formulas (3)*. For all systems (7) beginning with a certain $r > R$ to admit of cubature formulas (3) exact for all ψ_j in given finite set (1) of analytic functions, it is sufficient that the following condition holds: there exists a domain $\Omega_0 \subset \Omega$ for which nodes* (7) *form an ε-net for any $\varepsilon > 0$, beginning with a certain $r > r(\varepsilon)$.*

Proof. The idea of the proof is to consider Kth-order determinants of the matrix A. These determinants are values of the function Δ of K variables:

$$\Delta(x^{(1)}, x^{(2)}, \ldots, x^{(K)}) = \det \begin{bmatrix} \psi_1(x^{(1)}) & \ldots & \psi_1(x^{(K)}) \\ \ldots & \ldots & \ldots \\ \psi_K(x^{(1)}) & \ldots & \psi_K(x^{(K)}) \end{bmatrix} \tag{8}$$

for different particular values of $x^{(1)}$, $x^{(2)}$, ..., $x^{(K)}$.

In the domain $\Omega_0 \times \cdots \times \Omega_0$ there exist points where determinant (8) is not zero. Since nodes (7) fall in an arbitrarily small neighborhood of any such point and $\Delta(x^{(1)}, x^{(2)}, \ldots, x^{(K)})$ is an analytic function, the matrix contains nonzero determinants. By virtue of our Remark, this implies Theorem 2. $\square$

As we have seen, the study of cubature formulas invariant under a group of rotations can be confined to those harmonics which are invariant under the same group. In trying to satisfy all conditions (5) we then have only the coefficients c_k for nonequivalent nodes. The number L of such nodes is greater than N/M, where N is the total number of nodes and M is the order of the group.

As n increases, the number $\sigma(n)$ or $\sigma^*(n)$ of invariant harmonics up to the given degree n grows slower than $L(n)$. Therefore, roughly speaking,

$$L(n) = \sigma(n) \quad \text{or} \quad L(n) = \sigma^*(n). \tag{9}$$

If the symmetry was not invoked and all the parameters taken into account were used, we would have in general the equality

$$(n+1)^2 = 3N. \tag{10}$$

Here $(n+1)^2$ is the total number of spherical harmonics up to degree n, and $3N$ is the number of degrees of freedom in formula (3): besides the coefficients c_k there are two parameters to specify each point $x^{(k)}$. For small N, formulas invariant under the icosahedron group do give such an advantage. For large N the advantage is less. It is convenient to estimate the advantage comparing the functions $N(L)$ and $n(\sigma)$ or $n(\sigma^*)$.

Let us compute $N(L)$ for the two nets obtained by projecting onto the sphere triangular nets symmetrically located on all the faces of the invariant polyhedron (see Figs. 1 and 2).

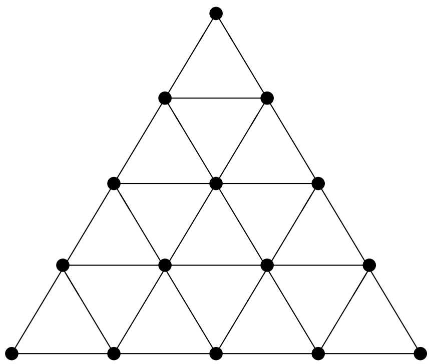

Fig. 1.

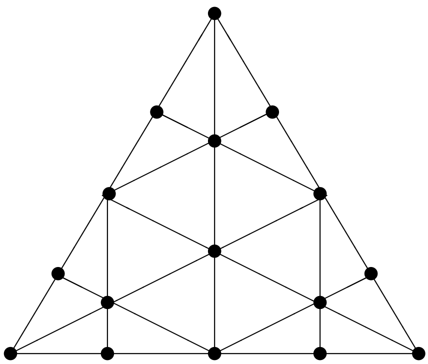

Fig. 2.

For the first type (see Fig. 1) there are k nodes on each side of the triangle. In this event, for the full rotation group and $k = 6s + r$, $r < 6$, we get

$$N = \frac{M}{6}k^2 + 2, \quad L = \begin{cases} (s+1)(k-3s)+1 & \text{for} \quad r = 0, \\ (s+1)(k-3s) & \text{for} \quad r > 0, \end{cases} \tag{11}$$

where M is the order of the rotation group of the polyhedron, equal to half the total order of the group of symmetries.

For the second type (see Fig. 2), with $k = 2s + r$ and $r < 2$:

$$N = \frac{M}{2}k^2 + 2, \quad L = \begin{cases} (s+1)^2 & \text{for} \quad r = 0, \\ (s+1)(s+2) & \text{for} \quad r = 1. \end{cases} \tag{12}$$

This yields, for example, for $k = 6s + 5$ in the first case,

$$N(L) = 2M\big(L - \sqrt{3}\,L^{1/2} + \dots\big), \tag{13}$$

and for $k = 2s + 1$ in the second case,

$$N(L) = 2M\bigl(L - 2\,L^{1/2} + \dots\bigr). \tag{14}$$

Computing σ and σ^* for

$$n = \frac{KM}{2} - 1 \tag{15}$$

in the manner indicated in preceding note [2], we get

$$\sigma(K\frac{M}{2} - 1) = K\sigma(\frac{M}{2} - 1) + \frac{K(K-1)}{2}\frac{M}{2}, \tag{16}$$

$$\sigma^*(K\frac{M}{2} - 1) = K\sigma^*(\frac{M}{2} - 1) + \frac{K(K-1)}{2}\frac{M}{4}, \tag{17}$$

$$\sigma(M/2-1) = M/4, \quad \sigma^*(M/2-1) = \begin{cases} 5 & \text{for} \quad G_{VIII}, \\ 11 & \text{for} \quad G_{XX}. \end{cases}$$

Hence, $\sigma(KM/2-1) = K^2M/4$, which means that $[n(\sigma)+1]^2 = M\sigma$, and further that

$$(n^*+1)^2 = 2M\bigl(\sigma^* - \sqrt{49/30}\,\sigma^{*\,1/2} + \dots\bigr) = 2M\bigl(\sigma^* - 1.3\,\sigma^{*\,1/2} + \dots\bigr) \tag{18}$$

for the icosahedron group and

$$(n^*+1)^2 = 2M\bigl(\sigma^* - \sqrt{4/3}\,\sigma^{*\,1/2} + \dots\bigr) = 2M\bigl(\sigma^* - 1.16\,\sigma^{*\,1/2} + \dots\bigr) \tag{19}$$

for the octahedron group.

The comparison of (13) and (14) with (18) and (19) for large L exhibits the gain obtained by using invariant formulas of the type described. It is apparent that this gain is not large.

Setting $L = \sigma^*$ for the small values of n, we tabulate the functions $n(N)$, $(n(N)+1)^2$, and $L(N)$ for the formulas of the first and second types which are invariant under the group of octahedron and icosahedron.

G^I_{VIII}				G^{II}_{VIII}				G^I_{XX}				G^{II}_{XX}			
N	n	$(n{+}1)^2$	L	N	n	$(n{+}1)^2$	L	N	n	$(n{+}1)^2$	L	N	n	$(n{+}1)^2$	L
6	3	16	1	14	5	26	2	12	5	36	1	32	9	100	2
18	5	32	2	50	9	100	4	42	9	100	2	122	15	256	4
38	7	64	3	110	11	144	6	92	11	144	3	272	19	400	6
66	9	100	4	194	15	256	9	162	15	258	4	482	25	676	9
102	11	144	5					262	17	324	5				
146	13	196	7												
198	15	256	8												
258	17	324	10												

Formulas with four points for the icosahedron group were given in the paper of V. A. Ditkin and L. A. Lyusternik [3].

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[1] Paper [9] of Part II of this book. – *Ed.*
[2] Paper [10] of Part II of this book. – *Ed.*

12. Certain Questions of the Theory of Cubature Formulas*

S. L. Sobolev

Let $l(x)$ be a generalized function such that

$$l(x) = \chi_\Omega(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}), \tag{1}$$

where $\chi_\Omega(x)$ is the characteristic function of the domain Ω. The values of the functional

$$(l, \varphi) = \int l(x)\varphi(x)\, dx \tag{2}$$

are the errors of a certain cubature formula, and we discuss this functional keeping in mind this formula. Suppose that

$$(l, x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{3}$$

and introduce the Lax norm

$$\|l \mid L_2^{(-m)}\| = \inf_{\varphi \neq 0} \frac{|(l, \varphi)|}{\|\varphi \mid L_2^{(m)}\|}. \tag{4}$$

Here $m > n/2$ and $\|\varphi \mid L_2^{(m)}\|^2 = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2\, dx$. The best is a cubature formula whose error functional has a lesser norm.

The explicit expression of the norm of $l(x)$ can be obtained by solving a variation problem for the Euler equation

$$\Delta^m u(x) = (-1)^m l(x) \tag{5}$$

with the corresponding boundary conditions. The equality holds

$$\|l \mid L_2^{(-m)}\| = \frac{|(l(x), u(x))|}{\|u \mid L_2^{(m)}\|}.$$

* Proc. Joint Soviet-American Sympos. Partial Differential Equations (Novosibirsk, 1963). Inst. Mat., Akad. Nauk SSSR Sibirsk. Otdel., Novosibirsk (1963), 8 p.

Theorem 1. *In the norm*

$$\left\|\varphi \mid V_2^{(m)}(\Omega)\right\| = \inf_{\substack{\widetilde{\varphi}(x)=\varphi(x)\\ x\in\Omega}} \left\|\widetilde{\varphi} \mid L_2^{(m)}(R^n)\right\| \tag{6}$$

the required solution of (5) *can be explicitly written as*

$$u_0(x) = G(x) * l(x), \tag{7}$$

where

$$G(x) = (-1)^m \varkappa_{m,n} |x|^{2m-n} \begin{cases} \ln|x|, & \text{if} \quad n \quad \text{even and} \quad n \le 2m, \\ 1, & \text{if} \quad n \quad \text{odd or} \quad n > 2m. \end{cases} \tag{8}$$

The proof of Theorem 1 is based on an estimate of convergence of the integrals

$$\int D^\alpha G(x-y)\,dx \quad \text{for} \quad |\alpha| < m.$$

In the periodic case the problems under study can be also solved explicitly. Let H be the matrix of periods,

$$H = \bigl(\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n\bigr), \tag{9}$$

$$\det\, H = |H| = 1. \tag{10}$$

Let us consider the set of functions such that

$$\varphi(x + H\beta) = \varphi(x), \tag{11}$$

where $x \in R^n$ is a coordinate column vector and β is an arbitrary integer column vector. We denote by Ω_0 the fundamental parallelohedron of the matrix H. It means that

$$\sum_\beta \chi_{\Omega_0}(x + H\beta) = 1. \tag{12}$$

Let the error functional of the cubature formula be defined as the generalized function

$$\widehat{l}(x) = \chi_{\Omega_0}(x) - \delta(x). \tag{13}$$

Then we have the next theorem.

Theorem 2. *The norm of* $\widehat{l}(x)$ *can be written as*

$$\|\widehat{l}(x) \mid \widetilde{L}_2^{(-m)}\| = \frac{|(\widehat{l}(x), u_0(x))|}{\|u_0(x) \mid \widetilde{L}_2^{(m)}\|}, \tag{14}$$

where $u_0(x)$ *is the periodic solution of equation* (5) *such that*

$$u_0(x) = -\left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma\neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}}\, e^{i2\pi H^{-1}x\cdot\gamma}. \tag{15}$$

From Theorem 2 it follows that

$$\|\widehat{l}(x) \mid \widetilde{L}_2^{(-m)}\|^2 = \frac{1}{(2\pi)^{2m}} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} = \frac{1}{(2m)!} B^2_{n,m}(H^{-1*}). \tag{16}$$

The vectors $H^{-1*}\gamma$ are the nodes of the lattice dual to the lattice with nodes $H\beta$, and $|H^{-1*}\gamma|$ is the distance of the nodes to the coordinate origin.

For large m, i.e., for many times differentiable functions, the term corresponding to the shortest of the distances mentioned dominates in (16):

$$B^2_{n,m}(H^{-1*}) \approx \frac{s}{r^{2m}_{\min}}. \tag{17}$$

Here s is the number of nodes of the lattice $H^{-1*}\gamma$ at minimal distance from the coordinate origin. Therefore, the optimal lattice is given by the nodes $H\beta$, for which the vectors $H^{-1*}\gamma$ constitute the lattice corresponding to the densest packing of balls in n-dimensional space.

For the bounded domain of integration and for the given lattice of nodes hH with a small mesh-size h, we construct cubature formulas with uniform boundary layer; they are obtained by summing cubature formulas for all elementary cells.

Assume that

$$l_0(x) = \chi_\Omega\left(\frac{x}{h}\right) - \sum_{|\beta'| \leq L} c[\beta']\delta\left(\frac{x}{h} - H\beta'\right), \tag{18}$$

and let $(l_0(x), x^\alpha) = 0$ for $|\alpha| \leq m$. Let B_1 be the set of all β such that $l_0(x - hH\beta)$ is supported in the interior of the domain Ω. We compose the sum

$$l_1(x) = \sum_{\beta \in B_1} l_0(x - hH\beta) = \chi_{\Omega^*}(x) - \sum_{\beta' \in B} c^*[\beta']\delta(x - hH\beta'), \tag{19}$$

where B is the set of all β' such that $hH\beta' \in \Omega$. The equality holds

$$\chi_\Omega(x) - \chi_{\Omega^*}(x) = \sum_{\beta \in B \setminus B_1} \chi_\Omega(x)\chi_{\Omega_0}(x - hH\beta). \tag{20}$$

For each β from $B \setminus B_1$ we consider a cubature formula with the error functional defined by

$$l_\beta(x) = \chi_\Omega(x)\chi_{\Omega_0}(x - hH\beta) - \sum_{\substack{|\beta'| \leq L \\ H(\beta+\beta') \in \Omega}} c^{\beta'}[\beta]\delta(x - H\beta - H\beta'). \tag{21}$$

Suppose that $\sup\limits_{\beta,\beta'} |c^{\beta'}[\beta]| \leq A$, and let

$$l_2(x) = \sum_{\beta \in B \setminus B_1} l_\beta(\frac{x}{h}). \tag{22}$$

We refer to the cubature formula with the error functional $l(x) = l_1(x) + l_2(x)$ as the *normal cubature formula*. Let

$$m(x) = \sum_{|\beta'| \le L} d[\beta']\delta(x - hH\beta'), \tag{23}$$

$$(m(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m. \tag{24}$$

Then we consider the sum

$$M(x) = \sum_{\beta \in B \setminus B_1} m(x - hH\beta) = \sum_{\beta} F[\beta]\delta(x - hH\beta). \tag{25}$$

This sum is equal to zero at all nodes $hH\beta$ of the lattice lying at a distance greater than $2Lh$ from the boundary of Ω. Hence, the discrete function $F[\beta]$ is supported inside a certain boundary layer of the boundary of Ω.

Theorem 3. *Let the generalized function* $M(x) = \sum_{\beta} F[\beta]\delta(x - hH\beta)$ *be equal to zero at all points* $hH\beta$ *lying at a distance greater than* $2Lh$ *from the boundary of* Ω*, and let* $(M(x), x^\alpha) = 0$ *for* $|\alpha| < m$*. Then* $M(x)$ *can be written as*

$$M(x) = \sum_{\beta \in B \setminus B_1} m_\beta(x), \tag{26}$$

where $m_\beta(x) = \sum_{\beta'} F^\beta[\beta']\delta(x - hH(\beta + \beta'))$ *for* $\beta \in B \setminus B_1$, $(m_\beta(x), x^\alpha) = 0$ *for* $|\alpha| < m$*, and the set* $B \setminus B_1$ *consists of the nodes in a certain expanded boundary layer with width* KL*.*

We call the function expanded like (26) the *normal homogeneous boundary layer with the order* m*.*

Corollary. *Two normal cubature formulas differ from each other by a normal boundary layer with the order* m*.*

Theorem 3 is proved by using a special technique of the partial summation over each variable in turn and the replacement of the integration domain by an approximate domain with the coordinate planes taken for its boundaries.

There is an analogy between operators orthogonal to x^α, $|\alpha| \le m$, and the differential operators with constant coefficients $L(D) = \sum_{\gamma} a_\gamma D^\gamma$, where $|\gamma| > m$. The integral over the volume of differential expressions with such operators is expressed as a surface integral containing derivatives of order higher than $m - 1$.

Theorem 4. *In the space* $V_2^{(m)}(\Omega)$ *the value of the extremal function of the normal cubature formula of order* m *at any interior point of* Ω *tends to the value of the periodic extremal function with the lattice* hH*. From above the errors of the normal cubature formula can be estimated as*

$$|(l, \varphi)| \le h^m B_{n,m}(H^{-1*})\sqrt{|\Omega|}\ \|\varphi \mid V_2^{(m)}(\Omega)\| + O(h^{m+1}).$$

Proof. The extremal function of a normal cubature formula can be expanded in the sum

$$G(x) * \Big(l_1(x) + l_2(x)\Big) = \sum_{\beta \in B_1} G(x) * l_0(x - hH\beta) + \sum_{\beta \in B \setminus B_1} G(x) * l_\beta(x - hH\beta).$$

Each term of the first sum on the right side decreases as $|x|/h \to \infty$ not slower than $|H\beta|^{-n-1}$, and each term of the second sum decreases not slower than $|H\beta|^{-n}$. Therefore, as $h \to 0$, the first sum converges to the periodic solution of the equation $\Delta^m u_0(x) = (-1)^m \big[1 - h^n \sum_\gamma \delta(x - hH\gamma)\big]$ absolutely, and the second sum tends to zero. □

From Theorem 4 it follows that the quality of normal cubature formulas is determined mainly by the properties of the lattice. Therefore, for large m, the optimal lattice is again dual to the lattice with nodes that are the centers of the balls constituting a densest packing in n-dimensional space.

It is convenient to construct normal cubature formulas using the Fourier transform.

Theorem 5. *For the given generalized function $l(x)$ to be orthogonal to all polynomials of degree $m-1$, it is necessary and sufficient that its Fourier image $\widetilde{l}(p)$ would have a zero of multiplicity m at the coordinate origin and the integrals $\displaystyle\int l(x) x^\alpha dx$ for $|\alpha| < m$ would have meaning.*

The proof is elementary.

Rejecting the second requirement of Theorem 5, we can construct normal cubature formulas for infinite domains such as a half-space and s-faced solid angles with rational faces. Also, we can construct boundary layers for polyhedral domains. In fact, such cubature formulas have been constructed in the cases listed.

Theorem 6. *In the case of a polyhedron, the cubature formula with boundary layer coinciding in a neighborhood of each s-faced solid angle of the polyhedron with the boundary layer constructed for the corresponding infinite solid angle is a normal cubature formula.*

13. A Method for Calculating the Coefficients in Mechanical Cubature Formulas*

S. L. Sobolev

For a given set of nodes, it is often possible to construct a mechanical cubature formula that is exact for arbitrary polynomials of a given degree by means of the Fourier transform. Let the problem require seeking for c_k such that the functional

$$(l, \varphi) = \int\limits_{\Omega} \varphi(x)\,dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}) \equiv \int l(x)\varphi(x)\,dx \tag{1}$$

vanishes at all polynomials of degree $m-1$. Here the generalized function $l(x)$ is defined by

$$l(x) = \chi_{\Omega}(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}), \tag{2}$$

where $\chi_{\Omega}(x)$ is the characteristic function of the domain Ω, and $\delta(x - x^{(k)})$ is the generalized Dirac delta function.

Theorem 1. *For the functional $l(x)$ to vanish at all polynomials of degree $m-1$, it is necessary and sufficient that its Fourier transform $\widetilde{l}(p)$ has a zero of multiplicity m at the origin.*

Proof. From the condition of the theorem it follows that for all α with $|\alpha| \leq m-1$, we have

$$l(x) * x^{\alpha} = 0, \tag{3}$$

where $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Since the Fourier transform of the convolution is transformed into a product of the Fourier images and the Fourier transform of x^{α} is $(2\pi)^{n/2} D^{\alpha}\delta(p)$, the equality holds

$$D^{\alpha}\widetilde{l}(p) = 0. \tag{4}$$

Formula (4) means that there exists a zero of multiplicity m of the function $\widetilde{l}(p)$ at the origin. □

* Dokl. Akad. Nauk SSSR, **150**, 1238–1241 (1963)

It is convenient to apply Theorem 1 to that domain Ω for which the Fourier transform of the characteristic function $\chi_\Omega(x)$ can be calculated in a finite form. For example, ellipsoids or polyhedrons possess this property. In this case, the use of Theorem 1 leads to a system of linear equations for c_k. This system is generally underdetermined, i.e., the number of its equations is less than the number of its unknowns. By finding the solutions of this system, we construct the required coefficients.

Theorem 1 was formulated for bounded domains. If the domain Ω is unbounded, then in general integrals (3) do not exist. However, the Fourier transform of the characteristic function $\chi_\Omega(x)$ sometimes also retains its sense in the case of unbounded domains.

The generalized function $l(x)$ is called a *function of order* m if it has a generalized Fourier transform $\widetilde{l}(p)$, which is an m times continuously differentiable function in a neighborhood about the coordinate origin, and there is a zero of order m of $\widetilde{l}(p)$ at the coordinate origin. Also an arbitrary linear combination of functions of order m is a function of order m.

In certain cases the use of generalized functions of order m makes it possible to calculate coefficients for cubature formulas of degree $m-1$. Let us give an example.

Let functions of one variable be defined on the infinite interval as follows:

$$\psi_0(x) = 1, \quad \Phi_0(x) = \sum_{k=-\infty}^{+\infty} \delta(x-k), \quad \psi_1(x) = \begin{cases} 1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \le 0, \end{cases}$$

$$\Phi_1(x) = \frac{1}{2}\delta(x) + \sum_{k=1}^{\infty} \delta(x-k),$$

$$\chi_0(x) = \sum_{k=-\infty}^{+\infty} \delta\Big(x-k-\frac{1}{2}\Big), \qquad \chi_1(x) = \sum_{k=0}^{\infty} \delta\Big(x-k-\frac{1}{2}\Big).$$

In n-dimensional space the characteristic functions of coordinate s-faced solid angles $\Omega^{(j_1,\ldots,j_s)}$ can be represented in the form

$$\chi^{(j_1,j_2,\ldots,j_s)}(x) = \psi_1(x_{j_1})\psi_1(x_{j_2})\ldots\psi_1(x_{j_s}). \tag{5}$$

Using a linear transformation we may write the characteristic functions of the s-faced solid angles between hyperplanes in arbitrary directions as a product like (5).

The characteristic function of the parallelepiped $\{x \mid 0 < x_j < a_j\}$ can be written as a sum of functions like (5). For example, in two dimensions we obtain

$$\chi_\Omega(x,y) = \chi^{(1,2)}(x,y) + \chi^{(1,2)}(a_1-x,y) + \chi^{(1,2)}(x,a_2-y)$$
$$+\chi^{(1,2)}(a_1-x,a_2-y) - \chi^{(1)}(x,y) - \chi^{(1)}(a_1-x,y)$$

$$-\chi^{(2)}(x, y) - \chi^{(2)}(x, a_2 - y) + 1.$$

There is an analogous formula for an arbitrary dimension n.

Let all differences

$$\chi^{(j_1, j_2, \dots, j_s)}(x) - \sum_{k \in K} c_k \delta(x - x^{(k)})$$

be generalized functions of order m. Then there exists a linear combinations of these functions such that it is an error functional of a certain cubature formula in a bounded parallelepiped. This cubature formula is exact for all polynomials of degree $m - 1$.

Instead of the parallelepiped, we can consider an arbitrary convex polyhedron with rational faces and seek for a cubature formula of order m for each unbounded s-faced solid angle $\Omega^{(j_1, j_2, \dots, j_s)}$ of the polyhedron in the form

$$\chi^{(j_1, j_2, \dots, j_s)}(x) - c \sum_{k \in K_1} \delta(x - x^{(k)}) - \sum_{k \in K_2} c_k \delta(x - x^{(k)}).$$

Here nodes $x^{(k)}$ are members of some parallelepipedal system of points. For $k \in K_1$ the nodes $x^{(k)}$ are all points of the lattice which lie in the interior of the domain $\Omega^{(j_1, j_2, \dots, j_s)}$, and for $k \in K_2$ the nodes $x^{(k)}$ are points of a boundary layer which lie in $\Omega^{(j_1, j_2, \dots, j_s)}$, at a finite distance from the boundary of the domain. In future publications we will establish that the choice of c and c_k may be realized in such a way that the corresponding cubature formula is close to an optimal formula in a known sense.

It is convenient to define the coefficients for the nodes in the boundary layer by distinguishing these nodes according to their order. We call a set $K_2^{(r)}$ a *boundary layer of order* r if it consists of points at distance r from the coordinate planes. We have:

$$K_2^{(n)} \cup \ldots \cup K_2^{(2)} \cup K_2^{(1)} = K_2.$$

The coefficients c_k for the nodes in $K_2^{(r)}$ are defined in such a way that they are common for all domains

$$\Omega^{(j_1, j_2, \dots, j_r)}, \ \Omega^{(j_1, j_2, \dots, j_{r+1})}, \ \dots, \ \Omega^{(1, 2, \dots, n)}.$$

The Fourier transform makes it possible to calculate all coefficients for nodes in boundary layers of order r. The cubature formulas so obtained are regular in a certain sense, which we shall indicate later.

Theorem 2. *Let Ω be an arbitrary bounded convex polyhedron with rational faces. For error functional*

$$\chi_\Omega(x) - c \sum_{k \in K_1} \delta(x - x^{(k)}) - \sum_{k \in K_2} c_k \delta(x - x^{(k)})$$

to vanish at all polynomials of degree $m-1$*, it is sufficient that the coefficients for nodes in the boundary layer* K_2 *are the same as the coefficients for nodes in unbounded boundary layers of* s*-faced solid angles formed by the boundaries of the polyhedron.*

Given an optimal periodic lattice in three-dimensional space, we have carried out the computation of the coefficients for nodes in the simplest boundary layers.

In [1] it was established that for a lattice defined by a lattice matrix H with $|H| = 1$ and for the space of periodic functions $\widetilde{L}_2^{(m)}$ the error has the bound

$$|(l,\varphi)|^2 \le \frac{1}{(2\pi)^{2m}} \sum_{\gamma\neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} \left\|\varphi \mid L_2^{(m)}\right\|^2 .$$

Hence, the constant

$$\sum_{\gamma\neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} \tag{6}$$

gives the quality measure of the lattice. In a later publication we will prove that the same constant (6) also estimates the quality of the lattice in the nonperiodic case.

From (6) it follows that at m large, only the first term $1/r_{\min}^{2m}$ of the total sum (6) is significant. Here, $r_{\min}$ is the shortest distance between points of the lattice with the lattice matrix H^{-1*}, i.e., $r_{\min}$ is the maximal diameter of the disjoint spheres centered at the nodes of the lattice with the matrix H^{-1*}. From this it follows that the optimal lattice is that for which the diameter $r_{\min}$ is the largest. Hence, the volumes of the spheres are the largest as well. In a large domain for different H with $|H| = 1$ the number of spheres is constant and numerically equal to the volume V of the domain. Hence, the optimal lattice H^{-1*} must be the lattice *with the closest packing of spheres* in n dimensions[1].

From this it follows that for $n = 3$ the optimal H^{-1} is the face-centered cubic lattice, and the optimal H is the centered cubic lattice. For the optimal lattice we have constructed formulas for the two-faced and three-faced solid angles.

In calculating the coefficients of such formulas, the method is based on the Fourier transform of the functions Φ_j, ψ_j, and χ_j under consideration. The formulas for these transforms are

$$\widetilde{\Phi}_0(p) = \sqrt{2\pi} \sum_{k=-\infty}^{+\infty} \delta(p - 2k\pi),$$

[1] The algorithm for finding the closest packing was proposed by G. F. Voronoi. For small dimensions the description of the closest packing can be found in Chap. II of the book: Sobolev, S. L.: Introduction to the Theory of Cubature Formulas. Nauka, Moscow (1974). – *Ed.*

$$\widetilde{\chi}_0(p) = \sqrt{2\pi} \sum_{k=-\infty}^{+\infty} (-1)^k \delta(p - 2k\pi),$$

$$\widetilde{\psi}_0(p) = \sqrt{2\pi}\delta(p),$$

$$\widetilde{\Phi}_1(p) = \sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{+\infty} \delta(p - 2k\pi) + \frac{i}{2\sqrt{2\pi}} \cot\frac{p}{2},$$

$$\widetilde{\chi}_1(p) = \sqrt{\frac{\pi}{2}} \sum_{k=-\infty}^{+\infty} (-1)^k \delta(p - 2k\pi) + \frac{i}{2\sqrt{2\pi}\sin\dfrac{p}{2}},$$

$$\widetilde{\psi}_1(p) = \sqrt{\frac{\pi}{2}}\delta(p) + \frac{i}{\sqrt{2\pi}p}.$$

L. V. Voitisek has carried out numerical calculations of the coefficients for the boundary layer in such formulas. For the three-faced solid angle $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, one may write the formula for the error functional in the following way:

$$l(x) = \chi^{(1,2,3)}(x) - \frac{1}{2}\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) - \frac{1}{2}\chi_1(x_1)\chi_1(x_2)\chi_1(x_3)$$

$$-\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) \sum_{j_3=1}^{3} \sum_{k=0}^{2} \frac{\alpha_{2k}}{\Phi_1(x_{j_3})} \delta(x_{j_3} - k)$$

$$-\chi_1(x_1)\chi_1(x_2)\chi_1(x_3) \sum_{j_3=1}^{3} \sum_{k=0}^{1} \frac{\alpha_{2k+1}}{\chi_1(x_{j_3})} \delta(x_{j_3} - k - \frac{1}{2})$$

$$-\Phi_1(x_1)\Phi_1(x_2)\Phi_1(x_3) \sum_{j_2=1}^{3} \sum_{j_1=j_2}^{3} \sum_{k,l=0}^{2} \frac{\alpha_{2k,2l}}{\Phi_1(x_{j_1})\Phi_1(x_{j_2})} \delta(x_{j_1} - k)\delta(x_{j_2} - l)$$

$$-\chi_1(x_1)\chi_1(x_2)\chi_1(x_3) \sum_{j_2=1}^{3} \sum_{j_1=j_2}^{3} \sum_{k,l=0}^{2} \frac{\alpha_{2k+1,2l+1}}{\chi_1(x_{j_1})\chi_1(x_{j_2})}\delta(x_{j_1} - k - \frac{1}{2})\delta(x_{j_2} - l - \frac{1}{2})$$

$$- \sum_{k_1,k_2,k_3=0}^{2} \alpha_{2k_1,2k_2,2k_3}\delta(x_1 - k_1)\delta(x_2 - k_2)\delta(x_3 - k_3)$$

$$- \sum_{k_1,k_2,k_3=0}^{1} \alpha_{2k_1+1,2k_2+1,2k_3+1}\delta(x_1 - k_1 - \frac{1}{2})\delta(x_2 - k_2 - \frac{1}{2})\delta(x_3 - k_3 - \frac{1}{2}).$$

It turns out that, for the coefficients α, we may take the following values:

$\alpha_0 = -0.850694444 \cdot 10^{-1}$	$\alpha_{000} = -0.335894549 \cdot 10^{-2}$
$\alpha_2 = -0.116666667$	$\alpha_{002} = \alpha_{020} = \alpha_{200} = 0.309787330 \cdot 10^{-2}$
$\alpha_4 = -0.93750000 \cdot 10^{-2}$	$\alpha_{022} = \alpha_{202} = \alpha_{220} = 0.527777778 \cdot 10^{-1}$
$\alpha_1 = +0.160416667$	$\alpha_{222} = \alpha_{224} = \alpha_{242} = \alpha_{442} = \alpha_{333} = 0$
$\alpha_3 = +0.506944444 \cdot 10^{-1}$	$\alpha_{004} = \alpha_{040} = \alpha_{400} = 0.969939054 \cdot 10^{-1}$
$\alpha_{00} = +0.318612557 \cdot 10^{-1}$	$\alpha_{024} = \alpha_{042} = \alpha_{204} = \alpha_{402} =$
$\alpha_{02} = \alpha_{20} = 0.633680555 \cdot 10^{-1}$	$\alpha_{240} = \alpha_{420} = 0.63964836 \cdot 10^{-2}$
$\alpha_{22} = +0.361111111 \cdot 10^{-1}$	$\alpha_{440} = \alpha_{404} = \alpha_{044} = 0.210458262 \cdot 10^{-3}$
$\alpha_{04} = \alpha_{40} = 0.372902200 \cdot 10^{-1}$	$\alpha_{442} = \alpha_{424} = \alpha_{244} = 0.969509547 \cdot 10^{-2}$
$\alpha_{24} = \alpha_{42} = 0.607638888 \cdot 10^{-1}$	$\alpha_{444} = 0.927847402 \cdot 10^{-2}$
$\alpha_{44} = +0.320818866 \cdot 10^{-2}$	$\alpha_{111} = 0.114626736$
$\alpha_{11} = -6.127097801$	$\alpha_{113} = \alpha_{131} = \alpha_{311} = 0.247395833 \cdot 10^{-1}$
$\alpha_{13} = \alpha_{31} = -0.444299768 \cdot 10^{-1}$	$\alpha_{133} = \alpha_{313} = \alpha_{331} = 0.742187498 \cdot 10^{-2}$
$\alpha_{33} = +0.484664352 \cdot 10^{-2}$	$\alpha_{333} = 0.$

By the same method a boundary layer is computed near a boundary separating domains with lattices of different densities, if such domains exist.

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[2] Paper [7] of Part II of this book. – *Ed.*

14. On the Rate of Convergence of Cubature Formulas*

S. L. Sobolev

The subject of the present note is the estimation of the norm of the error of cubature formulas in a domain Ω of n independent variables. We consider the error of a cubature formula as a linear functional of the form

$$l(x) = \chi_\Omega(x) - \sum_{t=1}^{N} c_t \delta(x - x^{(t)}) \tag{1}$$

in the space $L_2^{(m)}(R^n)$. The function $\varphi(x)$ with domain R^n is a member of $L_2^{(m)}(R^n)$ provided that it has all derivatives up to order m locally integrable and the norm

$$\|\varphi \mid L_2^{(m)}\| = \left\{ \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx \right\}^{1/2} \tag{2}$$

is finite. The space $L_2^{(m)}(R^n)$ is the quotient space of $W_2^{(m)}$ over the space of polynomials of degree $m-1$. By $\chi_\Omega(x)$ in (1) we denote the characteristic function of the domain Ω, the symbol $|\alpha|$ for a vector α with integer entries means $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$. Here it is necessary to assume that

$$m > n/2 \tag{3}$$

and

$$(l(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m. \tag{4}$$

Let $|\Omega|$ be the volume of the domain Ω and let

$$\frac{|\Omega|}{N} = h^n. \tag{5}$$

* Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)

Theorem 1. *There exists a constant* K_1 *depending only on* m *and* n *such that*

$$\|l \mid L_2^{(m)*}\| \geq K_1\sqrt{|\Omega|}h^m. \tag{6}$$

Proof. The proof of this theorem is based on estimates similar to those given by N. S. Bakhvalov in the proof of theorems like Theorem 1 in other functional spaces.

Let

$$Q_j = \left\{x \mid \left|x_s - x_s^{(j,0)}\right| < k/2,\ s = 1, 2, \ldots, n\right\}$$

be a cube with the edge length k, and require that Q_j does not contain the nodes $x^{(t)}$ of the error (1) on its boundary or near it to a distance ηk, where $\eta > 0$. If

$$(l(x), \chi_{Q_j}(x)) > \eta_2 k^n, \tag{7}$$

where $\eta_2 > 0$, then Q_j is called a *cube with insufficient data for the error* $l(x)$.

Lemma 1. *Let* Q_j, $j = 1, 2, \ldots, N_1$, *be a system of disjoint cubes with insufficient data for the error* $l(x)$ *in the domain* Ω, *and let the sum of volumes* Q_j *be greater than some positive constant*

$$\sum_{j=1}^{N_1} |Q_j| > |\Omega_1|.$$

Then the norm of the error $l(x)$ *satisfies the inequality*

$$\|l(x) \mid L_2^{(m)*}\| \geq K_1\sqrt{|\Omega_1|}k^m. \tag{8}$$

Proof. As in the paper by N. S. Bakhvalov, the proof of (8) consists of a direct estimate of the error $l(x)$ on a certain function consisting of a sum of "hats" over each cube of a system with insufficient data for the error $l(x)$. □

Theorem 1 follows from Lemma 1. Let us cover the domain Ω with a system of cubes generating a cubic lattice with side $k_1 = 2^{-1/n}h$ and consider all cubes with the edge length $k < (1 - \eta_3)k_1$, concentric with the cubes of this lattice. The number of these concentric cubes is obviously no less than $2N$. Since they do not contain common points, at least half of them contain none of the nodes $x^{(t)}$, and hence they are cubes with insufficient data for the error $l(x)$. Their total volume $|\Omega_1|$ is greater than $(1 - \eta_3)^n|\Omega|/2$, from which Theorem 1 follows. □

Lemma 1 also shows that the main source of the error of the cubature formula is the irregularity of distribution of the nodes $x^{(t)}$, and this distribution cannot be made perfect.

The estimate given by Theorem 1 is attainable, as the following theorem shows.

Theorem 2. *Let the error $l(x)$ be written as the sum*

$$l(x) = \sum_{\gamma} l_\gamma \left(\frac{x}{h} - \gamma\right), \tag{9}$$

where γ ranges over the points of the integral lattice, and let each $l_\gamma(y)$ satisfy the conditions

$$(l_\gamma(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \le m, \tag{10}$$

$$\|l_\gamma(y) \mid L_2^{(m)*}\| \le A, \tag{11}$$

$$\operatorname{supp} l_\gamma(y) \subset \{y \mid |y| \le L\}, \tag{12}$$

where $\operatorname{supp} l_\gamma(y)$ *denotes the support of $l_\gamma(y)$. Then for the norm of $l(x)$ the following inequality is valid:*

$$\|l \mid L_2^{(m)*}\| \le K_2 h^m. \tag{13}$$

Here the constant K_2 depends on the domain Ω, the numbers A and L, but K_2 does not depend on the functionals $l_\gamma(y)$.

Proof. Before proving this theorem, it is useful to note that for a domain with piece-wise smooth boundaries and given numbers A and L, it is always possible to construct for sufficiently small h an infinite set of functionals permitting representation (9). Indeed, we may always decompose the domain Ω in the union of cells

$$\Omega = \bigcup_{\gamma} \Omega_\gamma, \tag{14}$$

where Ω_γ is a cell lying at a distance not greater than Lh from the point $x = h\gamma$:

$$\operatorname{dist}(\Omega_\gamma, h\gamma) \le Lh. \tag{15}$$

The characteristic function $\chi_{\Omega_\gamma}(x)$ may be written as

$$\chi_{\Omega_\gamma}(x) = \chi_{\Omega^*_\gamma}(\frac{x}{h} - \gamma), \tag{16}$$

where $\chi_{\Omega^*_\gamma}(y)$ is the characteristic function of some bounded domain Ω^*_γ. By the classical method of extrapolation, we can construct in the domain Ω^*_γ a cubature formula, which is exact for all polynomials of degree $m-1$ and has the error functional

$$l_\gamma(y) = \chi_{\Omega^*_\gamma}(y) - \sum_{|\gamma'| \le L} c_\gamma^{(\gamma')} \delta(y - \gamma'). \tag{17}$$

The nodes of $l_\gamma(y)$ are those points of the lattice where

$$h(\gamma + \gamma') \in \Omega. \tag{18}$$

In this case, the error $l(x)$ defined by (9) satisfies all the conditions of Theorem 2.

Let us point out the idea of the proof of Theorem 2. As has been established [1–3], the norm of the error $l(x)$ may be expressed by means of a solution of the polyharmonic equation

$$\varDelta^m u = (-1)^m l(x). \tag{19}$$

For the norm of $l(x)$ the equality holds

$$\|l \mid L_2^{(m)*}\| = \frac{|(l,u)|}{\|u \mid L_2^{(m)}\|} = \|u \mid L_2^{(m)}\|. \tag{20}$$

To find the solution u of (19) it is convenient to use the elementary solution of the polyharmonic equation

$$G(x) = (-1)^m \varkappa_{m,n} |x|^{2m-n} \begin{cases} 1, & \text{for} \quad n \quad \text{odd or} \quad n > 2m, \\ \ln|x|, & \text{for} \quad n \quad \text{even and} \quad n \le 2m. \end{cases} \tag{21}$$

In this case, we apply the known formula for the inner product:

$$(\phi, \psi) = \Big(\phi(x) * \psi(-x)\Big)\Big|_{x=0}. \tag{22}$$

From (20) and (22) it follows that

$$\|l \mid L_2^{(m)*}\|^2 = (l,u) = l(x) * G(x) * l(-x)\Big|_{x=0}. \tag{23}$$

By virtue of the fact that $l(x)$ and $l(-x)$ are finite generalized functions, the triple convolution on the right side of (23) is associative and commutative. Substituting in (23) the expressions for $l(x)$ and $l(-x)$ from (9), we have

$$\begin{aligned} \|l \mid L_2^{(m)*}\|^2 &\le \sum_{\gamma_1}\sum_{\gamma_2} \left| l_{\gamma_1}\left(\frac{x}{h} - \gamma_1\right) * G(x) * l_{\gamma_2}\left(-\frac{x}{h} + \gamma_2\right)\Big|_{x=0} \right| \\ &= \sum_{\gamma_1}\sum_{\gamma_2} \left| G(x) * \left(l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(-\frac{x}{h}\right)\right)\Big|_{x=h(\gamma_1+\gamma_2)} \right|. \end{aligned} \tag{24}$$

It is not difficult to establish the equalities

$$l_1\left(\frac{x}{h}\right) * l_2\left(\frac{x}{h}\right) = h^n l_3\left(\frac{x}{h}\right), \tag{25}$$

where $l_3(y) = l_1(y) * l_2(y)$, and

$$\left\| l\left(\frac{x}{h}\right) \mid L_2^{(m)*} \right\| = h^{n/2+m} \|l(y) \mid L_2^{(m)*}\|, \tag{26}$$

$$\|\varphi(x) \mid L_2^{(m)}(hx \in \varOmega)\| = h^{n/2-m} \|\psi(y) \mid L_2^{(m)}(y \in \varOmega)\|, \tag{27}$$

where $\psi(y) = \varphi(hy)$, i.e., $\varphi(x) = \psi(x/h)$. The convolution $l_3(y)$ possesses the properties

$$\|l_3(y) \mid L_2^{(m)*}\| \le \|l_1(y) \mid L_2^{(m)*}\| \|l_2(y) \mid L_2^{(m)*}\| \le A^2; \tag{28}$$

$$(l_3(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \le 2m+1; \tag{29}$$

$$\operatorname{supp} l_3(y) \subset \{y \mid |y| \le 2L\}. \tag{30}$$

Lemma 2. *The following estimate holds:*

$$\left| G(x) * l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(\frac{x}{h}\right) \right| \le K \, \frac{A^2 h^{2n+2m+2}}{(h^2+|x|^2)^{n/2+1}}, \tag{31}$$

where the constant K does not depend on h, l_{γ_1}, and l_{γ_2}.

Proof. For $|x| \le 3Lh$ inequality (31) follows from (25)–(30). In order to prove (31) for $|x| \ge 3Lh$, we expand $G(x-y)$ in a power series in y in a neighborhood about the point $y = 0$:

$$G(x-y) = \sum_{|\alpha|<2m+2} \frac{(-y)^\alpha}{\alpha!} D^\alpha G(x) + R_{2m+2}(x,y). \tag{32}$$

It is obvious that for $|y| \le 2L$ the function $R_{2m+2}(x,y)$ satisfies the inequality

$$|D_y^\alpha R_{2m+2}(x,y)| \le K \, |x|^{-n-2}. \tag{33}$$

Hence,

$$\|R_{2m+2}(x,y) \mid L_2^{(m)*}(|y| \le 2Lh)\| \le K \, h^{m-n/2} |x|^{-n-2}. \tag{34}$$

Since

$$G(x) * l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(\frac{x}{h}\right) = \int l_3(\frac{y}{h}) G(x-y)\, dy = (l_3(\frac{y}{h}), G(x-y)),$$

where $l_3 = l_{\gamma_1} * l_{\gamma_2}$, (31) follows from (26)–(30) and (34). □

Theorem 2 is obtained from Lemma 2 and (24). The double sum (24) may be estimated from above by applying the integral criterion. □

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[1] Paper [7] of Part II of this book. – *Ed.*

15. Theory of Cubature Formulas*

S. L. Sobolev

The connection between functional analysis and computational mathematics, completely realized in the recent two decades, is too broad to discuss it as a whole. Therefore, in this report I restrict myself only to one aspect of this connection, the theory of approximate integration of functions of many independent variables. In this area, it is possible to obtain a number of important results by applying functional analytic methods.

To each formula of mechanical cubature used for integration of the function φ from X, where X is a certain Banach space, there corresponds the linear error functional defined by

$$(l, \varphi) = \int_{\Omega} \varphi(x)\, dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}), \tag{1}$$

$$l(x) = \chi_{\Omega}(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}). \tag{2}$$

The subject of our study is the norm of this functional:

$$\|l \mid X^*\|. \tag{3}$$

To the different nodes $(x^{(k)})$ and the coefficients (c_k) there correspond different cubature formulas. It is important to study them, and to minimize the norm $\|l \mid X^*\|$.

Of course, in practical questions the value of the error (l, φ) for each individual function is more important than the norm of $l(x)$. For any continuous function this error tends to zero. There is always a weak convergence. However, it is difficult to estimate the error, and therefore it is useful to apply the formula with the least norm of the error functional in the space X^*.

* Wiss. Z. Hochsch. Architektur Bauwesen Weimar. Jahrgang 12 (1965). Heft 5/6, S. 537–546. (III International Colloquium on Application of Mathematics in Engineering, June 27 – July 4, 1965, Weimar)

The choice of the space X, where we consider a cubature formula, is the matter of the insight of a researcher and his intuition pointing to what properties of the function, e.g., its smoothness, we should pay the greatest attention. This choice also determines the quality of chosen cubature formulas.

However, following the traditions accepted in mathematics from A. M. Lyapunov's times, a problem stated mathematically has to be mathematically solved in strict terms.

The norm of the error functional $l(x)$ characterizes the degree of approximation of the functional $\chi_\Omega(x)$, the characteristic function of the domain Ω, by the functionals of specific form

$$\sum_{k=1}^{N} c_k \delta(x - x^{(k)}) = R_N(x), \tag{4}$$

as N increases. The study of $l(x)$ reduces to the study of different functionals of form (4). To minimize the norm of the error functional, we can change:

a) the coefficients c_k for given nodes $x^{(k)}$ and N;
b) the disposition of the nodes $x^{(k)}$ for fixed N;
c) the number N of the nodes.

These three problems, composing three steps of the search for the best integration formulas, are the particular problems of such general problems of functional analysis as approximations in the functional space. It is difficult to point out traditional approaches to solving this problem because of its enormous complexity, and generally speaking, choice of a solution method is significantly based on the intuition of the researcher.

Many scientists have offered different approaches. The study of this branch of mathematics has often resembled a list of more or less successful prescriptions such as the formulas of Simpson, Gregory, Gauss, Chebyshev, and others. It seems that in our time the state of the problem has begun to change, and the general methods of functional analysis are changing the theory of cubature formulas in front of our eyes.

As we have already mentioned, the main problem in question is the problem about the error functionals with the least norms. However, somewhat later we will also consider certain questions about the rate of convergence to zero of the error of the formula for individual functions. We will show that there is an essential difference in the estimates obtained under the two approaches to the problem.

It turns out that $|(l, \varphi)|$ for an individual function φ is significantly less than $\|l \mid X^*\| \|\varphi \mid X\|$ in a large number of cases, and in particular, in the Hilbert spaces that we study.

Besides the Banach spaces X, it is also convenient to consider approximate integration in certain countably normed spaces, for example, in the spaces of infinitely differentiable functions, which often appear in applications. We will also discuss this question.

Recently, approximate integration in different functional spaces X has been studied in the literature. Many papers are devoted to approximate integration over cubes of periodic functions that have some derivatives integrable with degrees exceeding 1. Also, there are some results in other spaces. However, it is not my task to discuss all such results here, and the major part of my presentation is devoted to the mechanical cubature formulas in the spaces $L_2^{(m)}$ of functions defined on the whole Euclidean n-dimensional space R^n, whose derivatives of order m are square integrable. The norm squares of such functions may be written as

$$\left\|\varphi \mid L_2^{(m)}\right\|^2 = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx, \tag{5}$$

with an integral that is invariant under orthogonal transformations of the variable x from R^n. In (5), as commonly accepted today, α is a vector with integer nonnegative entries, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, and $D^\alpha \varphi$ stands for the derivative

$$\frac{\partial^m \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

It is surely assumed that the functional $l(x)$ has to be defined on $L_2^{(m)}$. Hence, for all polynomials of degree $m-1$ the values of $l(x)$ must be equal to zero.

Also, the number m must satisfy the inequality $m > n/2$, for the value of function $\varphi(x)$ at a fixed point to be the linear functional in $L_2^{(m)}$ which is defined on the whole space. In other words, for $m > n/2$ the following embedding holds

$$L_2^{(m)} \subset C, \quad \|\varphi \mid C\| \le K \|\varphi \mid L_2^{(m)}\|. \tag{6}$$

Besides the functions φ with domain R^n, we also consider the periodic functions φ defined in the bounded domain Ω_0 with fixed periods.

Assuming that x is a column vector, and writing the periods of the function $\varphi(x)$ in the form of columns of the square matrix H of periods, we write the periodicity condition for the function $\varphi(x)$ as

$$\varphi(x + H\gamma) = \varphi(x), \quad \forall x \in R^n, \tag{7}$$

with γ an arbitrary column vector with integer entries. In this case the integration domain is chosen as the fundamental domain Ω_0, i.e., in such a way that

$$\sum_\gamma \chi_{\Omega_0}(x + H\gamma) = 1. \tag{8}$$

It is often assumed that the volume of the fundamental domain is equal to 1:

$$\det H = |H| = 1. \tag{9}$$

For a domain with the volume h^n, the matrix of periods may be written as hH.

We consider the approximate integration formula over an arbitrary domain Ω as the approximation of the functional $\chi_\Omega(x)$ in the space $L_2^{(m)*}$. As we know from the Calderon theory, an arbitrary function from $L_2^{(m)}(\Omega)$ can be continued to the whole space R^n. Hence, in $L_2^{(m)}(\Omega)$ we can introduce the new norm

$$\left\|\varphi \mid V_2^{(m)}(\Omega)\right\| = \inf_{\substack{\overline{\varphi}(x)=\varphi(x)\\ x\in\Omega}} \left\|\overline{\varphi} \mid L_2^{(m)}(R^n)\right\|. \tag{10}$$

It is easy to see that

$$\left\|l \mid V_2^{(m)*}(\Omega)\right\| = \left\|l \mid L_2^{(m)*}(R^n)\right\|, \tag{11}$$

which simplifies our problem.

The computation of the $L_2^{(m)*}$-norm of a given functional is the simplest problem of variations calculus on the minimum of the quadratic form $\left\|l \mid \ L_2^{(m)*}\right\|^2$. It can be reduced in a classical fashion to the solution of the partial differential equation

$$\Delta^m u(x) = (-1)^m l(x) \tag{12}$$

in $L_2^{(m)}$. Problem (12) turns out to be solvable because of the conditions

$$(l, x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{13}$$

where x^α denotes the product $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Equality (13) expresses the orthogonality of $l(x)$ to all polynomials of degree $m-1$, which we mentioned above.

Under condition (13) the solution of equation (12) may be explicitly expressed through the fundamental solution

$$G_{m,n}(x) = \varkappa_{m,n} |x|^{2m-n} \begin{cases} \ln|x|, & \text{if} \quad n \quad \text{even and} \quad n \le 2m, \\ 1, & \text{if} \quad n \quad \text{odd or} \quad n > 2m, \end{cases} \tag{14}$$

of the polyharmonic equation. The equality holds

$$u(x) = (-1)^m G_{m,n}(x) * l(x) \equiv G(x) * l(x). \tag{15}$$

As usual, the symbol $*$ stands for the convolution of generalized (or regular) functions

$$\varphi(x) * \psi(x) = \int \varphi(x-y)\psi(y)\,dy. \tag{16}$$

Using the convolution of two regular functions (or, the convolution of a generalized function with a regular one), their inner product can be written as

$$(\varphi, \psi) = \left[\varphi(x) * \overline{\psi}(-x)\right]\Big|_{x=0}. \tag{17}$$

From (16) we obtain the explicit expression for the norm of $l(x)$:

$$\left\| l \mid L_2^{(m)*} \right\|^2 = (l, u) = (l(x) * G(x), l(x)) = \left[l(x) * G(x) * l(-x) \right] \Big|_{x=0}. \tag{18}$$

Let us consider the periodic case with the matrix hH of periods in more detail. Put

$$l_0(x) = 1 - h^n \sum_{\gamma} \delta(x - hH\gamma), \tag{19}$$

where γ ranges over the set of all possible vectors with integer entries. Such error functional is convenient for integration of functions with periods $hH\Lambda$ over the fundamental domain Ω_0, where Λ is a diagonal integer matrix. It is convenient to map the fundamental domain Ω_0 onto the torus in R^n. In this case the periods of the function $\varphi(x)$ are multiples of periods of the generalized function $l_0(x)$. Moreover, for integration of compactly-supported functions in R^n, it is convenient to use the same error functionals, and hence, the same cubature formula.

In the case of periodic functions, generating on the torus Ω_0 the space $\widetilde{L}_2^{(m)}(\Omega_0)$, the norm of $l_0(x)$ is again computed using the solution of equation (13). This is convenient to do, using the Fourier expansions for generalized functions, studied by L. Schwartz, and also by I. M. Gelfand and G. E. Shilov. The solution of the equation

$$\Delta^m u_0(x) = (-1)^m l_0(x) \tag{20}$$

in the class of functions that are members of $\widetilde{L}_2^{(m)}(\Omega_0)$ and have a zero average over Ω_0, can be written as the Fourier series

$$u_0(x) = -\left(\frac{h}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}} e^{i2\pi H^{-1} x \cdot \gamma / h}. \tag{21}$$

From (21) it follows that in the periodic case the norm of the functional $l_0(x)$ may be written as

$$\left\| l_0 \mid \widetilde{L}_2^{(m)*} \right\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega_0|}, \tag{22}$$

where $B_{2m}(H^{-1*})$ stands for the expression

$$\begin{aligned} B_{2m}(H^{-1*}) &= (-1)^{m-1} \frac{(2m)!}{(2\pi)^{2m}} \zeta(H^{-1*} \mid 2m) \\ &= (-1)^{m-1} (2m)! \left(\frac{1}{2\pi}\right)^{2m} \sum_{\gamma \neq 0} \frac{1}{|H^{-1*}\gamma|^{2m}}. \end{aligned} \tag{23}$$

Here $\zeta(H^{-1*} \mid 2m)$ is the known Epstein zeta function of the quadratic form $\psi(\gamma) = |H^{-1*}\gamma|^{2m}$.

The same parameters are involved in the estimate of the values of (l_0, φ) for an arbitrary compactly-supported function $\varphi(x)$ from $L_2^{(m)}(R^n)$. Using the Green identity and the Cauchy–Bunyakovskii–Schwarz inequality, we obtain the estimate

$$\big|(l_0, \varphi)\big| \le h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{S(\varphi)}\big(1 + O(h)\big), \tag{24}$$

where $S(\varphi)$ is a finite volume of a support of the function φ.

The main idea of our theory is to connect the value of the norm of the linear functional with the uniformity of its distribution. In what follows we expose the specifics of this idea and systematically perform it. Let

$$h = \left(\frac{|\Omega|}{N}\right)^{1/n}. \tag{25}$$

Theorem 1. *Let a system of cubes Ω_j in the domain Ω be of total volume $|\Omega'|$, the length of the edge of Ω_j be equal to K, and the part in Ω_j of the functional $l(x)$ be insufficiently defined, i.e., let this part integrate the identity over Ω_j with the significant nonpositive error*

$$\int \chi_{\Omega_j}(x) l(x)\, dx > qK^n, \quad q > 0. \tag{26}$$

Then the functional $l(x)$ has a norm satisfying the condition

$$\big\| l \mid L_2^{(m)*} \big\| \ge \eta K^n \sqrt{|\Omega'|}, \tag{27}$$

where η is a positive constant.

Since the cubes with edges $h/2$, not containing the nodes $x^{(k)}$ of $l(x)$, always occupy more than half of the volume Ω and the functional $l(x)$ is insufficiently defined in each of these cubes, then for no cubature formula in $L_2^{(m)}$ is it possible to obtain the norm of the functional $l(x)$ less than Kh^m:

$$\big\| l \mid L_2^{(m)*} \big\| \ge Kh^m. \tag{28}$$

We omit the proof. It consists of the construction of a certain special function φ from $L_2^{(m)}$, for which the value (l, φ) is greater than $\eta K h^m \sqrt{|\Omega'|} \big\| \varphi \mid L_2^{(m)} \big\|$.

By definition, the functional $l(x)$ is *completely equidistributed over Ω*, if it may be written as

$$l(x) = \sum_{hH\gamma \in \Omega} l_\gamma\left(\frac{x}{h} - H\gamma\right), \tag{29}$$

where the support of $l_\gamma(y)$ lies in the ball of radius L,

$$\operatorname{supp} l_\gamma(y) \subset \{y : |y| \le L\}, \tag{30}$$

$l_\gamma(y)$ is orthogonal to all polynomials of degree $m-1$,

$$(l_\gamma(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{31}$$

and the norm of $l_\gamma(y)$ in C^* is bounded by a constant A, the same for all γ,

$$\|l_\gamma(y) \mid C^*\| \le A. \tag{32}$$

Theorem 1 solves the question about the order of the norm of the error, while the following theorem establishes the attainability of this order.

Theorem 2. *The norm of the completely equidistributed functional $l(x)$ satisfies the inequality*

$$\left\| l(x) \mid L_2^{(m)*} \right\| \le K h^m. \tag{33}$$

Proof. The proof is based on the estimate of the quadratic form

$$\left\| l \mid L_2^{(m)*} \right\|^2 = \sum_{hH\gamma, hH\gamma' \in \Omega} \left(l_\gamma \left(\frac{x}{h} - H\gamma \right), l_{\gamma'} \left(\frac{x}{h} - H\gamma' \right) * G(x) \right). \tag{34}$$

It turns out that under the conditions

$$(l_\gamma(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \le s_1, \quad (l_{\gamma'}(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| \le s_2, \tag{35}$$

where

$$s_1 + s_2 > 2m - n, \tag{36}$$

the following estimate holds:

$$\left| G(x) * l_\gamma \left(\frac{x}{h} \right) * l_{\gamma'} \left(\frac{x}{h} \right) \right| \le K \frac{A^2 h^{2n+s_1+s_2}}{(h^2 + |x|^2)^{-m+(n+s_1+s_2)/2}}, \tag{37}$$

where the constant K does not depend on h, l_γ, $l_{\gamma'}$. Formula (37) means that the functionals $l_\gamma\left(x/h - H\gamma\right)$ and $l_{\gamma'}\left(x/h - H\gamma'\right)$ become "more orthogonal" in the sense of the inner product $\left(l_\gamma\left(x/h - H\gamma\right), l_{\gamma'}\left(x/h - H\gamma'\right) * G(x)\right)$ when their supports move away from each other. Using (37) and applying the integral majorant we estimate the right side of (34) and prove (33). □

Theorems 1 and 2 establish the order of convergence of cubature formulas. In the next part of the presentation we establish the principal term in the expansion of the norm of the error functional with the given lattice of nodes $hH\gamma$. Also, we consider cubature formulas where this optimal value of the norm is attained. The corresponding conclusions are based on several theorems.

Theorem 3. *Let $l_*(y)$ satisfy conditions* (30), (32), *and*

$$(l_*(y), y^\alpha) = 0 \quad \textit{for} \quad |\alpha| < 2m + 2. \tag{38}$$

Also, let

$$\sum_\gamma l_* \left(\frac{x}{h} - H\gamma\right) = l_0(x) = 1 - h^n \sum_\gamma \delta(x - hH\gamma). \tag{39}$$

Then the solution of equation (20) *can be written as*

$$u_0(x) = \sum_\gamma \left(l_*(\frac{y}{h}), G(x - hH\gamma - y)\right) + c. \tag{40}$$

Proof. For example, the functional $l_*(y)$ with the conditions of Theorem 3 may be constructed by integration of the interpolation formula.

To prove (40), it suffices to establish the estimate

$$\left| \left(l_*(\frac{y}{h}), G(x-y)\right) \right| = \left|G(x) * l_*(\frac{x}{h})\right| \le K|x|^{-n-1}, \tag{41}$$

from where the convergence of the series on the right side of (40) follows, and to use the uniqueness of solution (20) to within an additive constant. □

The functional of the form

$$k(y) = \sum_{|\gamma|<L} c_\gamma \delta(y - H\gamma) \tag{42}$$

with bounded coefficients and a bounded support, consisting of points of the lattice, is called the *point functional of order s* provided that

$$(k(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < s. \tag{43}$$

Theorem 4 (Summation by parts). *Let Ω be a domain with a smooth boundary and let $k(y)$ be a point functional of order s. Then the sum*

$$m(x) = \sum_{hH\gamma\in\Omega} k\left(\frac{x}{h} - H\gamma\right) \tag{44}$$

may be written as

$$m(x) = \sum_{hH\gamma\in B_2} k_\gamma\left(\frac{x}{h} - H\gamma\right), \tag{45}$$

where B_2 is the set of points that lie at a distance less than Lh from the boundary Γ of Ω, and each functional $k_\gamma(y)$ is the point functional of order $s-1$.

Proof. The proof is based on an easily established expansion of an arbitrary point functional of order s in a sum of differences in each variable of certain functionals of order $s-1$:

$$k(y) = \sum_{j=1}^n \widehat{\Delta}_j k_j(y). \tag{46}$$

□

By definition, the error functional $l(x)$ is an *error functional with regular boundary layer, width* L*, order* m*, and estimate* A, if it may be written down as

$$l(x) = \sum_{hH\gamma\in B_1} l_*\left(\frac{x}{h} - H\gamma\right) + \sum_{hH\gamma\in B_2} l_\gamma\left(\frac{x}{h} - H\gamma\right), \tag{47}$$

in which $B_1 = \Omega \setminus B_2$, all l_γ and l_* satisfy (30)–(32), and l_* satisfies (38).

Let

$$(l_1(y), y^\alpha) = 0 \quad \text{for} \quad |\alpha| < m+1. \tag{48}$$

Whence and from Theorem 4 it follows that

$$\sum_{hH\gamma\in B_1} l_1\left(\frac{x}{h} - H\gamma\right) - \sum_{hH\gamma\in B_1} l_*\left(\frac{x}{h} - H\gamma\right) = \sum_{hH\gamma\in B_2} l_\gamma\left(\frac{x}{h} - H\gamma\right), \tag{49}$$

where l_γ has order m. Hence, in (47) we could use just (48) instead of (38) for l_*.

From (47) and (19) it follows that $l(x) = l_0(x) + l_2(x)$, where

$$l_2(x) = \sum_{hH\gamma\in B_2} l_\gamma\left(\frac{x}{h} - H\gamma\right) - \sum_{hH\gamma\notin\Omega} l_*\left(\frac{x}{h} - H\gamma\right). \tag{50}$$

All coefficients c_γ of the error functional with regular boundary layer are equal to h^n at the points that lie at a distance greater than $2Lh$ from the boundary Γ of Ω, since in this condition $c_\gamma = \sum_{\gamma'} c^*_{\gamma'}$, where $c^*_{\gamma'}$ are the coefficients of $l_*(y)$.

Theorem 5. *The extremal function* $u(x)$ *of the functional* $l(x)$ *with regular boundary layer may be written as*

$$u(x) = G(x) * l(x) + P_{m-1}(x),$$

where $P_{m-1}(x)$ *is a polynomial of degree* $m-1$ *and*

$$G(x) * l(x) = \sum_{hH\gamma\in B_1} G(x) * l_*\left(\frac{x}{h} - H\gamma\right)$$

$$+ \sum_{hH\gamma\in B_2} G(x) * l_\gamma\left(\frac{x}{h} - H\gamma\right) = u_0(x) + \sum_{hH\gamma\in B_3} l'_\gamma\left(\frac{x}{h} - H\gamma\right) * G(x)$$

$$+ \sum_{hH\gamma\in B_4} l_*\left(\frac{x}{h} - H\gamma\right) * G(x) = u_0(x) - w(x). \tag{51}$$

The set B_3 *consists of points that lie at a distant less than* Lh *from the boundary* Γ *of* Ω*, and* $B_4 = (R^n \setminus \Omega) \setminus B_3$.

Theorem 6. *The norm of the error functional with regular boundary layer of order m is expressed as*

$$\left\| l \mid L_2^{(m)*} \right\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega|} + O(h^{m+1}). \tag{52}$$

Proof. By (51) we have:

$$\left\| l \mid L_2^{(m)*} \right\|^2 = (l, u) = (l, u_0 - w) = (l, u_0) - (l, w). \tag{53}$$

The direct computation of (l, u_0) gives

$$(l, u_0) = h^{2m} \frac{|B_{2m}(H^{-1*})|}{(2m)!} |\Omega_0| (1 + O(h)). \tag{54}$$

The second term (l, w) may be written as

$$(l, w) = \sum_{hH\gamma \in B_1 \cup B_2} \sum_{hH\gamma' \in B_3 \cup B_4} \left[l_\gamma \left(\frac{x}{h} - H\gamma \right) * G(x) * l'_{\gamma'} \left(\frac{x}{h} - H\gamma' \right) \right], \tag{55}$$

where $hH\gamma$ ranges over the nodes lying in Ω, i.e., the set of nodes from $B_1 \cup B_2$, and $hH\gamma'$ ranges over the set of nodes from $B_3 \cup B_4$.

Using (37) and applying the integral majorant, we have the final result, i.e., the estimate

$$|(l, w)| \leq K h^{2m+1}, \tag{56}$$

where K depends only on L and A.

Theorem 6 immediately follows from (54) and (56). □

In Theorem 6 we establish the principal term of the norm of the error functional with regular boundary layer. As we see in the proof, we cannot change the value of this principal term by increasing the order of $l_*(y)$ from m up to any other number. In the following theorems we establish directly that the norm of the optimal error functional with the given lattice of nodes $hH\gamma$ has the same principal term.

Theorem (Babuška). *For the given lattice of nodes $hH\gamma$, coefficients $c^{(0)}[\gamma]$ of optimal error functional*

$$l^{(0)}(x) = \chi_\Omega(x) - \sum_{hH\gamma \in \Omega} c^{(0)}[\gamma] \delta(x - hH\gamma) \tag{57}$$

are such that the solution $u(x)$ of (12) vanishes at all nodes of the formula:

$$u(hH\gamma) = 0 \quad \textit{for} \quad hH\gamma \in \Omega. \tag{58}$$

Proof. Equalities (58) are equivalent to the fact that the convolution

$$G(x) * l^{(0)}(x) \tag{59}$$

coincides with a certain polynomial of degree $m-1$ at all nodes $hH\gamma$ from Ω. If (58) would not hold, then on the set of all $c[\gamma]$, subject to the conditions

$$\sum_{hH\gamma\in\Omega} c[\gamma]\,(hH\gamma)^{\alpha} = \int\limits_{\Omega} x^{\alpha}\,dx \quad \text{for} \quad |\alpha| < m, \tag{60}$$

there would exist directions such that the directional derivative of the polynomial

$$\psi(c) = [l(x) * G(x) * l(-x)]\Big|_{x=0} \tag{61}$$

of the second degree with respect to the variables $c[\gamma]$ would be nonzero, which is impossible at the minimum point. □

Theorem 7. *The difference of the norm square of an arbitrary error functional $l(x)$ in $L_2^{(m)*}$ and the norm square of the optimal error functional $l^{(0)}(x)$ is expressed as*

$$\begin{aligned} &\left\| l \mid L_2^{(m)*} \right\|^2 - \left\| l^{(0)} \mid L_2^{(m)*} \right\|^2 \\ &= \sum_{\substack{hH\beta\in\Omega \\ hH\beta'\in\Omega}} G\big(hH(\beta-\beta')\big)\big(c[\beta]-c^{(0)}[\beta]\big)\big(c[\beta']-c^{(0)}[\beta']\big). \end{aligned} \tag{62}$$

In other words, this difference is a quadratic form with respect to the differences of coefficients of the cubature formulas under consideration, and the matrix of this quadratic form has the elements $G\left(hH(\beta-\beta')\right)$.

Theorem 7 means that the difference between $\psi(c)$ and its minimum is the second-order value expressed by the quadratic form $\Xi(c[\beta]-c^{(0)}[\beta])$ of increments of the independent variables.

In view of Theorem 7, the deviation of the norm of the given error functional from the least possible one reduces to the study of the quadratic form of a large number of independent variables.

We use (62) to show that the norm of the optimal error functional differs from the norm of an error functional with regular boundary layer by a value of higher order of smallness.

Let us assume that the coefficients $c[\beta]$ and $c^{(0)}[\beta]$ are given not only for $hH\beta \in \Omega$, but also for all β, moreover, let

$$c[\beta] = c^{(0)}[\beta] = 0 \quad \text{for} \quad hH\beta \notin \Omega. \tag{63}$$

Then, as it is easy to show, the form under consideration is reduced to the infinite convolution

$$\left\| l \mid L_2^{(m)*} \right\|^2 - \left\| l^{(0)} \mid L_2^{(m)*} \right\|^2 = \Xi\big(c[\beta]-c^{(0)}[\beta]\big)$$

$$= \Big[(c[\beta] - c^{(0)}[\beta]) * G(hH\beta) * (c[-\beta] - c^{(0)}[-\beta]) \Big] \Big|_{\beta=0} \tag{64}$$

with respect to the discrete argument β. The quadratic form $\Xi(c[\beta] - c^{(0)}[\beta])$ is the generalization of the form studied above,

$$\int \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u(x) D^\alpha v(x)\, dx = \left[l_1\left(\frac{x}{h}\right) * G(x) * l_2\left(-\frac{x}{h}\right) \right] \Big|_{x=0}, \tag{65}$$

and it is the discrete analogue of (65).

Theorem 4 serves as the beginning of the theory of functions defined on a lattice. Let us continue this theory. For our goals it is necessary to develop the theory of the form $\Lambda(U, V)$, in the same way by which we develop the theory of the corresponding integral form (65) using the polyharmonic equation.

First we consider the discrete potential

$$U^*[\beta] = \sum_{\beta'} G[hH(\beta - \beta')]c[\beta'] = G[hH\beta] * c[\beta], \tag{66}$$

which is similar to the polyharmonic potential. It is useful to note that, by Babuška's theorem, for the given functional $l(x)$, values of the potential

$$U[\beta] = G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) \tag{67}$$

are known for $hH\beta \in \Omega$.

Indeed, it is easy to see that for $hH\beta \in \Omega$,

$$G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) = \Big[G(x) * \Big(l(x) - l^{(0)}(x) \Big) \Big] \Big|_{x=hH\beta}. \tag{68}$$

However, $G(x) * l^{(0)}(x) \Big|_{x=hH\beta} = P_{m-1}(hH\beta)$, and therefore

$$G[hH\beta] * (c[\beta] - c^{(0)}[\beta]) = u(hH\beta) - P_{m-1}(hH\beta) \quad \text{for} \quad hH\beta \in \Omega. \tag{69}$$

Theorem 8. *The operator of convolution with $G[hH\beta]$, i.e., the discrete potential, has a difference inverse operator $L_h[\beta]$ such that*

$$L_h[\beta] * G[hH\beta] = \delta[\beta] = \begin{cases} 1 & \textit{for} \quad \beta = 0, \\ 0 & \textit{for} \quad \beta \neq 0; \end{cases} \tag{70}$$

$$L_h[\beta] * [\beta]^\alpha = 0 \quad \textit{for} \quad |\alpha| < 2m - 1; \tag{71}$$

$$|L_h[\beta]| \leq e^{-\eta|\beta|/h}, \tag{72}$$

where η is a positive constant.

Thus, the operation $L_h[\beta]*$ is an analogue of the usual discrete difference polyharmonic operator $\widehat{\Delta}^m$ with the only difference that the function $L_h[\beta]$ is not compactly-supported, but rather decreases exponentially at infinity.

The proof of Theorem 8 is based on an application of the Fourier transform, and in the case of functions of a discrete argument it reduces to the search for such periodic functions of the variable x with periods $2\pi h^{-1}H^{-1}$, for which $G[hH\beta]$ and $L_h[\beta]$ are the Fourier coefficients.

Let $U[\beta]$ and $V[\beta]$ be compactly-supported functions. Let us compose the bilinear form

$$\Lambda(U,V) = U[\beta] * L_h[\beta] * V[-\beta]\Big|_{\beta=0}. \tag{73}$$

Theorem 9. *The form $\Lambda(U,V)$ can be extended by continuity on the space $l_2^{(m)}$ of the functions $U[\beta]$ and $V[\beta]$ of the discrete argument with square summable differences of order m. The corresponding quadratic form $\Lambda(U,U)$ satisfies the inequalities*

$$0 < M_1\|U \mid l_2^{(m)}\| \le \Lambda(U,U) \le M_2\|U \mid l_2^{(m)}\| < \infty. \tag{74}$$

The proof of this theorem can be conducted by study of the Fourier transform in detail. We can also prove it directly using the following expansion of the operator $L_1[\beta]$:

$$L[\beta] \equiv L_1[\beta] = (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} L^\alpha[\beta] * L^\alpha[-\beta]. \tag{75}$$

Equality (75) is the generalization of the known formula

$$\Delta^m = (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha(x) * D^\alpha(-x). \tag{76}$$

From (75) it follows that the form $\Lambda(U,U)$ expands in the following sum of squares:

$$\Lambda(U,U) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \big|L^\alpha(U,U)\big|^2. \tag{77}$$

First the expansion (77) is established for all compactly-supported functions, and then for all functions from $l_2^{(m)}$. The operator $L^\alpha[\beta]*$ turns out to be an analogue of the finite difference Δ^α of order α.

The proof of (77) is based on the following lemma, first proved for differentiable functions from $L_2^{(m)}$, and then for difference functions from $l_2^{(m)}$.

Lemma (on the density of compactly-supported functions). *Each function φ from $L_2^{(m)}$ can be expressed as the limit of a sequence of compactly-supported functions:*

$$\varphi(x) + P_{m-1}(x) = \lim_{\substack{\eta\to\infty \\ L_2^{(m)}}} \Big[\varphi(x) + P_{m-1}(x)\Big]\xi\left(\frac{\ln|x|}{\ln\eta}\right), \tag{78}$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$ and $\xi(\lambda)$ is a truncator, i.e., a function with the properties

$$\xi(\lambda) = \begin{cases} 0 & \text{for} \quad \lambda > 1, \\ 1 & \text{for} \quad \lambda < 1/2, \end{cases} \tag{79}$$

$|\xi(\lambda)| \le 1$, *and moreover $\xi(\lambda)$ has continuous derivatives of all orders. A property like (78) holds for the members of $l_2^{(m)}$ as well.*

Using the form Λ, let us introduce the Hilbert space $\mathfrak{G}$ with the inner product

$$\{U, V\} = U[\beta] * L_h[\beta] * V[-\beta]\Big|_{\beta=0}, \tag{80}$$

defined for every pair of compactly-supported functions. Applying the lemma on the density of compactly-supported functions we establish the formula

$$\{U, V\} = \lim_{\eta\to\infty} \{U_\eta, V_\eta\} = (U[\beta], L_h * V[\beta]) \tag{81}$$

in the case when at least one of $U[\beta]$ and $V[\beta]$ is compactly-supported.

Theorem 10. *Let $\varrho[\beta]$ relate with $V[\beta]$ by*

$$V[\beta] = G[hH\beta] * \varrho[\beta], \tag{82}$$

and $\varrho[\beta] * [\beta]^\alpha = 0$ for $|\alpha| < m$. Then[1]

$$\Xi(\varrho[\beta]) = \Lambda(V[\beta], V[\beta]). \tag{83}$$

Equality (83) is the generalization of the usual equality

$$xFx^* = yF^{-1}y^*, \quad y = xF, \tag{84}$$

from the theory of quadratic forms on a finite number of variables. Equality (84) is proved by accurate testing of the validity of the associative law in the formula

$$yF^{-1}FF^{-1}y^* = yF^{-1}y^* \tag{85}$$

for symmetric finite matrices.

The continuous analogue of (83) may be written as

$$\int\!\!\int \varrho(x)\varrho(y)G(x-y)\,dxdy = \int \sum_{|\alpha|=m} \frac{m!}{\alpha!}\Big|D^\alpha u(x)\Big|^2 dx, \tag{86}$$

[1] Here $\Xi(\varrho[\beta]) = \varrho[\beta] * G[hH\beta] * \varrho[-\beta]\Big|_{\beta=0}$. – *Ed.*

where

$$u(x) = G(x) * \varrho(x). \tag{87}$$

Equality (86) is valid for any compactly-supported function $\varrho(x)$ such that

$$(\varrho(x), x^\alpha) = 0 \quad \text{for} \quad |\alpha| < m, \tag{88}$$

and we used this when deriving formula (18) for the norm of the error functional.

Theorem 11. *Let $U^*[\beta]$ be a discrete potential with the compactly-supported density $\varrho[\beta]$,*

$$U^*[\beta] = G[hH\beta] * \varrho[\beta], \tag{89}$$

$$\operatorname{supp} \varrho[\beta] = \Omega. \tag{90}$$

The $\Lambda(U^*, U^*)$ is less than $\Lambda(U, U)$ for all those functions $U[\beta]$, which take the same values as $U^*[\beta]$ at the points of Ω.

Proof. Let $W[\beta]$ vanish at the points of Ω and be a member of $l_2^{(m)}$. Then this function is orthogonal to $U^*[\beta]$ in the inner product of $\mathfrak{S}$. Indeed, for $W[\beta] \in l_2^{(m)}$ we have

$$\Lambda(U^*, W) = (U^*[\beta] * L_h[\beta], W[\beta]) = \sum_\beta (U^*[\beta] * L_h[\beta]) \cdot W[\beta].$$

In the last sum all terms are equal to zero, since the function $W[\beta]$ vanishes at the points of Ω, and the convolution $L_h[\beta] * U^*[\beta]$ is equal to zero at the others. Hence, $\Lambda(U^*, W) = 0$ and

$$\Lambda(U^* + W, U^* + W) = \Lambda(U^*, U^*) + \Lambda(U^*, W) + \Lambda(W, U^*) + \Lambda(W, W). \tag{91}$$

From this it follows that

$$\Lambda(U^* + W, U^* + W) \geq \Lambda(U^*, U^*). \tag{92}$$

However, $U^* + W$ is an arbitrary function from $l_2^{(m)}$ coinciding with U^* at the points of Ω. □

In order to complete the estimate of the norm of the optimal error functional for the given lattice of nodes, we note that by Theorem 11 and (69) the following inequality holds:

$$\Lambda(U^*, U^*) \leq \Lambda(U[\beta], U[\beta]), \tag{93}$$

where $U[\beta] = u(hH\beta)$, and $u(x)$ is the extremal function of a cubature formula with regular boundary layer of order m. This very important inequality allows us to prove that

$$\left\| l \mid L_2^{(m)*} \right\|^2 - \left\| l^{(0)} \mid L_2^{(m)*} \right\|^2 = O(h^{2m+1}), \tag{94}$$

where $l^{(0)}(x)$ is the optimal error functional, and $l(x)$ is an arbitrary error functional with regular boundary layer of order m.

In many respects the derivation of estimate (94) is similar to the one carried out above when estimating the norm $\left\| l \mid L_2^{(m)*} \right\|$ of an error functional with regular boundary layer. From (52) it follows that it suffices to establish (94) for the functional with regular boundary layer of order $2m+2$, which we are going to do.

Studying the form $\Lambda(U,U)$, let us make use of (51):

$$\Lambda(U,U) = (L_h[\beta] * U[\beta], U[\beta]) = \sum_{\beta} \Big(\sum_{\beta'} L_h[\beta - \beta'] U[\beta'] \Big) U[\beta]$$

$$= \sum_{\beta} \Big(\sum_{\beta'} L_h[\beta - \beta'] (u_0(hH\beta') - w(hH\beta')) \Big) U[\beta]. \tag{95}$$

Since the function $u_0(hH\beta')$ is constant at the points $hH\beta'$, and the operator $L_h[\beta - \beta']$ is orthogonal to the constant, we obtain

$$\Lambda(U,U) = - \sum_{\beta} \Big(\sum_{\beta'} L_h[\beta - \beta'] w(hH\beta') \Big) U[\beta]$$

$$= -\Lambda(U,W) = -\{U,W\}. \tag{96}$$

The inner product $\{U,W\}$ can be estimated in the same way as the corresponding inner product in $L_2^{(m)}$. We obtain

$$\{U,W\} = \sum_{\gamma} \sum_{\gamma'} L_h[\beta] * U_\gamma[\beta] * W_{\gamma'}[-\beta] \Big|_{\beta=0}, \tag{97}$$

where

$$U_\gamma[\beta] = G(x) * l_\gamma(\frac{x}{h}) \Big|_{x=hH\beta}, \quad W_{\gamma'}[\beta] = G(x) * l_{\gamma'}(\frac{x}{h}) \Big|_{x=hH\beta}, \tag{98}$$

and

$$\begin{aligned} (l_\gamma\ (x), x^\alpha) &= 0 \quad \text{for} \quad |\alpha| < 2m+1, \\ (l_{\gamma'}(x), x^\alpha) &= 0 \quad \text{for} \quad |\alpha| \le 2m+1. \end{aligned} \tag{99}$$

The convolution $L_h[\beta] * U_\gamma[\beta]$ can be written as

$$L_h[\beta] * U_\gamma[\beta] = \sum_{\beta'} L_h[\beta - \beta'] \Big(l_\gamma(\frac{y}{h}), G(hH\beta' - y) \Big)$$

$$= \Big(l_\gamma(\frac{y}{h}), \sum_{\beta'} L_h[\beta - \beta'] G(hH\beta' - y) \Big).$$

It turns out that the function $\psi(y) = \sum_{\beta'} L_h[\beta - \beta'] G(hH\beta' - y)$ is regular in y and decreases exponentially at infinity. Hence,

$$|L_h[\beta] * U_\gamma[\beta]| \leq K h^n e^{-\eta|\beta|/h}. \tag{100}$$

By the assumption that the order of the boundary layer is equal to $2m+1$, we obtain

$$|W_{\gamma'}(x)| \leq K \begin{cases} h^{2m+n+1}|x|^{-n-1} \quad \text{for} \quad |x| \geq L'h, \\ h^{2m} \quad \text{for} \quad |x| \leq Lh. \end{cases} \tag{101}$$

Comparing (100) and (101), and performing some natural computations, we obtain an estimate similar to (37):

$$\big|U_\gamma[\beta] * L_h[\beta] * W_{\gamma'}[-\beta]\big| \leq K \begin{cases} h^{2m+n} \quad \text{for} \quad |hH\beta| \geq L'h, \\ h^{2m+2n+1} \quad \text{for} \quad |hH\beta| \leq Lh. \end{cases} \tag{102}$$

In the same way as above, we obtain

$$\big|U[\beta] * L_h[\beta] * W[-\beta]\big|\Big|_{\beta=0} \leq K h^{2m+1}. \tag{103}$$

The next theorem follows from (103).

Theorem 12. *For the given lattice of nodes the norm of the optimal error functional is expressed as*

$$\big\|l^{(0)} \mid L_2^{(m)*}\big\| = h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}} \sqrt{|\Omega|} + O(h^{m+1}). \tag{104}$$

In particular, Theorem 12 tells us that the formulas with regular boundary layer are asymptotically optimal.

For each particular computation a practical error of the cubature formula can happen to be far away from the estimate that follows from (104). The fact is that for each given value of the mesh-size h of the lattice, there is an extremal function $u_0(x/h)$ such that

$$(l, u_0) = \|l \mid X^*\| \cdot \|u_0 \mid X\|. \tag{105}$$

In this case, the sequence $\dfrac{1}{\|u_0(x/h) \mid X\|} u_0(x/h)$ is noncompact, and moreover it has no condensation point $w(x)$ in $L_2^{(m)}$. On the contrary, we establish that for every φ from $L_2^{(m)}$ the estimate $|(l,\varphi)|$ as $h \to 0$ is always substantially better than the one that follows from the inequality $|(l,\varphi)| \leq \|l \mid L_2^{(m)*}\| \cdot \|\varphi \mid L_2^{(m)}\|$.

For function φ from $L_2^{(m)}$ we have

$$(l, \varphi) = h^m \int \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u_0(\frac{x}{h}) D^\alpha \varphi(x)\, dx. \tag{106}$$

The right side of this formula can be conveniently estimated by considering the integrals over each cell Ω_γ, corresponding to the periods $hH\gamma$ of the function u_0. Using the Cauchy–Bunyakovskii–Schwarz inequality, we obtain

$$|(l, \varphi)| \leq h^m \sum_{hH\gamma \in \Omega} \Big(\int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha u_0(\frac{x}{h}) \Big|^2 dx \Big)^{1/2}$$

$$\times \Big(\int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 dx \Big)^{1/2}. \tag{107}$$

One can show that the sum on the right side of (107) has a limit as $h \to 0$. Passing to the limit, we obtain the estimate

$$|(l, \varphi)| \leq h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}}\, \|\varphi \mid L_1^{(m)}(\Omega)\| \Big(1 + \eta(h) \Big), \tag{108}$$

where $\eta(h) \to 0$ as $h \to 0$, and $\|\varphi \mid L_1^{(m)}(\Omega)\|$ denotes the integral

$$\|\varphi \mid L_1^{(m)}(\Omega)\| = \int_\Omega \Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 \Big)^{1/2} dx. \tag{109}$$

Estimate (108) is stronger than when it follows from (104). Indeed, from the Cauchy–Bunyakovskii–Schwarz inequality we obtain

$$\|\varphi \mid L_1^{(m)}(\Omega)\| \leq \Big\{ \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 dx \Big\}^{1/2} \Big\{ \int_\Omega dx \Big\}^{1/2}$$

$$= \sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}(\Omega)\|. \tag{110}$$

The equality in (110) can only occur in such a case if

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 = 1. \tag{111}$$

Since the extremal function of $l(x)$ does not satisfy (111), we see that the maximum of $|(l, \varphi)|$ taken over the solutions of (111) is less than the maximum of $|(l, \varphi)|$ taken over $L_2^{(m)}(\Omega)$. Hence, estimate (108) is always more advantageous than the estimate in the norm.

It is possible that estimate (108) is also nonoptimal. It would be quite interesting from the theoretical point of view to find the exact estimate of the functional $K(\varphi) = \lim\limits_{h \to 0} h^{-m} |(l, \varphi)|$ in the whole $L_2^{(m)}(\Omega)$. As we have seen,

$$K(\varphi) \leq \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}}\,\|\varphi \mid L_1^{(m)}(\Omega)\|. \tag{112}$$

Now we study the question about the rate of convergence to zero of the error of cubature formulas in classes of infinitely differentiable functions. Let us consider the set of such functions, which are periodic with the matrix H of periods and define a space with a sequence of norms:

$$\|\varphi \mid L_2^{(m)}(\Omega_0)\|, \quad m = \left[\frac{n}{2}\right] + 1, \left[\frac{n}{2}\right] + 2, \ldots. \tag{113}$$

As we observed above, on this space the simplest error functional

$$l_0(x) = 1 - h^n \sum_{\gamma} \delta(x - hH\gamma) = 1 - \Phi_0(h^{-1}H^{-1}x),$$

where

$$\Phi_0(h^{-1}H^{-1}x) = h^n \sum_{\gamma} \delta(x - hH\gamma), \tag{114}$$

has an infinite order of exactness, and its norm in any space $L_2^{(m)*}(\Omega_0)$ can be explicitly estimated. Thus, for each fixed h and for all m the following inequalities hold:

$$|(l_0, \varphi)| \leq h^m \sqrt{\frac{|B_{2m}(H^{-1*})|}{(2m)!}}\,\sqrt{|\Omega|}\,\|\varphi \mid L_2^{(m)}(\Omega_0)\|. \tag{115}$$

Estimating the order of the right side in (115) for different classes of functions, and finding every time the best value of m, we obtain the estimate $|(l_0, \varphi)|$ in terms of h.

Let us consider the Gevrey classes of functions, where the growth of the mth-order derivatives obeys the following conditions

$$\left\|\frac{D^\alpha \varphi}{\alpha!}\right\| \leq K e^{A|m|^\beta} \quad \text{for} \quad |\alpha| = m, \tag{116}$$

where K, A, and β are independent of m, and $m = 1,\ 2,\ \ldots$ There are particular cases of Gevrey's classes such as quasi-analytic functions, functions regular in a certain strip of the complex plane surrounding the set of real values of x, and, finally, entire functions of a particular order and type. As we have seen, these classes are characterized by the pair of numbers A and β. For the Gevrey classes we have the inequality

$$\|\varphi \mid L_2^{(m)}(\Omega_0)\| \leq K e^{A|m|^\beta}. \tag{117}$$

Further, from the explicit expression for $\dfrac{|B_{2m}(H^{-1*})|}{(2m)!}$ we obtain

$$\frac{|B_{2m}(H^{-1*})|}{(2m)!} = \left(\frac{1}{2\pi}\right)^{2m} \frac{1}{r_{\min}^{2m}} \Big(1 + \eta(m)\Big), \tag{118}$$

where $r_{\min}$ denotes the shortest distance between the nodes of the lattice with the matrix H^{-1*} of periods. Formulas (117) and (118) lead to the following corollaries:

a) for sufficiently large m the best optimal lattice with the matrix H^{-1*} is the one whose nodes coincide with the centers of balls from the densest packing. The theory of such lattices has been a subject of numerous studies in the geometry of numbers;
b) from (115) it follows that the best estimate has the form

$$|(l_0, \varphi)| \leq K e^{-Bh^{-\sigma}}, \tag{119}$$

where B and σ are expressed through A and β by formulas, which we do not discuss now. For example, we obtain the estimate of form (119) if $\varphi(x)$ is an analytic function with a given radius of convergence at each point x of the integration domain.

From (119) it follows that beginning with a certain h the corresponding cubature formulas are completely exact for each trigonometric polynomial of a given degree.

In conclusion, let us discuss certain practical methods of the construction of formulas with regular boundary layer.

If Ω is a polyhedron with the rational faces, then we can construct formulas with regular boundary layer by using the Fourier transform of the error functional $l(x)$. The following theorem holds.

Theorem 13. *For the error functional $l(x)$ in a bounded domain Ω to be orthogonal to all polynomials of degree $m-1$, it is necessary and sufficient for its Fourier transform $\widetilde{l}(p)$ to have a zero of multiplicity m at the origin.*

By using Theorem 13, the boundary layer can be found with the aid of the method of undetermined coefficients. For certain cases these layers were calculated in the Novosibirsk Computer Center of the Siberian Division of the USSR Academy of Sciences.

In practice there are cases such that for a given function it is convenient to choose a grid with different mesh-sizes in different parts of its domain. For example, this occurs, when the integrand is changing rapidly in some subdomains, and slowly in others. In order to keep a good order of accuracy, it suffices to trace the fact that in different parts the coefficients of the formulas would be equal to h_1^n and h_2^n, respectively, and to introduce the boundary layer of the required order on the boundary between the subdomains. The coefficients of the cubature formulas in this layer are computed by the same method of the Fourier transform, provided that the boundary is formed by parts of the rational planes.

Conclusion

The problems of the theory of cubature formulas, when we study their error functionals in the corresponding functional spaces, can be treated as problems of functional analysis. In particular, applying certain Hilbert metrics and solving the appearing problems by the methods of variations calculus, we can obtain exact estimates of the norms of error functionals, find an optimal lattice, and find asymptotically optimal coefficients. The formulas with such nodes and coefficients are convenient for practical applications. These formulas are called formulas with regular boundary layer, and they are generalizations of the Gregory quadrature formulas for one independent variable. Apparently, in a number of cases they are also sufficiently convenient for practical applications.

16. Convergence of Approximate Integration Formulas for Functions from* $L_2^{(m)}$

S. L. Sobolev

In the papers by the author [1,2] it was established that an extremal function $u(x)$ for which the error functional attains its maximum value on the unit sphere in $L_2^{(m)}$ is a solution of the polyharmonic equation with a right side:

$$\Delta^m u = (-1)^m l(x). \tag{1}$$

Let the nodes of a cubature formula be of the form

$$x^{(\gamma)} = hH\gamma, \tag{2}$$

where $x^{(\gamma)}$ is a column vector, γ is a column vector with integer entries, H is a matrix with the unit determinant, and h is a small positive parameter. In what follows we consider periodic functions of n variables defined on a torus Ω. Let the periods of the torus Ω be multiples of the columns of the matrix hH, i.e., periods of the lattice.

Theorem 1. *All coefficients C_γ of the error*

$$l(x) = 1 - \sum_\gamma h^n C_\gamma \delta(x - x^{(\gamma)}) \tag{3}$$

with minimal $\widetilde{L}_2^{(m)*}$-norm are given by

$$C_\gamma = 1. \tag{4}$$

Proof. The proof and the formulation of Theorem 1 are known, although for other spaces. It is also known that the $\widetilde{L}_2^{(m)*}$-norm of $l(x)$ is a strictly convex function of C_γ. It means that for $l_1 \neq l_2$ and $\|l_1 \mid \widetilde{L}_2^{(m)*}\| = \|l_2 \mid \widetilde{L}_2^{(m)*}\| = C$ the inequality holds

$$\left\| \frac{l_1(x) + l_2(x)}{2} \mid \widetilde{L}_2^{(m)*} \right\| < C. \tag{5}$$

* Dokl. Akad. Nauk SSSR, **162**, 1259–1261 (1965)

If the coefficients of some functional of form (3) are not all equal, then $l(x)$ and $l(x - hH\gamma)$ do not coincide. At the same time their half-sum is an error functional with the same nodes (2), but smaller in norm. Therefore the $\widetilde{L}_2^{(m)*}$-norm of $l(x)$ cannot be minimal. The proof of Theorem 1 is complete. □

By the Fourier method, we obtain an explicit expression for the extremal function of the error

$$l_0(x) = 1 - \sum_{\gamma} h^n \delta(x - hH\gamma). \tag{6}$$

This function is

$$u(x) = -\left(\frac{h}{2\pi}\right)^{2m} \sum_{\beta \neq 0} \frac{e^{-i2\pi\beta h^{-1}H^{-1}x}}{[A(\beta)]^m}. \tag{7}$$

Here $A(\beta) = (A\beta, \beta)$ is a quadratic form with the matrix

$$A = H^{-1}H^{-1^*}. \tag{8}$$

From (7) it follows that

$$\|l_0(x) \mid \widetilde{L}_2^{(m)*}\| = \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)}, \tag{9}$$

where

$$\zeta(H^{-1*} \mid 2m) = \sum_{\gamma \neq 0} \frac{1}{[A(\gamma)]^m}. \tag{10}$$

By (9), the following estimate for the error of the cubature formula holds:

$$|(l_0, \varphi)| \leq \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)}\, \|\varphi \mid \widetilde{L}_2^{(m)}\|. \tag{11}$$

The purpose of the present note is to prove the following theorem.

Theorem 2. *For every individual function φ in $\widetilde{L}_2^{(m)}$ the estimate holds*

$$|(l_0, \varphi)| \leq \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\, \|\varphi \mid L_1^{(m)}(\Omega)\| + o(h^m), \tag{12}$$

where $o(h^m)$ depends on the function $\varphi(x)$, $h \to 0$, and

$$\|\varphi \mid L_1^{(m)}(\Omega)\| = \int_{\Omega} \Big(\sum_{|\alpha| = m} \frac{m!}{\alpha!} \Big| D^\alpha \varphi(x) \Big|^2 \Big)^{1/2} dx. \tag{13}$$

Estimate (12) *is sharper than* (11) *as $h \to 0$.*

As follows from Theorem 2, estimate (11), which is exact for the total class of periodic functions in $\widetilde{L}_2^{(m)}$, is not exact for every individual function. For each given h the equality in (11) holds for the function $\varphi(x)$ depending on h. However, for all functions except solutions of

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha\varphi)^2 = \text{const}, \tag{14}$$

we have the strict inequality

$$\|\varphi \mid L_1^{(m)}(\Omega)\| < \sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}(\Omega)\|, \tag{15}$$

and extremal function (7) does not satisfy equation (14). Hence estimate (12) is stronger than estimate (11) for every individual function and sufficiently small h.

Proof. Let us present the idea of the proof of Theorem 2. For any function $\varphi(x)$ in $\widetilde{L}_2^{(m)}$ the error (l,φ) is given by the formula

$$\begin{aligned}(l,\varphi) &= \big((-1)^m \Delta^m u, \varphi\big)\\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha u, D^\alpha \varphi) \equiv \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u D^\alpha \varphi \, dx. \end{aligned}\tag{16}$$

We cover the domain Ω with a system of disjoint parallelepipeds Ω_γ. The sides of each Ω_γ are given by the columns of the matrix hH, and the beginning of Ω_γ is $hH\gamma$. Using the Cauchy–Bunyakovskii–Schawrz inequality, we have

$$\begin{aligned}(l,\varphi) &= \sum_\gamma \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha \varphi D^\alpha u \, dx\\ &\leq \sum_\gamma \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx \Big\}^{1/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u)^2 \, dx \Big\}^{1/2}\\ &= \Big(\frac{h}{2\pi}\Big)^m \sqrt{\zeta(H^{-1*} \mid 2m)} \sum_\gamma h^{n/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx \Big\}^{1/2}. \end{aligned}\tag{17}$$

To prove estimate (12) it remains to show that

$$\begin{aligned}&\sum_\gamma h^{n/2} \Big\{ \int_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 dx \Big\}^{1/2}\\ &= \int_\Omega \Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \Big\}^{1/2} dx + o(1). \end{aligned}\tag{18}$$

Consider the function

$$f_\gamma(\lambda) = \int\limits_{\Omega_\gamma} \left(\left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2} - \lambda \right)^2 dx. \tag{19}$$

Clearly, the function $f_\gamma(\lambda)$ is a positive quadratic trinomial in λ, and

$$f_\gamma(\lambda) = h^n \lambda^2 - 2\lambda \int\limits_{\Omega_\gamma} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2} dx + \int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx. \tag{20}$$

For every positive trinomial $f(\lambda) = a\lambda^2 - 2b\lambda + c$ the equality holds

$$a \min f(\lambda) = ac - b^2. \tag{21}$$

Let

$$\min f_\gamma(\lambda) = f_\gamma(\lambda_\gamma) = \varepsilon_\gamma. \tag{22}$$

Clearly, the function $\left[\sum\limits_{|\alpha|=m} \dfrac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2}$ is square integrable over Ω. Hence, it can be approximated in norm by a step function

$$\psi(x) = \lambda_\gamma \quad \text{for} \quad x \in \Omega_\gamma. \tag{23}$$

From this it follows that the sum

$$\sum_\gamma \varepsilon_\gamma = \varepsilon = \tau_\varphi^{(m)}(h) \tag{24}$$

tends to zero as $h \to 0$. However, from (21) and (22) it follows that

$$\varepsilon_\gamma = \int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx - \frac{1}{h^n} \left(\int\limits_{\Omega_\gamma} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2, \tag{25}$$

and hence

$$\begin{aligned}
&\sum_\gamma h^{n/2} \left\{ \int\limits_{\Omega_\gamma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \, dx \right\}^{1/2} \\
&= \sum_\gamma h^{n/2} \left\{ \frac{1}{h^n} \left(\int\limits_{\Omega_\gamma} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 + \varepsilon_\gamma \right\}^{1/2} \\
&= \sum_\gamma \left\{ \left(\int\limits_{\Omega_\gamma} \left[\sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha \varphi)^2 \right]^{1/2} dx \right)^2 + h^n \varepsilon_\gamma \right\}^{1/2}.
\end{aligned} \tag{26}$$

Finally, from the triangle inequality it follows that

$$\left\{\left(\int\limits_{\Omega_\gamma}\left[\sum_{|\alpha|=m}\frac{m!}{\alpha!}(D^\alpha\varphi)^2\right]^{1/2}dx\right)^2+h^n\varepsilon_\gamma\right\}^{1/2}$$

$$-\int\limits_{\Omega_\gamma}\left[\sum_{|\alpha|=m}\frac{m!}{\alpha!}(D^\alpha\varphi)^2\right]^{1/2}dx\le h^{n/2}\sqrt{\varepsilon_\gamma}, \tag{27}$$

and further

$$\sum_\gamma h^{n/2}\sqrt{\varepsilon_\gamma}\le\left(\sum_\gamma h^n\right)^{1/2}\left(\sum_\gamma\varepsilon_\gamma\right)^{1/2}=\sqrt{|\Omega|}\sqrt{\tau_\varphi^{(m)}(h)}. \tag{28}$$

Summing (27) over all γ, and using (26) and (28), we obtain (18). The proof of Theorem 2 is complete. □

References

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[1] Paper [7] of Part II of this book. – *Ed.*

17. Evaluation of Integrals of Infinitely Differentiable Functions*

S. L. Sobolev

In previous note [1] of the author it was shown that for each individual function in the class of periodic functions from $L_2^{(m)}$, the error estimate for the numerical evaluation of the integral using the method of nets is always better than the error estimate by means of the norm of the error functional. Clearly, if we consider the whole space $L_2^{(m)}$, the estimate of accuracy of a formula by means of the norm of its error functional is unimprovable. Studying countably-normed spaces of infinitely differentiable functions, we come to another similar case. Each such space is the intersection of the sequence of spaces $L_2^{(m)}$, $m = 1, 2, \dots$. The cubature formulas with equal coefficients mentioned in the note [1] are applicable for periodic functions which are members of all spaces $L_2^{(m)}$ simultaneously. Hence, for any function belonging to all of $L_2^{(m)}$ simultaneously, we get the following sequence of estimates[1]:

$$|(l, \varphi)| \leq \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\sqrt{|\Omega|}\, \|\varphi \mid L_2^{(m)}\|, \tag{1}$$

$$m = [\frac{n}{2}] + 1,\ [\frac{n}{2}] + 2,\ \dots .$$

Given some law of growth of $L_2^{(m)}$-norm of φ we may choose for each given h the best estimate as the lower bound of all estimates (1). By this way we can, in a number of cases, establish that the convergence of (l, φ) to zero as $h \to 0$ is far more rapid than polynomial decay of errors. The present note is devoted to this question.

Lemma 1. *The function $\zeta(A \mid m)$ at large m admits the estimate*

$$\zeta(H^{-1*} \mid 2m) \leq \frac{K_{\min}}{r_{\min}^{2m}} \Big(1 + o(1)\Big), \tag{2}$$

* Dokl. Akad. Nauk SSSR, **163**, 33–35 (1965)

[1] Here Ω is an integration domain. – *Ed.*

where $r_{\min}$ *is the shortest distance between the nodes of the integer lattice* βH^{-1}, β *is an integer row vector, and* K_{min} *is the number of such lattice nodes which are at the distance* $r_{\min}$ *from a given one.*

The proof of Lemma 1 is easy from the definition of the Epstein zeta function $\zeta(H^{-1*} \mid 2m)$ (see [1]).

We consider classes of real infinitely differentiable periodic functions whose derivatives of order α are subject to the inequalities

$$|\frac{D^\alpha \varphi(x)}{\alpha!}| \leq K A^{|\alpha|} |\alpha|^{(\beta-1)|\alpha|}. \tag{3}$$

Here, $\alpha!$ denotes $\alpha_1! \dots \alpha_n!$ and the remaining notations are the same as in [1,2]. We call the number β the *order of growth* of the derivatives of φ, and the number A the *type* of this growth. When (3) is satisfied, we write

$$\varphi \in \mathfrak{K}(A, \beta). \tag{4}$$

A long time ago, the basic properties of such classes were established for functions of one independent variable, and we may translate these properties almost without any change to functions of several independent variables.

We distinguish five cases, not all of which are of interest: a) $\beta < 0$; b) $\beta = 0$; c) $0 < \beta < 1$; d) $\beta = 1$; and e) $\beta > 1$.

In case a), the class $\mathfrak{K}(A, \beta)$ contains no periodic function besides a constant function, and we shall not consider it.

In case b), i.e., for $\beta = 0$, the class $\mathfrak{K}(A, \beta)$ of periodic functions may contain only polynomials whose degree n depends on the size of the constant A: $n \leq K_1 A$.

In case c), for $0 < \beta < 1$, $\mathfrak{K}(A, \beta)$ contains entire functions of order

$$\varrho = \frac{1}{1-\beta} \tag{5}$$

and of type

$$\sigma = \frac{1-\beta}{e} A^{1/(1-\beta)}. \tag{6}$$

In case d), for $\beta = 1$, the class $\mathfrak{K}(A, \beta)$ consists of functions, analytic with a radius of convergence at each point determined by the constant A,

$$\operatorname{Im}\{x_j\} < e^{-K_2 A}, \tag{7}$$

where K_2 is some constant.

Finally, for $\beta > 1$, the class $\mathfrak{K}(A, \beta)$ consists of infinitely differentiable non-analytic functions, and for each A the compactly-supported functions belong to it. Various quasi-analytic functions belong to the intersection $\cap_{A>0} \mathfrak{K}(A, 1)$.

Lemma 2. *Let periodic function φ of the variable x in n dimensions be in the class $\mathfrak{K}(A,\beta)$. Then the following estimates of the $L_2^{(m)}$-norms are valid:*

$$\|\varphi \mid L_2^{(m)}\| \leq K_3 m^{\beta m+1/2}\left(\frac{A}{e}\right)^m, \quad m = 0, 1, 2, \ldots. \tag{8}$$

Here K_3 is some constant depending on K and Ω.

Proof. The proof of this lemma is based on simple estimates. We may always assume that $\sum\limits_{|\alpha|=m} f(\alpha)$ denotes a summation as though the function $f(\alpha)$ does not possess symmetry with respect to permutations of the entries of the integer vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $f(\alpha)$ possesses such symmetry, then the sum can be taken with the corresponding repetitions. Hence, the norm of the function $\|\varphi \mid L_2^{(m)}\|$ may be written as

$$\|\varphi \mid L_2^{(m)}\|^2 = \int\limits_{\Omega} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx = \int\limits_{\Omega} \sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!}\right)^2 dx. \tag{9}$$

Since

$$\sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!}\right)^2 \leq A^{2m} m^{2m(\beta-1)} \sum_{|\alpha|=m} (\alpha!)^2, \tag{10}$$

we obtain

$$\|\varphi \mid L_2^{(m)}\| \leq A^m m^{m(\beta-1)} \sqrt{|\Omega|} \left\{\sum_{|\alpha|=m} (\alpha!)^2\right\}^{1/2}. \tag{11}$$

The inequality is valid

$$\sum_{|\alpha|=m} (\alpha!)^2 \leq K_h e^{-2m} m^{2m+1}. \tag{12}$$

It is proved by rearranging the sum

$$\sum_{|\alpha|=m} (\alpha!)^2 = m! \sum_{|\alpha^{(j)}|=m} (\alpha^{(j)})!. \tag{13}$$

The sum on the right side of (13) is taken over distinct integer vectors $\alpha^{(j)}$ without repetitions. Further, one establishes

$$\sum_{|\alpha^{(j)}|=m} \frac{(\alpha^{(j)})!}{(m-1)!} = n + O\left(\frac{1}{m}\right). \tag{14}$$

Inequality (12) follows immediately from (13), (14), and the Stirling formula. Finally, from (11) and (12) inequality (8) follows. □

Lemmas 1 and 2 allow us to prove the main theorem.

Theorem 1. *For each periodic function φ of the class $\mathfrak{K}(A, \beta)$, $\beta > 0$, and for the cubature formula with nodes at the points $hH\gamma$ the following estimate of the error of this cubature formula holds*

$$|(l, \varphi)| \leq K h^{-1/2} \exp\left[-\frac{\beta}{e}\left(\frac{2\pi e r_{\min}}{Ah}\right)^{1/\beta}\right]. \tag{15}$$

Proof. The proof of Theorem 1 is based on determining the minimum with respect to m of the function on the right side of the inequality

$$\begin{aligned} &\|l \mid L_2^{(m)*}\| \cdot \|\varphi \mid L_2^{(m)}\| \\ &\leq \left(\frac{h}{2\pi}\right)^m \sqrt{|\Omega|}\sqrt{\zeta(H^{-1*} \mid 2m)} K m^{\beta m + 1/2}\left(\frac{A}{e}\right)^m, \end{aligned} \tag{16}$$

which is easy to carry out in an elementary way. Incidentally, one establishes which m is optimal for a given value of h, namely,

$$m = \frac{1}{e}\left(\frac{2\pi e r_{\min}}{Ah}\right)^{1/\beta}, \tag{17}$$

as required. □

It is interesting to note the case which does not follow directly from Theorem 1, namely, when $\beta = 0$. A direct estimate in this case gives Theorem 2.

Theorem 2. *For a sufficiently small mesh-size h, the cubature formula with a fixed lattice of nodes $hH\gamma$ and equal coefficients is exact for any trigonometric polynomial.*

References

1. Sobolev, S. L.: Convergence of approximate integration formulas for functions from $L_2^{(m)}$. Dokl. Akad. Nauk SSSR, **162**, 1259–1261 (1965)[2]
2. Sobolev, S. L.: On the rate of convergence of cubature formulas. Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)[3]

[2] Paper [16] of Part II of this book. – *Ed.*
[3] Paper [14] of Part II of this book. – *Ed.*

18. Cubature Formulas with Regular Boundary Layer*

S. L. Sobolev

In one of the previous notes [1] we considered error functionals of cubature formulas of the form[1]

$$l(x) = \chi_\Omega(x) - \sum_{j=1}^{N} c_j \delta(x - x^{(j)}) = \sum_\gamma l_\gamma \left(\frac{x}{h} - \gamma\right), \tag{1}$$

where $l_\gamma(x)$ are functionals with bounded supports and finite norms in $L_2^{(m)}$, and $l_\gamma(x)$ are orthogonal to all polynomials of degree m. The set of $l_\gamma(x)$, which satisfy conditions (10), (11), and (12) of [1], is denoted by $\mathfrak{R}(L, A, m+1)$.

Let H be the matrix of periods of some lattice with determinant equal to 1, $|H| = 1$. Formula (1) is evidently equivalent to the equality

$$l(x) = \sum_\gamma l_\gamma \left(\frac{x}{h} - H\gamma\right). \tag{2}$$

In what follows, we consider formulas for which:

a) the nodes of all $l_\gamma(x)$ are located at the points $hH\gamma'$,

$$l_\gamma(x) = \chi_\gamma(x) - \sum_{\gamma'} c_\gamma^{\gamma'} \delta(x - hH\gamma'), \quad \sum \chi_\gamma \left(\frac{x}{h} - H\gamma\right) = \chi_\Omega(x); \tag{3}$$

b) all errors $l_\gamma(x)$ are members of $\mathfrak{R}(L, A, s)$,

$$l_\gamma(x) \in \mathfrak{R}(L, A, s); \tag{4}$$

c) for all points $hH\gamma$ such that $\operatorname{dist}(hH\gamma, \Gamma) > 2Lh$ errors $l_\gamma(x)$ coincide,

$$l_\gamma(x) = l_0(x). \tag{5}$$

Under these conditions we call $l(x)$ the *error with regular boundary layer of order m.*

The purpose of our note is to establish the following theorem.

* Dokl. Akad. Nauk SSSR, **163**, 587–590 (1965)

[1] Here Ω is a domain with piece-wise smooth boundary. – *Ed.*

Theorem. *Let $l(x)$ be an error with regular boundary layer of order m. Then the following equality holds:*

$$\|l \mid L_2^{(m)}\| = \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1*} \mid 2m)}\sqrt{|\Omega|} + O(h^{m+1}) \quad as \quad h \to 0. \tag{6}$$

The proof is based on a series of auxiliary lemmas, which are given here.

Lemma 1. *Let $l(x)$ be an error with regular boundary layer. Then for all nodes $hH\gamma$ in Ω at a distance not less than Lh from the boundary Γ of Ω, coefficients c_γ are all equal to h^n.*

Proof. Indeed, for the node $hH\gamma$ under consideration we have

$$c_\gamma = h^n \sum_{|\gamma'|<L} c^{(0)}_{\gamma-\gamma'}. \tag{7}$$

Here $c^{(0)}_\gamma$ are coefficients of $l_0(x)$. From the conditions that the volume of Ω_0 equals 1 and $(l_0(x), 1) = 0$ it follows that

$$c_\gamma = h^n, \tag{8}$$

as required. □

We call the set of nodes $hH\gamma$ at which $c_\gamma \neq h^n$ the *boundary layer*. If only nodes in the interior of Ω are used in integration, then the boundary layer is *interior*. If we also use nodes in the exterior of Ω in approximating the functional $\chi_\Omega(x)$ in $L_2^{(m)}$, then we can get a two-sided boundary layer which is comprised by nodes $hH\gamma$ at a distance not greater than Lh from the boundary Γ, or else an exterior boundary layer which is comprised by nodes $hH\gamma$ in the exterior of Ω at a distance not greater than $2Lh$ from the boundary Γ. Of course, for a function $\varphi(x)$ in $L_2^{(m)}(\Omega)$ only formulas with interior boundary layer make sense. We call the number $2L$ the *width of the boundary layer.*

Let $m_\gamma(x) = \sum c_\gamma^{\gamma'}\delta(x - hH\gamma')$. We call such functionals the *narrow-like functionals.* Further, let $m_\gamma(x) \in \mathfrak{R}(L, A, s)$ and

$$m(x) = \sum_{\gamma\in B_j} m_\gamma(x - hH\gamma), \tag{9}$$

where γ ranges over some set B_j, and $\{hH\gamma |\ \gamma \in B_j\}$ is a boundary layer of width L. Then we call functional (9) a *zero's error with a boundary layer of order s.* A zero's error with a boundary layer is exterior, interior, or two-sided according to the location of its support. The width of this functional is introduced analogously to the width of a boundary layer, and it is, generally speaking, equal to $3L$, but it may also be less. In all that follows it may be taken equal to $2L$.

Lemma 2. *Let $l^{(1)}(x)$ and $l^{(2)}(x)$ be errors with regular boundary layer of orders $s^{(1)}$ and $s^{(2)}$, respectively. The difference of $l^{(1)}(x)$ and $l^{(2)}(x)$ is a zero's error with a boundary layer of order*

$$\min(s^{(1)}, s^{(2)}) - 1. \tag{10}$$

The proof of Lemma 2 is based on an auxiliary lemma.

Lemma 3. *Let $m(x)$ be a compactly-supported functional of the form*

$$m(x) = \sum_{|H\gamma|<L} c[\gamma]\delta(x - hH\gamma), \tag{11}$$

and let $m(x)$ be orthogonal to all polynomials of degree s:

$$(m(x), x^\alpha) = \sum c[\gamma](hH\gamma)^\alpha = 0 \quad \textit{for} \quad |\alpha| \le s.$$

Then $m(x)$ admits the equivalent representation as follows[2]*:*

$$m(x) = \sum_{j=1}^{n} \big(M_j(x + hH\delta_j) - M_j(x)\big), \tag{12}$$

where $(M_j(x), x^\alpha) = 0$ for $|\alpha| \le s - 1$ and $\operatorname{supp} M_j(x)$ is a subset of the smallest parallelepiped, with edges parallel to the columns of H, containing $\operatorname{supp} m(x)$.

Proof. Lemma 3 is proved by the method of induction on the number n of independent variables. We have to establish that the coefficients $c[\gamma]$ may be written as

$$c[\gamma] = \sum_{j=1}^{n} \widehat{\Delta}_j c_j[\gamma], \tag{13}$$

where $\sum c_j[\gamma]\gamma^\alpha = 0$ for $|\alpha| \le s - 1$ and $\widehat{\Delta}_j \varphi[\gamma] = \varphi[\gamma + \delta_j] - \varphi[\gamma]$.

Let us show that for $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ the equality holds

$$c[\gamma] = c_n[\gamma_1, \dots, \gamma_{n-1}, \gamma_n + 1] - c_n[\gamma_1, \dots, \gamma_{n-1}, \gamma_n] + c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}], \tag{14}$$

where the function $c^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ is orthogonal to all polynomials in the variables $(\gamma_1, \gamma_2, \dots, \gamma_{n-1})$ of degree s, and the function $c_n[\gamma_1, \gamma_2, \dots, \gamma_n]$ is orthogonal to polynomials of degree $s - 1$ and $\operatorname{supp} c_n[\gamma]$ is a subset of the smallest parallelepiped containing the support of $c[\gamma]$. From this Lemma 3 follows. □

[2] Here $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0)$. – *Ed.*

Functions $c_n[\gamma_1, \gamma_2, \ldots, \gamma_n]$ and $c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]$ may be written as

$$c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}] = \sum_{\gamma_n'=-L}^{L} c[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n']; \tag{15}$$

$$c_n[\gamma] = \begin{cases} 0 \quad \text{for} \quad |\gamma_n| \geq L, \\ \sum\limits_{\gamma_n'=-L}^{\gamma_n-1} c[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n'] - (\gamma_n + L)c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]. \end{cases} \tag{16}$$

Hence, formula (14) and the orthogonality of $c^*[\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]$ to polynomials in the variables $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ of degree s are clear. The orthogonality of $c_n[\gamma_1, \gamma_2, \ldots, \gamma_n]$ to all polynomials of degree $s-1$ follows from the well-known formula for summation by parts:

$$\sum_\gamma [\varphi(\gamma + \delta_n) - \varphi(\gamma)]\psi(\gamma) = \sum_\gamma \varphi(\gamma)[\psi(\gamma) - \psi(\gamma - \delta_n)]. \tag{17}$$

It suffices to note that $x_n^{\alpha_n} = \dfrac{1}{\alpha_n + 1}\widehat{\Delta}_n B_{\alpha_n+1}(x_n)$, where B_{α_n+1} is the Bernoulli polynomial of degree $\alpha_n + 1$, and use (14).

Lemma 3 can be also proved in a different way, namely, by passing to the Fourier transform. This is its dual statement.

Lemma 3a. *Let Z be the class of rational functions $\Psi(z)$ of the form $\dfrac{P(z)}{z^{\mathbf{k}}}$, where $P(z)$ is a polynomial in $z = (z_1, \ldots, z_n)$, and $z^{\mathbf{k}} = z_1^{k_1} \ldots z_n^{k_n}$; $\mathbf{k} = (k_1, k_2, \ldots, k_n)$. Every function $\varphi(z)$ with a zero of multiplicity m at the point $(1, 1, \ldots, 1)$ may be written down as*

$$\varphi(z) = \sum_{j=1}^{n} (z_j - 1)\varphi_j(z), \tag{18}$$

where the functions φ_j are members of the same class Z with the polynomials $P_j(z)$, and they have zeros of multiplicity $m-1$ at the point $(1, 1, \ldots, 1)$. Also, the degree of the polynomial $P_j(z)$ in the variable z_j does not exceed the degree of $P(z)$ and $\mathbf{k}_j \leq \mathbf{k}$.

It seems the proof in the above text is no longer than any possible proof of Lemma 3a, especially if we take into account the necessity to establish their equivalence.

Lemma 3a and Lemma 3 are the particular examples of lemmas on the expansion of an analytic function with a root of multiplicity m at a given point $z^{(0)}$ into the sum

$$\varphi(z) = \sum (z_j - z^{(0)})\varphi_j(z), \tag{19}$$

and of dual lemmas on the corresponding expansion of generalized functions ψ in $K^{(s)}$, where $(\psi(x) * x^\alpha) = 0$ for $|\alpha| \le s$.

Essentially, the theorem of L. Schwartz on the representation of every generalized function in the form of a differential operator on a continuous function is like Lemma 3.

Corollary. *Let $m_0(x)$ be a zero's error from $\mathfrak{R}(L, A, s+1)$. Then the sum*

$$\sum_{hH\gamma \in \Omega} m_0(x - hH\gamma) = M_0(x), \tag{20}$$

is a zero's error with boundary layer of order s.

This corollary is obtained immediately, if we replace m_0 in the left side of (20) by its expansion into the sum (11) and change the order of summation.

Lemma 2 follows immediately from Lemma 3.

Corollary to Lemma 2. *Each functional $l(x)$ with a boundary layer of order $s \ge m$ may be written as*

$$l(x) = \sum_{hH\gamma \in \Omega} l^* \left(\frac{x}{h} - H\gamma\right) + \sum_{\gamma \in B} l^{**}_\gamma \left(\frac{x}{h} - H\gamma\right), \tag{21}$$

where B is a boundary layer of width L,

$$l^* \left(\frac{x}{h} - H\gamma\right) \in \mathfrak{R}(L, A, s_1). \tag{22}$$

The order s_1 in (22) can be each number greater than s.

To prove (21) for given s_1 it suffices to expand the difference between $l(x)$ and any given error functional with regular boundary layer of order s_1 into a sum like (9).

Let $l(x)$ be a linear functional with regular boundary layer of order $s \ge m$. The equality is valid,

$$l(x) = 1 - \Phi_0(h^{-1}H^{-1}x) - l^{(1)}(x), \tag{23}$$

where $l^{(1)}(x)$ is a functional with exterior regular boundary layer for the domain $\overline{\Omega} = R^n \backslash \Omega$. As was established in [1], the extremal function for $l(x)$ has the form

$$u(x) = l(x) * G(x). \tag{24}$$

From (23) and (24) it follows that

$$u(x) = u_0(x) + C - l^{(1)}(x) * G(x), \tag{25}$$

where $u_0(x)$ is the elementary solution of the extremal problem in the periodic case.

Let us write down the norm of $l(x)$ explicitly,

$$\|l(x)\|^2 = \big(l(x), u(x)\big) = \big(l(x), u_0(x)\big) - l(x) * G(x) * l^{(1)}(-x)\Big|_{x=0}. \tag{26}$$

Replacing $l(x)$ and $l^{(1)}(x)$ by their expansions into sums like (21) and repeating the estimates mentioned in [1], we obtain

$$l(x) * G(x) * l^{(1)}(-x)\Big|_{x=0} = O(h^{2m+1}). \tag{27}$$

By a direct calculation one can also show that

$$(l(x), u_0(x)) = \frac{h^{2m}}{(2\pi)^{2m}}\zeta(H^{-1*} \mid 2m)|\Omega| + O(h^{2m+1}). \tag{28}$$

The main theorem follows from (26)–(28).

References

1. Sobolev, S. L.: On the rate order of convergence of cubature formulas. Dokl. Akad. Nauk SSSR, **162**, 1005–1008 (1965)[3]

[3] Paper [14] of Part II of this book. – *Ed.*

19. A Difference Analogue of the Polyharmonic Equation*

S. L. Sobolev

Let x be a column vector in the n-dimensional space R^n, and let β be a column vector with integer entries. The generalized function

$$\psi(x) = \sum_{\beta} \varphi[\beta]\delta(x-\beta)$$

is called a *lattice* function, and the function $\varphi[\beta]$ of integer variables β is called a *discrete* function. We denote the set of lattice functions by P, and the set of discrete functions by R. To an arbitrary continuous function $\varphi(x)$, there correspond lattice and discrete functions defined by

$$\begin{gathered} \Omega^{CR}[\beta \mid \varphi] = \varphi(\beta); \\ \Omega^{CP}(x \mid \varphi) = \sum_{\beta} \varphi[\beta]\delta(x-\beta) = \varphi(x)\Phi_0(x) = \psi(x), \end{gathered} \tag{1}$$

where $\Phi_0(x) = \sum_{\beta} \delta(x-\beta)$. Equality (1) may be also written as

$$\psi(x) = \Omega^{RP}(x \mid \varphi) \quad \text{and} \quad \varphi[\beta] = \Omega^{PR}[\beta \mid \psi]. \tag{2}$$

Also let us introduce the mappings

$$\Omega^{PC}(x \mid \varphi) \quad \text{and} \quad \Omega^{RC}(x \mid \varphi), \tag{3}$$

where

$$\Omega^{PC} = (\Omega^{CP})^{-1} \quad \text{and} \quad \Omega^{RC} = (\Omega^{CR})^{-1}. \tag{4}$$

Mappings (3) and (4) are not uniquely defined. Indeed, we may explicitly express Ω^{PC} as a convolution

$$\Omega^{PC}(x \mid \varphi) = \varphi(x) * \Lambda(x),$$

* Dokl. Akad. Nauk SSSR, **164**, 54–57 (1965)

where $\Lambda(x)$ is an arbitrary solution of the equation $\Lambda(x)\Phi_0(x) = \delta(x)$; i.e.,

$$\Lambda(\beta) = 1 \quad \text{for} \quad \beta = 0 \quad \text{and} \quad \Lambda(\beta) = 0 \quad \text{for} \quad \beta \neq 0.$$

It is surely assumed that the convolution of $\Lambda(x)$ with the function $\varphi(x)$ exists.

Below we expose the spaces containing the product and the convolution of two functions from C or P. As regards the inner product

$$(\varphi, \psi) = c \in R^1, \tag{5}$$

it exists provided that φ and ψ are not simultaneously members of P.

a) $\varphi \cdot \psi = \chi$

$\varphi \backslash \psi$	C	P
C	C	P
P	P	

b) $\varphi * \psi = \chi$

$\varphi \backslash \psi$	C	P
C	C	C
P	C	P

(6)

On the set of functions in R there are defined conventional operations. Obviously, convolution is applicable not for an arbitrary pair of functions, but only for sufficiently rapidly decreasing ones. The mappings Ω of the spaces C, P, and R preserve the validity of (5) and (6a) in all cases, and the validity of (6b) everywhere except for the case $\varphi \in C$ and $\psi \in C$.

Also let us introduce the space Φ of functions decreasing at infinity sufficiently rapidly. For example, compactly-supported functions are members of Φ. Let Π be the space of periodic functions with integer periods, and let T be the space of functions defined on the torus Ω_0 obtained by identifying all points of R^n that differ by an integer vector. Thus, we can consider the mappings

$$\Xi^{\Phi\Pi}, \quad \Xi^{\Pi\Phi}, \quad \Xi^{T\Pi}, \quad \Xi^{\Pi T}, \quad \Xi^{\Phi T}, \quad \text{and} \quad \Xi^{T\Phi}.$$

The mapping $\Xi^{\Pi T}$ sends a periodic function $\varphi(p)$ with domain R^n into a function defined on the torus with the same values. On the other hand, the mapping $\Xi^{T\Pi}$ sends T into Π.

The mapping $\Xi^{\Phi\Pi}$ sends $\varphi(p) \in \Phi$ into the function

$$\psi(p) \equiv \Xi^{\Phi\Pi}(p \mid \varphi) = \sum_{\gamma} \varphi(p - \gamma) = \varphi(p) * \Phi_0(p),$$

where the vector γ has integer entries. Further, $\Xi^{\Phi T} = \Xi^{\Phi\Pi}\Xi^{\Pi T}$.

The inner product (φ, ψ) of the functions $\varphi \in \Phi \cup \Pi$ and $\psi \in \Phi \cup \Pi$ has sense provided that φ and ψ are not simultaneously members of Π.

Below we expose the spaces containing the product and the convolution of functions under consideration.

a) $\varphi \cdot \psi = \chi$

$\varphi \backslash \psi$	Φ	Π
Φ	Φ	Φ
Π	Φ	Π

b) $\varphi * \psi = \chi$

$\varphi \backslash \psi$	Φ	Π
Φ	Φ	Π
Π	Π	

Certainly, the product of functions in Π is defined only in the case when neither φ nor ψ is a generalized function. Otherwise, we need the special hypotheses for defining the product, and we shall not consider this case.

The mappings Ξ of the spaces Φ, Π, and P preserve the binary operations and the inner product

$$(\varphi, \psi) = \int \varphi\psi \, dp,$$

except for the cases when the operation is the product $\varphi\psi = \chi$ or the operation is the inner product (φ, ψ) with $\varphi \in \Phi$ and $\psi \in \Phi$.

The Fourier transform and the inverse Fourier transform of functions with domain R^n are defined by the formulas

$$\tilde{f}(p) = \int e^{i2\pi px} f(x) \, dx \quad \text{and} \quad \overset{\frown}{f}(x) = \int e^{-i2\pi px} f(p) \, dp, \tag{7}$$

respectively. In particular, (7) holds for an arbitrary function $f(x)$ in $L_2^{(m)}$. The Fourier transform defined by (7) is a unitary operator. As well known, using the weak continuity of the Fourier transform, we may extend it to the space of generalized functions.

Theorem 1. *The following equalities are valid:*

$$\overset{\frown}{\tilde{f}}(x) = \tilde{\overset{\frown}{f}}(x) \quad \textit{and} \quad \tilde{\tilde{f}}(x) = \overset{\frown}{\overset{\frown}{f}}(x) = f(-x).$$

Theorem 2 (Parseval's identity). $\big(\tilde{f}(p), \tilde{\varphi}(p)\big) = \big(f(x), \varphi(x)\big)$.

Using Theorem 2, we may define the Fourier transforms for generalized functions.

Theorem 3. *The duality of the multiplication and convolution holds, i.e.,*

$$\Big(\widetilde{f\varphi}\Big)(p) = \tilde{f}(p) * \tilde{\varphi}(p) \quad \textit{and} \quad \Big(\overset{\frown}{f\varphi}\Big)(p) = \overset{\frown}{f}(p) * \overset{\frown}{\varphi}(p).$$

The Fourier images of some simplest functions are given by the formulas:

$$\tilde{\delta}(x) = 1, \ \tilde{1} = \delta(p), \ \widetilde{e^{-\pi x^2}} = e^{-\pi p^2}, \ \widetilde{\Phi_0(x)} = \Phi_0(p), \ \widetilde{D^\alpha(x)} = (i2\pi p)^\alpha.$$

By definition, the convolution of a function with the generalized function $D^\alpha(x)$ is the derivative of order α for the function. The equality for the Fourier transform of $\Phi_0(x)$ is the well-known Poisson formula [1].

The Fourier transform maps periodic functions into lattice functions. On the other hand, it sends lattice functions into periodic functions. The Fourier transform extends to discrete functions if we suppose that

$$\widetilde{\varphi}[\beta] = \widetilde{\varphi}(p) \equiv \Omega^{\Pi T}\big(\widetilde{\Omega^{RP}}(x \mid \varphi)\big).$$

Theorems 1, 2, and 3 remain valid for discrete functions.

Theorem 4. *The Fourier transform maps the spaces C, P, and R in a one-to-one fashion onto Φ, Π, and T, respectively. Correspondences established above for the operators Ω become analogous correspondences established above for the operators Ξ.*

All the theorems that we state above may be translated to the case of lattice functions which have singularities at the nodes $A\beta$ like the corresponding translations of the Dirac delta function. To these lattice functions there correspond periodic functions with periods γA^{-1}. In this event the definitions are as follows.

Let P_A be the space of lattice functions of the form

$$\psi(x) = \sum_\beta \varphi[\beta]\delta(x - A\beta).$$

The spaces C, P_A, and R are mapped into each other by means of the mappings

$$\Omega_A^{CP}, \quad \Omega_A^{PC}, \quad \Omega_A^{PR}, \quad \Omega_A^{RP}, \quad \Omega_A^{CR}, \quad \text{and} \quad \Omega_A^{RC},$$

whose definitions are similar to the definitions of the mappings Ω considered above. In this case,

$$\Omega_A^{CP}(x \mid \varphi) = \varphi(x)|A|^{-1}\Phi_0(A^{-1}x) \text{ and } \Omega_A^{CR}[\beta \mid \varphi] = \varphi(A\beta).$$

The remaining Ω_A are formed similarly. All Ω_A are easily expressed by means of the corresponding Ω. The mappings Ω again preserve the elementary binary operations.

Also let us consider the space Π_B of periodic functions $\varphi(p)$ with periods $B\gamma$, i.e.,

$$\varphi(p) = \varphi(p + B\gamma),$$

where B is a nonsingular matrix. The spaces Φ, Π_B, and T_B are mapped into each other by means of the mappings Ξ_B. To the space Π_B there corresponds the space T_B of functions defined on the torus. For $\Xi_B^{\Phi\Pi}$ we use the formula

$$\Xi_B^{\Phi\Pi}(p \mid \varphi) = |B| \sum_\gamma \varphi(p - B\gamma) = \varphi(p) * \Phi_0(B^{-1}p).$$

The remaining Ξ_B are formed similarly. All Ξ_B are easily expressed in elementary form by means of the corresponding Ξ.

Theorem 5. *The Fourier transform maps the spaces C, P_A, and R in a one-to-one fashion onto Φ, $\Pi_{A^{-1}}$, and $T_{A^{-1}}$, respectively. The image of the mapping Ω_A under this transform is the mapping $\Xi_{A^{-1}}$, and vice versa.*

Theorem 6. *Let the inner product and the convolution of two functions in T_B be given by the formulas*[1]

$$(\varphi(p), \psi(p)) = \frac{1}{|B|} \int \varphi(p)\overline{\psi(p)}\, dp, \quad \varphi(p) * \psi(p) = \frac{1}{|B|} \int \varphi(p-q)\psi(q)\, dq.$$

In this event, the inner product is invariant under the Fourier transform; the image of the convolution under the Fourier transform is the product of the images, and vice versa.

The polyharmonic equation

$$\Delta^m u = f \tag{8}$$

with the inverse operator $G(x) * f(x) = u(x)$, where

$$G(x) = \varkappa_{m,n}|x|^{2m-n} \begin{cases} 1, & \text{if } n \text{ odd or } n > 2m, \\ \ln|x|, & \text{if } n \text{ even and } n \leq 2m, \end{cases} \tag{9}$$

is often studied using the generalized inner product

$$D(\varphi, \psi) = \int \sum_{|\alpha|=m} D^\alpha \varphi D^\alpha \psi\, dx = (-1)^m \int \varphi \Delta^m \psi\, dx = (-1)^m \int \psi \Delta^m \varphi\, dx.$$

There are various analogues of Δ^m, G, and $D(\varphi, \psi)$ for discrete functions $\varphi[\beta]$. The finite differences $\widehat{\Delta}^m[\beta]$ are often used, where[2]

$$\widehat{\Delta} * \varphi[\beta] = \sum_{j=1}^{n} \big(\delta[\beta + \delta_j] + \delta[\beta - \delta_j] - 2\delta[\beta]\big) * \varphi[\beta]$$

$$\equiv \sum_{j=1}^{n} \big[\varphi[\beta + \delta_j] + \varphi[\beta - \delta_j]\big] - 2n\varphi[\beta].$$

The inverse operator to $\widehat{\Delta}^m$ is a convolution with some discrete function which behaves like $G[\beta]$ at infinity. As regards the sum

$$\varDelta(\varphi, \psi) = \sum_\beta (\widehat{\Delta}^\alpha \varphi[\beta], \widehat{\Delta}^\alpha \psi[\beta]),$$

[1] Here the integration is carried out over the domain of functions $\varphi(p)$ and $\psi(p)$, i.e., over the torus. – *Ed.*

[2] Here, $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0)$; $\delta[0] = 1$ and $\delta[\beta] = 0$ for $\beta \neq 0$. – *Ed.*

it plays the role of the inner product $D(\varphi, \psi)$. We give here one more generalization of these notions.

Let $G(x)$ be the fundamental solution of the polyharmonic equation, i.e., let G be given by (9). Then we assume that

$$\overset{\sqcap}{G}_{hH}(x) = \Omega_{hH}^{CP}(x \mid G) \text{ and } G_{hH}[\beta] = \Omega_{hH}^{CR}[\beta \mid G] = G(hH\beta), \tag{10}$$

where $|H| = 1$. We call the convolution $G_{hH}[\beta] * \varrho[\beta] = U[\beta]$ a *discrete potential*. This is the natural generalization of the convolution $G(x) * \varrho(x)$.

For the inverse operator $L_{hH}[\beta]$ of the convolution with $G_{hH}[\beta]$ the equality holds

$$G_{hH}[\beta] * L_{hH}[\beta] = \delta[\beta].$$

Passing to lattice functions and using the Fourier transform, we find the function $\widetilde{L}_{hH}(p)$ as

$$\widetilde{L}_{hH}(p) = \left(\left(\frac{1}{2\pi} \right)^{2m} \sum_{\gamma} \frac{h^{-n}}{\left\{ \sum\limits_{j=1}^{n} [p_j - (h^{-1}H^{-1*}\gamma)_j]^2 \right\}^m} \right)^{-1}. \tag{11}$$

From (11) a number of theorems follow.

Theorem 7. *The convolution with $L_{hH}[\beta]$ is an orthogonal operator to all polynomials of degree $2m-1$, i.e.,*

$$L_{hH}[\beta] * \beta^{\alpha} = 0 \quad \textit{for} \quad |\alpha| < 2m.$$

Theorem 8. *The discrete function $L_{hH}[\beta]$ can be written as*

$$L_{hH}[\beta] = h^{n-2m} L_H[\beta],$$

where $L_H[\beta]$ decreases exponentially with $[\beta]$ growing, i.e., $|L_H[\beta]| \le e^{-\eta|\beta|}$.

Theorem 9. *For an arbitrary $2m$-times continuously differentiable function φ we have*

$$h^{-n} L_{hH}(x) * \varphi(x) \overset{\textit{weakly}}{\Longrightarrow} \Delta^m \varphi \quad \textit{as} \quad h \to 0.$$

Theorem 10. *The convolution $\tau_{hH} = L_{hH}(x) * G(x) = \tau_H(h^{-1}\gamma)$ decreases exponentially at infinity, i.e., there exist positive constants K and η such that*

$$|\tau_{hH}(x)| = |\tau_H(h^{-1}x)| \le K e^{-\eta|x|/h}.$$

Theorem 11. *For any two real discrete functions $\varphi[\beta]$ and $\psi[\beta]$, decreasing exponentially at infinity, the bilinear form*

$$D_{hH}(\varphi, \psi) = (\varphi[\beta], L_{hH}[\beta] * \psi[\beta]) = \varphi[\beta] * L_{hH}[\beta] * \psi[-\beta] \Big|_{\beta=0}$$

is symmetric and non-negative; moreover,

$$D_{hH}(\varphi(hH\beta), \psi(hH\beta)) \overset{weakly}{\Longrightarrow} D(\varphi, \psi) \quad as \quad h \to 0.$$

There exist positive constants m and M such that

$$m\Delta(\varphi, \varphi) \leq D_{hH}(\varphi, \varphi) \leq M\Delta(\varphi, \varphi),$$

and m and M are independent of φ.

Theorems 6–10 follow from the fact that $\widetilde{L}_{hH}(p)$ is an analytic function which is non-negative for all real p. Obviously, $\widetilde{L}_{hH}(p) = h^{n-2m}\widetilde{L}_H(hp)$. The product of $\widetilde{L}_{hH}(p)$ and $\widetilde{G}(p)$ is a regular function which is equal to h^n at the coordinate origin.

Theorem 11 follows from the fact that the ratio

$$\widetilde{L}_{hH}(p)/\left(\sum_{j=1}^{n} \sin^2 \frac{(hH^*p)_j}{2}\right)^m$$

is bounded by finite positive limits, and the forms $\Delta(\varphi, \psi)$ and $D_{hH}(\varphi, \psi)$ are reduced by the Fourier transform to the integrals

$$D_{hH}(\varphi, \psi) = \int \widetilde{L}_{hH}(p)\varphi(p)\overline{\psi}(p)\, dp$$

and

$$\Delta(\varphi, \psi) = \int \left(\sum_{j=1}^{n} \sin^2 \frac{(hHp)_j}{2}\right)^m \varphi(p)\overline{\psi}(p)\, dp.$$

From the boundedness of the ratio of the integrands it follows that the ratio of these integrals is also bounded by finite positive limits.

References

1. Titchmarsh, E. C.: Introduction to the Theory of Fourier integrals. Clarendon Press, Oxford (1937).

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